Weyl orbits of embeddings of root systems

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Abstract

In this note, which is an appendix to the article [CT26], we classify the Weyl orbits of embeddings of an ADE root system into another.

For a group G acting on a set X (left action), denote by $[G \curvearrowright X]$ the associated groupoid. For ϕ an ADE root system, normalized so that each root has norm 2, we denote by $\mathrm{Q}(\phi)$ the associated root lattice, and by $\mathrm{W}(\phi)$ the Weyl group of ϕ, a subgroup of the orthogonal group $\mathrm{O}(\phi)$ of $\mathrm{Q}(\phi)$. For ϕ and ψ root systems of type ADE, we denote by $\mathrm{Emb}(\phi, \psi)$ the set of (isometric) embeddings of lattices $\mathrm{Q}(\phi) \to \mathrm{Q}(\psi)$. Our aim is to determine the groupoid

$$\mathcal{G}(\phi, \psi) := [\mathrm{W}(\psi) \curvearrowright \mathrm{Emb}(\phi, \psi)].$$

We may assume ϕ and ψ irreducible, as well as $m \geq n$, otherwise this groupoid is empty. For any isometric embedding of lattices $\iota : U \to V$ we define the *orthogonal of* ι as the sublattice $\iota(U)^\perp$ of V.

Theorem 1. *The number of* $\mathrm{W}(\psi)$*-orbits of proper embeddings* $\mathrm{Q}(\phi) \to \mathrm{Q}(\psi)$ *(i.e. isomorphism classes in* $\mathcal{G}(\phi, \psi)$*), and the root system* R_i *of the orthogonal of an embedding in the* i*-th orbit (ordered arbitrarily), are given in Table 1.*

This result is presumably folklore, but we could not find any reference. The $\mathrm{O}(\psi)$-orbits of sublattices of $\mathrm{Q}(\psi)$ which are isometric to $\mathrm{Q}(\phi)$, or equivalenty the $\mathrm{O}(\phi) \times \mathrm{O}(\psi)$-orbits in $\mathrm{Emb}(\phi, \psi)$, are given in [Ki03, Table 4].[1] Our proof below does not rely on King's table and provides quite more detail, including representatives for each $\mathrm{W}(\psi)$-orbit. It follows from the proof that two embeddings $\mathrm{Q}(\phi) \to \mathrm{Q}(\psi)$ whose orthogonals have isomorphic root systems are in the same $\mathrm{O}(\phi) \times \mathrm{O}(\psi)$-orbit.

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[1] We mention an inaccuracy in the line $S = \mathrm{A}_3$, $T = \mathrm{D}_4$ of that table: the 12 sublattices of D_4 isometric to A_3 form a single $\mathrm{O}(\mathrm{D}_4)$-orbit.

ϕ	ψ	R_1	R_2	R_3
$\mathbf{A}_1$	$\mathbf{A}_m$, $m > 1$	$\mathbf{A}_{m-2}$		
$\mathbf{A}_n$, $n \geq 2$	$\mathbf{A}_m$, $m > n$	$\mathbf{A}_{m-n-1}$	$\mathbf{A}_{m-n-1}$	
$\mathbf{A}_1$	$\mathbf{D}_m$, $m \geq 4$	$\mathbf{A}_1\mathbf{D}_{m-2}$		
$\mathbf{A}_n$, $n \geq 2$, $n \neq 3$	$\mathbf{D}_m$, $m > n+1$	$\mathbf{D}_{m-n-1}$		
$\mathbf{A}_n$, $n \geq 4$	$\mathbf{D}_{n+1}$	$\emptyset$	$\emptyset$	
$\mathbf{A}_3$	$\mathbf{D}_4$	$\emptyset$	$\emptyset$	$\emptyset$
$\mathbf{A}_3$	$\mathbf{D}_m$, $m > 4$	$\mathbf{D}_{m-4}$	$\mathbf{D}_{m-3}$	
$\mathbf{D}_4$	$\mathbf{D}_m$, $m > 4$	$\mathbf{D}_{m-4}$	$\mathbf{D}_{m-4}$	$\mathbf{D}_{m-4}$
$\mathbf{D}_n$, $n > 4$	$\mathbf{D}_m$, $m > n$	$\mathbf{D}_{m-n}$		
$\mathbf{A}_1$	$\mathbf{E}_6$	$\mathbf{A}_5$		
$\mathbf{A}_2$	$\mathbf{E}_6$	$\mathbf{A}_2\mathbf{A}_2$		
$\mathbf{A}_3$	$\mathbf{E}_6$	$\mathbf{A}_1\mathbf{A}_1$		
$\mathbf{A}_4$ or $\mathbf{A}_5$	$\mathbf{E}_6$	$\mathbf{A}_1$	$\mathbf{A}_1$	
$\mathbf{D}_4$	$\mathbf{E}_6$	$\emptyset$		
$\mathbf{D}_5$	$\mathbf{E}_6$	$\emptyset$	$\emptyset$	
$\mathbf{A}_1$	$\mathbf{E}_7$	$\mathbf{D}_6$		
$\mathbf{A}_2$	$\mathbf{E}_7$	$\mathbf{A}_5$		
$\mathbf{A}_3$	$\mathbf{E}_7$	$\mathbf{A}_1\mathbf{A}_3$		
$\mathbf{A}_4$	$\mathbf{E}_7$	$\mathbf{A}_2$		
$\mathbf{A}_5$	$\mathbf{E}_7$	$\mathbf{A}_2$	$\mathbf{A}_1$	
$\mathbf{A}_6$, $\mathbf{E}_6$ or $\mathbf{A}_7$	$\mathbf{E}_7$	$\emptyset$		
$\mathbf{D}_4$	$\mathbf{E}_7$	$\mathbf{A}_1\mathbf{A}_1\mathbf{A}_1$		
$\mathbf{D}_5$	$\mathbf{E}_7$	$\mathbf{A}_1$		
$\mathbf{D}_6$	$\mathbf{E}_7$	$\mathbf{A}_1$	$\mathbf{A}_1$	
$\mathbf{A}_1$	$\mathbf{E}_8$	$\mathbf{E}_7$		
$\mathbf{A}_2$	$\mathbf{E}_8$	$\mathbf{E}_6$		
$\mathbf{A}_3$	$\mathbf{E}_8$	$\mathbf{D}_5$		
$\mathbf{A}_4$	$\mathbf{E}_8$	$\mathbf{A}_4$		
$\mathbf{A}_5$	$\mathbf{E}_8$	$\mathbf{A}_1\mathbf{A}_2$		
$\mathbf{A}_6$	$\mathbf{E}_8$	$\mathbf{A}_1$		
$\mathbf{A}_7$	$\mathbf{E}_8$	$\mathbf{A}_1$	$\emptyset$	
$\mathbf{A}_8$	$\mathbf{E}_8$	$\emptyset$		
$\mathbf{D}_4$	$\mathbf{E}_8$	$\mathbf{D}_4$		
$\mathbf{D}_5$	$\mathbf{E}_8$	$\mathbf{A}_3$		
$\mathbf{D}_6$	$\mathbf{E}_8$	$\mathbf{A}_1\mathbf{A}_1$		
$\mathbf{D}_7$	$\mathbf{E}_8$	$\emptyset$		
$\mathbf{D}_8$	$\mathbf{E}_8$	$\emptyset$	$\emptyset$	
$\mathbf{E}_6$	$\mathbf{E}_8$	$\mathbf{A}_2$		
$\mathbf{E}_7$	$\mathbf{E}_8$	$\mathbf{A}_1$		

