Akizuki-Witt maps and Kaletha's global rigid inner forms

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January 12, 2018

Abstract

We give an explicit construction of global Galois gerbes constructed more abstractly by Kaletha to define global rigid inner forms. This notion is crucial to formulate Arthur's multiplicity formula for inner forms of quasi-split reductive groups. As a corollary, we show that any global rigid inner form is almost everywhere unramified, and we give an algorithm to compute the resulting local rigid inner forms at all places in a given finite set. This makes global rigid inner forms as explicit as global pure inner forms, up to computations in local and global class field theory.

Contents

1 Introduction

Let F be a number field, and G a connected reductive group over F. Following seminal work of Labesse-Langlands [LL79] and Shelstad, Langlands, Kottwitz and Arthur [Art89] conjectured a multiplicity formula for discrete automorphic representations for G, in terms of Arthur-Langlands parameters $\psi : L_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$. The formulation of this conjecture on automorphic multiplicities requires a precise version of the local Arthur-Langlands correspondence for $G_{F_v} := G \times_F F_v$ at all places v of F, describing individual elements of local packets using the theory of endoscopy. For this it is necessary to endow each G_{F_v} with a *rigidifying datum*. For places v such that G_{F_v} is quasi-split, that is for all but finitely many places of F, this can take the form of a Whittaker datum $\mathfrak{w}_v$. If G is quasi-split, then one can choose a global Whittaker datum $\mathfrak{w}$, and it is expected that taking localizations $\mathfrak{w}_v$ of $\mathfrak{w}$ yields a coherent family of precise versions of the local Arthur-Langlands correspondence. This coherence is crucial for the automorphic multiplicity formula to hold. For example this is the setting used in [Art13] and [Mok15]. Note that even though a choice of global Whittaker datum is necessary to express the formula for automorphic multiplicities, these multiplicities are canonical, as one can easily deduce from [Kal13, Theorem 4.3].

In general the connected reductive group G might not be quasi-split, and G is only an inner form of a unique quasi-split group. Recall (see [Bor79]) that two connected reductive groups have isomorphic Langlands dual groups if and only if they are inner forms of each other. Vogan [Vog93] and Kottwitz conjectured a formulation of the local Langlands correspondence in the case where G_{F_v} is a *pure* inner form of a quasi-split group. In this case a rigidifying datum is a quadruple $(G_v^*, \Xi_v, z_v, \mathfrak{w}_v)$ where G_v^* is a connected reductive quasi-split group over F_v, $\Xi_v : (G_v^*)_{\overline{F_v}} \to G_{\overline{F_v}}$ is an isomorphism, and $z_v \in Z^1(F_v, G_v^*)$ is such that for any $\sigma \in \mathrm{Gal}(\overline{F_v}/F_v)$ we have $\Xi_v^{-1}\sigma(\Xi_v) = \mathrm{Ad}(z_v(\sigma))$. If globally G is a pure inner form of a quasi-split group, one can choose a similar global quadruple $(G^*, \Xi, z, \mathfrak{w})$, and localizing at all places of F seems to yield a coherent family of rigidifying data. Away from a finite set S of places of F, the restriction z_v of z to a decomposition group $\mathrm{Gal}(\overline{F_v}/F_v)$ is cohomologically trivial, and writing it as a coboundary yields an isomorphism $\Xi'_v : G^*_{F_v} \simeq G_{F_v}$ well-defined up to conjugation by $G(F_v)$, which endows G_{F_v} with a Whittaker datum $(\Xi'_v)_*(\mathfrak{w}_v)$ in a canonical way. Furthermore, up to enlarging S this can be done integrally, that is over a finite