Table 1: Weyl orbits of proper embeddings and root systems of their orthogonal

Remark 1. *(i) For an object $\iota : \mathrm{Q}(\phi) \to \mathrm{Q}(\psi)$ of this groupoid its automorphism group is equal to the pointwise stabilizer of $\iota(\phi)$ in $\mathrm{W}(\psi)$. By* [Bou81, Ch. V, §3.3, Prop. 2]*, this stabilizer coincides with the Weyl group $\mathrm{W}(R)$ of the root system R of the orthogonal of ι. In particular, the $\mathrm{W}(\psi)$-orbit of ι has cardinality $|\mathrm{W}(\psi)|/|\mathrm{W}(R)|$, a quantity which can be deduced from Table 1.*

(ii) The case where ι is not proper (i.e. is an isomorphism) is easy: we may assume $\psi = \phi$ and then $\mathrm{Emb}(\phi,\phi) = \mathrm{O}(\phi)$ and isomorphism classes in $\mathcal{G}(\phi,\phi)$ are in bijection with the reduced isometry group $\mathrm{O}(\phi)/\mathrm{W}(\phi)$, which is also the group of automorphisms of the Dynkin diagram of ϕ. Explicitly, this group is trivial for $\mathbf{A}_1$, $\mathbf{E}_7$ and $\mathbf{E}_8$, has two elements for $(\mathbf{A}_n)_{n>1}$, $(\mathbf{D}_n)_{n>4}$ and $\mathbf{E}_6$, and is isomorphic to $\mathfrak{S}_3$ for $\mathbf{D}_4$.

Notation: Our convention is that bold letters denote root systems and roman letters denote root lattices. We denote either by $\phi\psi$ or $\phi \sqcup \psi$ the disjoint (orthogonal) union of the root systems ϕ and ψ. For $n \geq 0$ we denote by $\mathrm{I}_n = \mathbb{Z}^n$ the cubic (or standard) lattice, with orthonormal standard basis $e_1, \ldots, e_n$. For $n \geq 1$ we realize A_n as $\{\sum_{i=1}^{n+1} x_i e_i \in \mathrm{I}_{n+1} \mid \sum_i x_i = 0\}$. We also define D_n as the largest even sublattice in I_n. For $n \geq 4$ this is the root lattice of $\mathbf{D}_n$ and we choose the simple roots $(e_i - e_{i+1})_{1 \leq i < n}$ and $e_{n-1} + e_n$. We have isometries $\mathrm{D}_1 \simeq (4)$, $\mathrm{D}_2 \simeq \mathrm{A}_1 \perp \mathrm{A}_1$ and $\mathrm{D}_3 \simeq \mathrm{A}_3$. It is thus convenient to set $\mathbf{D}_2 = \mathbf{A}_1\mathbf{A}_1$ and $\mathbf{D}_3 = \mathbf{A}_3$. In the last three columns of Table 1, we use the conventions $\mathbf{A}_0 = \mathbf{D}_0 = \mathbf{D}_1 = \emptyset$, $\mathbf{D}_2 = \mathbf{A}_1\mathbf{A}_1$ and $\mathbf{D}_3 = \mathbf{A}_3$.

We now start the proof of Theorem 1, and fix irreducible root systems ϕ and ψ of respective ranks n and m with $n \leq m$. We consider first the case $\phi = \mathbf{A}_n$. It is well-known that $\mathcal{G}(\mathbf{A}_1, \psi)$ has one isomorphism class for any ψ.

- $\phi = \mathbf{A}_1$ and $\psi = \mathbf{A}_m$. We have $|\mathcal{G}(\mathbf{A}_1, \mathbf{A}_m)| = 1$ with representative $\alpha^{\mathbf{A}_m}_{\mathbf{A}_1} : e_1 - e_2 \mapsto e_1 - e_2$. The root system of the orthogonal of $\alpha^{\mathbf{A}_m}_{\mathbf{A}_1}$ is empty if $m \leq 2$, isomorphic to $\mathbf{A}_{m-2}$ if $m > 2$.

- $\phi = \mathbf{A}_2$ and $\psi = \mathbf{A}_m$. Consider $\iota \in \mathrm{Emb}(\phi, \psi)$. We may assume $\iota(e_1 - e_2) = e_1 - e_2$. Then ι maps $e_2 - e_3$ to $e_2 - e_j$ with $j > 2$ or to $e_j - e_1$ with $j > 2$. In both cases we may assume $j = 3$. The two cases define distinct $\mathrm{W}(\mathbf{A}_m)$-orbits, with representatives $\alpha^{\mathbf{A}_m}_{\mathbf{A}_2} : e_i - e_j \mapsto e_i - e_j$ and $\beta^{\mathbf{A}_m}_{\mathbf{A}_2} : e_i - e_j \mapsto e_j - e_i$. They are swapped by $-\mathrm{id}$ (source or target). The orthogonal of these embeddings has empty root system if $m \leq 3$, isomorphic to $\mathbf{A}_{m-3}$ if $m > 3$.

- $\phi = \mathbf{A}_n$ for $n \geq 2$ and $\psi = \mathbf{A}_m$. We proceed by induction on $n \geq 2$ to show that $\mathcal{G}(\mathbf{A}_n, \mathbf{A}_m)$ has two isomorphism classes $\alpha^{\mathbf{A}_m}_{\mathbf{A}_n}$ mapping $e_i - e_j$ to $e_i - e_j$ and $\beta^{\mathbf{A}_m}_{\mathbf{A}_n}$, characterized by their restriction to $\mathbf{A}_2$ and swapped by $-\mathrm{id}$ (source or target). The initial case $n = 2$ is the previous point. So assume $n > 2$ and try to extend $\alpha^{\mathbf{A}_m}_{\mathbf{A}_{n-1}}$ to $\mathbf{A}_n$. We see that we have to map $e_n - e_{n+1}$ to $e_n - e_j$ with $j > n$, and letting $\mathrm{Aut}_{\mathcal{G}(\mathbf{A}_{n-1},\mathbf{A}_m)}(\alpha^{\mathbf{A}_m}_{\mathbf{A}_{n-1}}) = \mathfrak{S}(\{n+1, \ldots, m+1\})$ act we see that we may assume $j = n+1$. For $\beta^{\mathbf{A}_m}_{\mathbf{A}_{n-1}}$ we reduce to the previous case by applying $-\mathrm{id}_{\mathrm{A}_m}$.

  The orthogonal of any $\iota \in \mathrm{Emb}(\mathbf{A}_n, \mathbf{A}_m)$ has empty root system if $m \in \{n, n+1\}$, isomorphic to $\mathbf{A}_{m-n-1}$ if $m > n+1$.

- $\phi = \mathbf{A}_1$ and $\psi = \mathbf{D}_m$ ($m \geq 4$). We have $|\mathcal{G}(\mathbf{A}_1, \mathbf{D}_m)| = 1$ with representative $\alpha_{\mathbf{A}_1}^{\mathbf{D}_m} : e_1 - e_2 \mapsto e_1 - e_2$, and orthogonal isometric to $\mathrm{A}_1 \perp \mathrm{D}_{m-2}$.

- $\phi = \mathbf{A}_2$ and $\psi = \mathbf{D}_m$. To extend $\alpha_{\mathbf{A}_1}^{\mathbf{D}_m}$ we have to map $e_2 - e_3$ to $e_2 \pm e_j$ or to $-e_1 \pm e_j$ for some $j \geq 3$. (Note that there is an element of $\mathrm{W}(\mathbf{D}_m)$ fixing e_1 and e_2 and mapping e_j to $-e_j$ because $m > 3$.) The reflection $s_{e_1+e_2}$ swaps these two cases, and letting $\mathrm{W}(\mathbf{D}_{m-2})$ act we conclude $|\mathcal{G}(\mathbf{A}_2, \mathbf{D}_m)| = 1$ with representative $\alpha_{\mathbf{A}_2}^{\mathbf{D}_m} : e_i - e_j \mapsto e_i - e_j$. Its orthogonal has empty root system if $m = 4$, isomorphic to $\mathbf{D}_{m-3}$ if $m > 4$.

- $\phi = \mathbf{A}_3$ and $\psi = \mathbf{D}_m$. To extend $\alpha_{\mathbf{A}_2}^{\mathbf{D}_m}$ we have to map $e_3 - e_4$ to $-e_2 - e_1$ or to $e_3 \pm e_j$ for some $j \geq 4$. In the latter case if $m > 4$ we let $\mathrm{W}(\mathbf{D}_{m-3})$ act to reduce to $e_3 - e_4$.

  It is clear from the case $\phi = \mathbf{A}_1$ that $\alpha_{\mathbf{A}_3}^{\mathbf{D}_m} : e_i - e_j \mapsto e_i - e_j$ and $\beta_{\mathbf{A}_3}^{\mathbf{D}_m} : e_3 - e_4 \mapsto -e_2 - e_1$ are not isomorphic in $\mathcal{G}(\mathbf{A}_3, \mathbf{D}_m)$ (they map the root $e_3 - e_4$ to different irreducible components of the root system orthogonal to the image of $e_1 - e_2$). The orthogonal of $\alpha_{\mathbf{A}_3}^{\mathbf{D}_m}$ has empty root system if $m \leq 5$, isomorphic to $\mathbf{D}_{m-4}$ if $m > 5$. The orthogonal of $\beta_{\mathbf{A}_3}^{\mathbf{D}_m}$ has empty root system if $m \leq 4$, isomorphic to $\mathbf{D}_{m-3}$ if $m > 4$. For $m > 4$ we thus have $|\mathcal{G}(\mathbf{A}_3, \mathbf{D}_m)| = 2$ and both isomorphism classes are fixed by the non-trivial outer automorphism of $\mathbf{D}_m$. For $m = 4$ we have a third class represented by $\gamma_{\mathbf{A}_3}^{\mathbf{D}_4} : e_3 - e_4 \mapsto e_3 + e_4$, and the natural morphism $\mathrm{Out}(\mathbf{D}_4) \to \mathfrak{S}(\mathcal{G}(\mathbf{A}_3, \mathbf{D}_4)/\sim)$ is an isomorphism.

- $\phi = \mathbf{A}_n$ for $n \geq 4$ and $\psi = \mathbf{D}_m$. We first consider the case where $m > n$. Denote by $\alpha_{\mathbf{A}_n}^{\mathbf{D}_m}$ the object of $\mathcal{G}(\mathbf{A}_n, \mathbf{D}_m)$ mapping $e_i - e_j$ to $e_i - e_j$. The orthogonal of $\alpha_{\mathbf{A}_n}^{\mathbf{D}_m}$ has empty root system if $m \leq n + 2$, isomorphic to $\mathbf{D}_{m-n-1}$ if $m > n + 2$. We prove by induction on n that we have just one isomorphism class if $m > n + 1$, and two isomorphism classes if $m = n + 1$, swapped by the non-trivial outer automorphism of $\mathbf{D}_m$.

  We first observe that $\beta_{\mathbf{A}_3}^{\mathbf{D}_m}$ cannot be extended to $\mathbf{A}_4$. For $n \geq 4$ and $m \geq n + 1$ to extend $\alpha_{\mathbf{A}_{n-1}}^{\mathbf{D}_m}$ to $\mathbf{A}_n$ we have to map $e_n - e_{n+1}$ to $e_n \pm e_j$ for some $j > n$. If $m > n + 1$ we let $\mathrm{Aut}_{\mathcal{G}(\mathbf{A}_{n-1}, \mathbf{D}_m)}(\alpha_{\mathbf{A}_{n-1}}^{\mathbf{D}_m}) \simeq \mathrm{W}(\mathbf{D}_{m-n})$ act to reduce to $e_n - e_{n+1}$. For $m = n+1$ the group $\mathrm{Aut}_{\mathcal{G}(\mathbf{A}_{n-1}, \mathbf{D}_m)}(\alpha_{\mathbf{A}_{n-1}}^{\mathbf{D}_m})$ is trivial so the two possibilities $e_n \pm e_{n+1}$ yield non-isomorphic objects of $\mathcal{G}(\mathbf{A}_n, \mathbf{D}_m)$, which are clearly swapped by the non-trivial outer automorphism of $\mathbf{D}_m$.

  Finally the groupoid $\mathcal{G}(\mathbf{A}_n, \mathbf{D}_m)$ is empty if $m = n$: it is enough to check that $\alpha_{\mathbf{A}_{n-1}}^{\mathbf{D}_n}$ cannot be extended to $\mathbf{A}_n$ (say along $\alpha_{\mathbf{A}_{n-1}}^{\mathbf{A}_n}$), which is elementary.

- $\phi = \mathbf{A}_1$ and $\psi = \mathbf{E}_8$. We represent the root lattice E_8 as

$$\mathrm{D}_8 \sqcup \left\{ (x_i)_{1 \leq i \leq 8} \,\middle|\, x_i \in \frac{1}{2}\mathbb{Z} \smallsetminus \mathbb{Z},\ \sum_i x_i \in 2\mathbb{Z} \right\}.$$

  and define $\mathbf{E}_8$ as its root system. The unique isomorphism class of $\mathcal{G}(\mathbf{A}_1, \mathbf{E}_8)$ is represented by $\alpha_{\mathbf{A}_1}^{\mathbf{E}_8} : e_1 - e_2 \mapsto e_1 - e_2$. It is well-known that the orthogonal of $\mathbf{A}_1$ in E_8 is a root lattice of type $\mathbf{E}_7$.

- $\phi = \mathbf{A}_2$ and $\psi = \mathbf{E}_8$. The root lattice A_2 has no strict overlattice which is integral, and the natural functor from $[\mathrm{W}(\mathbf{E}_8) \curvearrowright *]$ to the genus of even unimodular lattices of rank 8 is an equivalence, so $\mathcal{G}(\mathbf{A}_2, \mathbf{E}_8)$ is equivalent to the groupoid of pairs (L, t) where L is an even lattice of rank 6 and[2] $t : \mathrm{qres}\ L \xrightarrow{\sim} -\mathrm{qres}\ \mathrm{A}_2$. We know that the genus of lattices L such that such a trivialization t exists has just one isomorphism class, the root lattice of $\mathbf{E}_6$, and we have a short exact sequence

$$1 \to \mathrm{W}(\mathbf{E}_6) \to \mathrm{O}(\mathrm{E}_6) \to \mathrm{O}(\mathrm{qres}\ \mathrm{E}_6) \to 1$$

so we conclude that $\mathcal{G}(\mathbf{A}_2, \mathbf{E}_8)$ has just one isomorphism class, represented by $\alpha_{\mathbf{A}_2}^{\mathbf{E}_8} : e_i - e_j \mapsto e_i - e_j$.

- $\phi = \mathbf{A}_3$ and $\psi = \mathbf{E}_8$. Again the root lattice A_3 has no strict overlattice which is integral and even and so $\mathcal{G}(\mathbf{A}_3, \mathbf{E}_8)$ is equivalent to the groupoid of pairs (L, t) where L is an even lattice of rank 5 and $t : \mathrm{qres}\ L \xrightarrow{\sim} -\mathrm{qres}\ \mathrm{A}_3$. The genus of such lattices L is equivalent to the genus of (odd) unimodular lattice of rank 5 (such an L admits a unique overlattice L' which is integral and unimodular, and L is the even part of L'), which has just one isomorphism class so L is isomorphic to the root lattice D_5. Again we have a short exact sequence
$$1 \to \mathrm{W}(\mathbf{D}_6) \to \mathrm{O}(\mathrm{D}_5) \to \mathrm{O}(\mathrm{qres}\ \mathrm{D}_5) \to 1$$
so we conclude that $\mathcal{G}(\mathbf{A}_3, \mathbf{E}_8)$ has just one isomorphism class, represented by $\alpha_{\mathbf{A}_3}^{\mathbf{E}_8} : e_i - e_j \mapsto e_i - e_j$.

- $\phi = \mathbf{A}_4$ and $\psi = \mathbf{E}_8$. The same argument as in the previous two points applies: $\mathcal{G}(\mathbf{A}_4, \mathbf{E}_8)$ is equivalent to the groupoid of pairs (L, t) where L is an even lattice of rank 4 and $t : \mathrm{qres}\ L \xrightarrow{\sim} -\mathrm{qres}\ \mathrm{A}_4$. We know that the genus of even lattices of rank 4 and determinant 5 has just one isomorphism class represented by the root lattice A_4, and we have a short exact sequence as in the previous two cases, so we conclude that $\mathcal{G}(\mathbf{A}_4, \mathbf{E}_8)$ has just one isomorphism class, represented by $\alpha_{\mathbf{A}_4}^{\mathbf{E}_8} : e_i - e_j \mapsto e_i - e_j$.

- $\phi = \mathbf{A}_5$ and $\psi = \mathbf{E}_8$. This case is similar to the previous three cases: the relevant genus is that of $\mathrm{A}_1 \perp \mathrm{A}_2$ which again has just one isomorphism class, and we conclude that $\mathcal{G}(\mathbf{A}_5, \mathbf{E}_8)$ has just one isomorphism class, represented by $\alpha_{\mathbf{A}_5}^{\mathbf{E}_8} : e_i - e_j \mapsto e_i - e_j$.

- $\phi = \mathbf{A}_6$ and $\psi = \mathbf{E}_8$. This case is similar to the previous four cases: the relevant genus is that of $\begin{pmatrix} 2 & -1 \\ -1 & 4 \end{pmatrix}$ (i.e. even lattices of rank 2 and determinant 7) which has root system isomorphic to $\mathbf{A}_1$), it has only one isomorphism class. Again the automorphism group of this lattice acts transitively on the set of trivializations of its quadratic residue (which has two elements) and we conclude that $\mathcal{G}(\mathbf{A}_6, \mathbf{E}_8)$ has just one isomorphism class, represented by $\alpha_{\mathbf{A}_6}^{\mathbf{E}_8} : e_i - e_j \mapsto e_i - e_j$.

- $\phi = \mathbf{A}_7$ and $\psi = \mathbf{E}_8$. Now $\mathbf{A}_7$ does admit a strict overlattice which is integral and even, so the argument of the previous points does not apply. We try to extend $\alpha_{\mathbf{A}_6}^{\mathbf{E}_8}$ to $\mathbf{A}_7$: we have to map $e_7 - e_8$ to a root of the form $(x, x, x, x, x, x, x+1, y)$, so we either have $x = 0$ and $y \in \{\pm 1\}$ or $x = -1/2$ and $y = 1/2$. The two cases $(x, y) \in \{(0, 1), (-1/2, 1/2)\}$ are swapped by

[2]See [CT26] for the notation qres.

the reflection defined by the root $(1/2,\ldots,1/2)$ orthogonal to $\alpha^{\mathbf{E}_8}_{\mathbf{A}_6}(\mathbf{A}_6)$, so $\mathcal{G}(\mathbf{A}_7,\mathbf{E}_8)$ has two isomorphism classes, represented by $\alpha^{\mathbf{E}_8}_{\mathbf{A}_7} : e_i-e_j \mapsto e_i-e_j$ and the extension $\beta^{\mathbf{E}_8}_{\mathbf{A}_7}$ of $\alpha^{\mathbf{E}_8}_{\mathbf{A}_6}$ sending $e_7-e_8 \mapsto e_7+e_8$.

The orthogonal of $\alpha^{\mathbf{E}_8}_{\mathbf{A}_7}$ is $\mathbb{Z}(1/2,\ldots,1/2) \simeq \mathrm{A}_1$ (in particular $\alpha^{\mathbf{E}_8}_{\mathbf{A}_7}(\mathrm{A}_7)$ is not saturated in $\mathbf{E}_8$, its saturation is isomorphic to E_7). The orthogonal of $\beta^{\mathbf{E}_8}_{\mathbf{A}_7}$ has no root and so it has a basis vector of length 8.

- $\phi = \mathbf{A}_8$ and $\psi = \mathbf{E}_8$. We try to extend $\alpha^{\mathbf{E}_8}_{\mathbf{A}_7}$ to $\mathbf{A}_8$: we have to map e_8-e_9 to a root orthogonal to $\alpha^{\mathbf{E}_8}_{\mathbf{A}_6}(\mathbf{A}_6)$, i.e. to $\pm(1/2,\ldots,1/2)$ which is orthogonal to $\alpha^{\mathbf{E}_8}_{\mathbf{A}_7}(e_7-e_8)$, a contradiction. Now $\beta^{\mathbf{E}_8}_{\mathbf{A}_7}$ admits a unique extension to $\mathbf{A}_8$, mapping e_8-e_9 to $(-1/2,\ldots,-1/2)$.

- $\phi = \mathbf{A}_1$ and $\psi = \mathbf{E}_7$. We realize $\mathbf{E}_7$ as the orthogonal of some (arbitrary) $\mathbf{A}_1$ in $\mathbf{E}_8$ and set $\mathrm{E}_7 = \mathrm{Q}(\mathbf{E}_7)$. For any ϕ will repeatedly use the equivalence

$$\mathcal{G}(\phi,\mathbf{E}_7) \simeq \mathcal{G}(\phi \sqcup \mathbf{A}_1,\mathbf{E}_8),$$

which follows from $|\mathcal{G}(\mathbf{A}_1,\mathbf{E}_8)| = 1$ and Remark 1 (i). The groupoid on the right is also equivalent to that of pairs $(L, \mathrm{Q}(\phi) \perp \mathrm{A}_1 \hookrightarrow L)$ where L is an even unimodular lattice of rank 8. For $\phi = \mathbf{A}_1$ this groupoid is equivalent to the groupoid of pairs (M,t) where M is an even lattice of rank 6 and t is an isometry $\mathrm{qres}\, M \xrightarrow{\sim} -\mathrm{qres}\,(\mathrm{A}_1 \perp \mathrm{A}_1)$. The genus of those M is that of D_6 and it is equivalent to that of (odd) unimodular lattices of rank 6, which has only one isomorphism class. The group $\mathrm{O}(\mathrm{D}_6)$ acts transitively on the set of trivializations of $\mathrm{qres}\,\mathrm{D}_6$, so $\mathcal{G}(\mathbf{A}_1,\mathbf{E}_7)$ has only one isomorphism class and the orthogonal of any object ι is isomorphic to the root lattice D_6.

- $\phi = \mathbf{A}_2$ and $\psi = \mathbf{E}_7$. We have seen that there is a unique Weyl-orbit of embeddings $\mathbf{A}_2 \to \mathbf{E}_8$, with orthogonals $\simeq \mathrm{E}_6$. We have thus

$$\mathcal{G}(\mathbf{A}_2,\mathbf{E}_7) \simeq \mathcal{G}(\mathbf{A}_2 \sqcup \mathbf{A}_1,\mathbf{E}_8) \simeq \mathcal{G}(\mathbf{A}_1,\mathbf{E}_6)$$

and so this groupoid has just one equivalence class. The orthogonal of any $\iota \in \mathrm{Emb}(\mathbf{A}_2,\mathbf{E}_7)$ is thus isomorphic to the root lattice A_5 (see the determination of $\mathcal{G}(\mathbf{A}_5,\mathbf{E}_8)$).

- $\phi = \mathbf{A}_3$ and $\psi = \mathbf{E}_7$. We have seen that there is a unique Weyl-orbit of embeddings $\mathbf{A}_3 \to \mathbf{E}_8$, with orthogonals $\simeq \mathrm{D}_5$. Arguing as above we have

$$\mathcal{G}(\mathbf{A}_3,\mathbf{E}_7) \simeq \mathcal{G}(\mathbf{A}_3 \sqcup \mathbf{A}_1,\mathbf{E}_8) \simeq \mathcal{G}(\mathbf{A}_1,\mathbf{D}_5)$$

so this groupoid has just one isomorphism class. The orthogonal of any $\iota \in \mathrm{Emb}(\mathbf{A}_3,\mathbf{E}_7)$ is thus isomorphic to that of any A_1 in D_5, *i.e.* to $\mathrm{A}_1 \perp \mathrm{A}_3$.

- $\phi = \mathbf{A}_4$ and $\psi = \mathbf{E}_7$. We have seen that there is a unique Weyl-orbit of embeddings $\mathbf{A}_4 \to \mathbf{E}_8$, with orthogonals $\simeq \mathrm{A}_4$. We have thus

$$\mathcal{G}(\mathbf{A}_4,\mathbf{E}_7) \simeq \mathcal{G}(\mathbf{A}_4 \sqcup \mathbf{A}_1,\mathbf{E}_8) \simeq \mathcal{G}(\mathbf{A}_1,\mathbf{A}_4)$$

so this groupoid has just one isomorphism class. The orthogonal of any $\iota \in \mathrm{Emb}(\mathbf{A}_4,\mathbf{E}_7)$ is even of determinant 10 and rank 3; there is a unique isomorphism class of such lattices, with root system $\simeq \mathbf{A}_2$.

- $\phi = \mathbf{A}_5$ and $\psi = \mathbf{E}_7$. We have seen that there is a unique Weyl-orbit of embeddings $\mathbf{A}_5 \to \mathbf{E}_8$, with orthogonals $\simeq \mathrm{A}_2 \perp \mathrm{A}_1$. We deduce

$$\mathcal{G}(\mathbf{A}_5, \mathbf{E}_7) \simeq \mathcal{G}(\mathbf{A}_5 \sqcup \mathbf{A}_1, \mathbf{E}_8) \simeq \mathcal{G}(\mathbf{A}_1, \mathbf{A}_1) \sqcup \mathcal{G}(\mathbf{A}_1, \mathbf{A}_2)$$

so we have two isomorphism classes, one with orthogonal $\simeq \mathrm{A}_2$ and one with orthogonal $\simeq \mathrm{A}_1 \perp (6)$ (the orthogonal of an A_1 in A_2 being $\simeq (6)$).

- $\phi = \mathbf{A}_6$ and $\psi = \mathbf{E}_7$. We have seen that there is a unique Weyl-orbit of embeddings $\mathbf{A}_6 \to \mathbf{E}_8$, with orthogonals isomorphic to the rank 2 even lattice with determinant 7 (and root system $\simeq \mathbf{A}_1$). We have

$$\mathcal{G}(\mathbf{A}_6, \mathbf{E}_7) \simeq \mathcal{G}(\mathbf{A}_6 \sqcup \mathbf{A}_1, \mathbf{E}_8) \simeq \mathcal{G}(\mathbf{A}_1, \mathbf{A}_1) \simeq *$$

and the orthogonal of $\iota \in \mathrm{Emb}(\mathbf{A}_6, \mathbf{E}_7)$ is isomorphic to (14). The restriction functor $\mathcal{G}(\mathbf{A}_6, \mathbf{E}_7) \to \mathcal{G}(\mathbf{A}_5, \mathbf{E}_7)$ has essential image the class of embeddings $\mathbf{A}_5 \to \mathbf{E}_7$ with orthogonal isomorphic to $\mathrm{A}_1 \perp (6)$ (we have $\mathbb{Z}_2 \otimes \mathrm{E}_7 \simeq \mathbb{Z}_2 \otimes \mathrm{A}_6 \perp (14)$ and so the orthogonal of $\mathbb{Z}_2 \otimes \mathrm{A}_5$ in $\mathbb{Z}_2 \otimes \mathrm{E}_7$ is not self-dual).

- $\phi = \mathbf{A}_7$ and $\psi = \mathbf{E}_7$. We have

$$\mathcal{G}(\mathbf{A}_7, \mathbf{E}_7) \simeq \mathcal{G}(\mathbf{A}_7 \sqcup \mathbf{A}_1, \mathbf{E}_8) \simeq \mathcal{G}(\mathbf{A}_1, \mathbf{A}_1) \simeq *$$

(only one of the two isomorphism classes in $\mathcal{G}(\mathbf{A}_7, \mathbf{E}_8)$ is such that the orthogonal has a root: the class of $\alpha^{\mathbf{E}_8}_{\mathbf{A}_7}$).

- $\phi = \mathbf{A}_1$ and $\psi = \mathbf{E}_6$. We realize $\mathbf{E}_6$ as the orthogonal of an arbitrary $\mathbf{A}_2$ in $\mathbf{E}_8$ and set $\mathrm{E}_6 = \mathrm{Q}(\mathbf{E}_6)$. For any ϕ will repeatedly use the equivalence

$$\mathcal{G}(\phi, \mathbf{E}_6) \simeq \mathcal{G}(\phi \sqcup \mathbf{A}_2, \mathbf{E}_8),$$

which follows from $|\mathcal{G}(\mathbf{A}_2, \mathbf{E}_8)| = 1$ and Remark 1 (i). The groupoid $\mathcal{G}(\mathbf{A}_1 \sqcup \mathbf{A}_2, \mathbf{E}_8)$ was already determined for the case $\phi = \mathbf{A}_2$ and $\psi = \mathbf{E}_7$.

- $\phi = \mathbf{A}_2$ and $\psi = \mathbf{E}_6$. We have

$$\mathcal{G}(\mathbf{A}_2, \mathbf{E}_6) \simeq \mathcal{G}(\mathbf{A}_2 \sqcup \mathbf{A}_2, \mathbf{E}_8)$$

and this groupoid is isomorphic to the groupoid of pairs (L, t) where L is an even lattice of rank 4 and $t : \mathrm{qres}\, L \xrightarrow{\sim} -\mathrm{qres}\,(\mathrm{A}_2 \perp \mathrm{A}_2)$. The corresponding genus has just one isomorphism class, represented by $\mathrm{A}_2 \perp \mathrm{A}_2$, and the automorphism group of this lattice acts transitively on the set of trivializations of its quadratic residue. Thus $\mathcal{G}(\mathbf{A}_2, \mathbf{E}_6)$ has only one isomorphism class[3] and the orthogonal of any $\iota \in \mathrm{Emb}(\mathbf{A}_2, \mathbf{E}_6)$ is isomorphic to $\mathrm{A}_2 \perp \mathrm{A}_2$.

- $\phi = \mathbf{A}_3$ and $\psi = \mathbf{E}_6$. We have seen that there is a unique Weyl-orbit of embeddings $\mathbf{A}_3 \to \mathbf{E}_8$, with orthogonals $\simeq \mathrm{D}_5$. We have thus

$$\mathcal{G}(\mathbf{A}_3, \mathbf{E}_6) \simeq \mathcal{G}(\mathbf{A}_3 \sqcup \mathbf{A}_2, \mathbf{E}_8) \simeq \mathcal{G}(\mathbf{A}_2, \mathbf{D}_5)$$

which has only one isomorphism class. By the determination of $\mathcal{G}(\mathbf{A}_2, \mathbf{D}_5)$, the orthogonal of any $\iota \in \mathrm{Emb}(\mathbf{A}_3, \mathbf{E}_6)$ has root system $\simeq \mathbf{D}_2 \simeq \mathbf{A}_1 \sqcup \mathbf{A}_1$.

[3]This can also be checked by direct computation in $\mathbf{E}_6$.

- $\phi = \mathbf{A}_4$ and $\psi = \mathbf{E}_6$. We have seen that there is a unique Weyl-orbit of embeddings $\mathbf{A}_4 \to \mathbf{E}_8$, with orthogonals $\simeq \mathrm{A}_4$. We have thus

$$\mathcal{G}(\mathbf{A}_4, \mathbf{E}_6) \simeq \mathcal{G}(\mathbf{A}_4 \sqcup \mathbf{A}_2, \mathbf{E}_8) \simeq \mathcal{G}(\mathbf{A}_2, \mathbf{A}_4)$$

  and this groupoid has two isomorphism classes. The orthogonal of any embedding in $\mathrm{Emb}(\mathbf{A}_4, \mathbf{E}_6)$ is isomorphic to $\begin{pmatrix} 2 & -1 \\ -1 & 8 \end{pmatrix}$, with root system $\simeq \mathbf{A}_1$. As we have $-\mathrm{id} \in \mathrm{W}(\mathbf{E}_8)$, the action of $-\mathrm{id}_{\mathrm{A}_4}$ on $\mathcal{G}(\mathbf{A}_4, \mathbf{E}_6)/\sim$ coincides under the isomorphisms above with that of $-\mathrm{id}_{\mathrm{A}_2}$ on $\mathcal{G}(\mathbf{A}_2, \mathbf{A}_4)/\sim$, so the two orbits are swapped by $-\mathrm{id}_{\mathrm{A}_4}$ by the case $\phi = \mathbf{A}_2$ and $\psi = \mathbf{A}_4$. We deduce that the outer automorphism of $\mathbf{E}_6$ (represented by $-\mathrm{id}_{\mathrm{E}_6}$) also swaps the two isomorphism classes in $\mathcal{G}(\mathbf{A}_4, \mathbf{E}_6)$.

- $\phi = \mathbf{A}_5$ and $\psi = \mathbf{E}_6$. We have seen that there is a unique Weyl-orbit of embeddings $\mathbf{A}_5 \to \mathbf{E}_8$, with orthogonals $\simeq \mathrm{A}_2 \perp \mathrm{A}_1$. We have thus

$$\mathcal{G}(\mathbf{A}_5, \mathbf{E}_6) \simeq \mathcal{G}(\mathbf{A}_5 \sqcup \mathbf{A}_2, \mathbf{E}_8) \simeq \mathcal{G}(\mathbf{A}_2, \mathbf{A}_1) \sqcup \mathcal{G}(\mathbf{A}_2, \mathbf{A}_2) \simeq \mathcal{G}(\mathbf{A}_2, \mathbf{A}_2)$$

  and this groupoid has two isomorphism classes swapped by both $-\mathrm{id}_{\mathrm{A}_5}$ and $-\mathrm{id}_{\mathrm{E}_6}$ (same argument as above). The orthogonal of any $\iota \in \mathrm{Emb}(\mathbf{A}_5, \mathbf{E}_6)$ is isomorphic to A_1.

- $\phi = \mathbf{A}_6$ and $\psi = \mathbf{E}_6$. The ranks are equal and the quotient of determinants is not the square of an integer ($7/3$ is not even an integer) so the groupoid $\mathcal{G}(\mathbf{A}_6, \mathbf{E}_6)$ is empty.

For $\phi = \mathbf{D}_n$ (with $n \geq 4$) we often consider extensions of (already classified) embeddings $\mathrm{A}_{n-1} \to \mathrm{Q}(\psi)$ along $\alpha^{\mathbf{D}_n}_{\mathbf{A}_{n-1}} : \mathrm{A}_{n-1} \to \mathrm{D}_n$.

- $\phi = \mathbf{D}_n$ and $\psi = \mathbf{A}_m$. It is elementary to check that one cannot extend $\alpha^{\mathbf{A}_n}_{\mathbf{A}_{n-1}}$ along $\alpha^{\mathbf{D}_n}_{\mathbf{A}_{n-1}}$, and so one cannot extend $\beta^{\mathbf{A}_n}_{\mathbf{A}_{n-1}}$ either, so $\mathcal{G}(\mathbf{D}_n, \mathbf{A}_m)$ is empty.

- $\phi = \mathbf{D}_n$ and $\psi = \mathbf{D}_m$ with $m > n$, in particular $m > 4$. Each class in $\mathcal{G}(\mathbf{D}_n, \mathbf{D}_m)$ contains ϕ such that $\phi \circ \alpha^{\mathbf{D}_n}_{\mathbf{A}_{n-1}}$ is equal to $\alpha^{\mathbf{D}_m}_{\mathbf{A}_{n-1}}$, or to $\beta^{\mathbf{D}_m}_{\mathbf{A}_3}$ if $n = 4$. Elementary computations show that $\alpha^{\mathbf{D}_m}_{\mathbf{A}_{n-1}}$ can be extended only:

  - by $e_{n-1} + e_n \mapsto e_{n-1} + e_n$,
  - or by $e_{n-1} + e_n \mapsto -e_2 - e_1$ if $n = 4$.

  Elementary computations show that $\beta^{\mathbf{D}_m}_{\mathbf{A}_3}$ can be extended by $e_3 + e_4 \mapsto e_3 \pm e_i$ for some $i > 3$, which are all in the same $\mathrm{W}(\mathbf{D}_m)$-orbit. In summary:

  - If $n > 4$ then we have a single class in $\mathcal{G}(\mathbf{D}_n, \mathbf{D}_m)$, with orthogonals isomorphic to D_{m-n},
  - $\mathcal{G}(\mathbf{D}_4, \mathbf{D}_m)$ has three classes, permuted transitively by $\mathrm{O}(\mathrm{D}_4)$, distinguished by which pair among the three distinguished simple roots of $\mathbf{D}_4$ are mapped to roots in $\mathbf{D}_m$ with the same support. In any case the orthogonal of an embedding is isomorphic to D_{m-4}.

- $\phi = \mathbf{D}_8$ and $\psi = \mathbf{E}_8$. The lattice D_8 has two overlattices isomorphic to E_8, so there are two classes in $\mathcal{G}(\mathbf{D}_8, \mathbf{E}_8)$, swapped by the action of $\mathrm{O}(\mathbf{D}_8)/\mathrm{W}(\mathbf{D}_8)$.

- $\phi = \mathbf{D}_n$ for $4 \leq n \leq 7$ and $\psi = \mathbf{E}_8$. The lattice D_n has no strict overlattice which is integral and even so $\mathcal{G}(\mathbf{D}_n, \mathbf{E}_8)$ is equivalent to the group of pairs (L, t) where L is an even lattice of rank $8-n$ and $t : \mathrm{qres}\, L \xrightarrow{\sim} -\mathrm{qres}\, \mathrm{D}_n$. This implies $L \simeq \mathrm{D}_{8-n}$, and $\mathrm{O}(\mathrm{D}_{8-n})$ surjects onto $\mathrm{O}(\mathrm{qres}\, \mathrm{D}_{8-n})$ so $\mathcal{G}(\mathbf{D}_n, \mathbf{E}_8)$ has one isomorphism class and any embedding has orthogonal $\simeq \mathrm{D}_{8-n}$.

- $\phi = \mathbf{D}_n$ (for $4 \leq n \leq 7$) and $\psi = \mathbf{E}_7$. By the same argument as for $\mathcal{G}(\mathbf{A}_n, \mathbf{E}_7)$ we have $\mathcal{G}(\mathbf{D}_n, \mathbf{E}_7) \simeq \mathcal{G}(\mathbf{D}_n \sqcup \mathbf{A}_1, \mathbf{E}_8)$ and the latter is equivalent to the groupoid of embeddings $\mathrm{D}_n \perp \mathrm{A}_1 \to L$ with $L \simeq \mathrm{E}_8$. By the previous point this groupoid is equivalent to $\mathcal{G}(\mathbf{A}_1, \mathbf{D}_{8-n})$, which is empty for $n = 7$, has two classes for $n = 6$ (as $\mathrm{D}_2 \simeq \mathrm{A}_1 \perp \mathrm{A}_1$), and one class if $4 \leq n \leq 5$. For $n \leq 6$, the orthogonal of any $\iota \in \mathrm{Emb}(\mathbf{D}_n, \mathbf{E}_7)$ is isomorphic to $\mathrm{A}_1 \perp \mathrm{D}_{6-n}$. For $n = 6$, a similar argument as above shows that the two orbits are permuted transitively by $\mathrm{O}(\mathrm{D}_6)$ (note that for any embedding $\mathbf{D}_6 \sqcup \mathbf{D}_2 \to \mathbf{E}_8$ there is an element in $\mathrm{W}(\mathrm{E}_8)$ inducing the outer automorphisms of $\mathbf{D}_6$ and $\mathbf{D}_2 = \mathbf{A}_1 \sqcup \mathbf{A}_1$).

- $\phi = \mathbf{D}_n$ (for $4 \leq n \leq 6$) and $\psi = \mathbf{E}_6$. Similarly we have $\mathcal{G}(\mathbf{D}_n, \mathbf{E}_6) \simeq \mathcal{G}(\mathbf{D}_n \sqcup \mathbf{A}_2, \mathbf{E}_8) \simeq \mathcal{G}(\mathbf{A}_2, \mathbf{D}_{8-n})$, and the latter groupoid
  - is empty if $n = 6$,
  - has two isomorphism classes if $n = 5$ (both $\mathrm{O}(\mathrm{E}_6)$ and $\mathrm{O}(\mathrm{D}_5)$ act transitively on the set of isomorphism classes, and the orthogonal of an isometry $\mathrm{D}_5 \to \mathrm{E}_6$ is isomorphic to (12)),
  - has one isomorphism class if $n = 4$ (and the orthogonal of an isometry $\mathrm{D}_4 \to \mathrm{E}_6$ has no root).

- $\phi = \mathbf{E}_n$ and $\psi = \mathbf{A}_m$ or $\mathbf{D}_m$. Let us check that there is no isometry $\mathrm{E}_6 \to \mathrm{D}_m$ for any m, and thus no isometry $\mathrm{E}_n \to \mathrm{D}_m$ or $\mathrm{E}_n \to \mathrm{A}_m$ either. We choose $\mathrm{D}_5 \to \mathrm{E}_6$ mapping simple roots to simple roots, such that $e_4 + e_5$ is mapped to the simple root of $\mathbf{E}_6$ connected to the simple root not in the image. We try to extend $\mathrm{D}_5 \to \mathrm{D}_m$ (up to $\mathrm{W}(\mathrm{D}_m)$, there is only one such isometry, say $\pm e_i \pm e_j \mapsto \pm e_i \pm e_j$) to E_6. The remaining simple root of $\mathbf{E}_6$ must be mapped to $\sum_{i=1}^m x_i e_i$ with $x_1 = \cdots = x_5$ and $x_4 + x_5 = -1$, so $x_4 = x_5 = -1/2$ which is not an integer, a contradiction.

- $\phi = \mathbf{E}_6$ and $\psi = \mathbf{E}_8$. As before $\mathcal{G}(\mathbf{E}_6, \mathbf{E}_8)$ is equivalent to the groupoid of pairs (L, t) where L is an even lattice and $t : \mathrm{qres}\, L \xrightarrow{\sim} -\mathrm{qres}\, \mathrm{E}_6$ (implying $L \simeq \mathrm{A}_2$). This groupoid has one class, and for $\iota \in \mathcal{G}(\mathbf{E}_6, \mathbf{E}_8)$ we have $\iota(\mathrm{E}_6)^\perp \simeq \mathrm{A}_2$.

- $\phi = \mathbf{E}_6$ and $\psi = \mathbf{E}_7$. By the last point we have
$$\mathcal{G}(\mathbf{E}_6, \mathbf{E}_7) \simeq \mathcal{G}(\mathbf{E}_6 \sqcup \mathbf{A}_1, \mathbf{E}_8) \simeq \mathcal{G}(\mathbf{A}_1, \mathbf{A}_2)$$
  so $\mathcal{G}(\mathbf{E}_6, \mathbf{E}_7)$ has one class, with orthogonals isomorphic to (6).

- $\phi = \mathbf{E}_7$ and $\psi = \mathbf{E}_8$. As before $\mathcal{G}(\mathbf{E}_7, \mathbf{E}_8)$ is equivalent to the groupoid of pairs (L, t) where L is an even lattice and $t : \mathrm{qres}\, L \xrightarrow{\sim} -\mathrm{qres}\, \mathrm{E}_7$ (implying $L \simeq \mathrm{A}_1$, and t is redundant as it is unique). This groupoid has one class, and the orthogonal of any $\iota \in \mathrm{Emb}(\mathbf{E}_7, \mathbf{E}_8)$ is $\simeq \mathrm{A}_1$.

This concludes the proof of Theorem 1. □

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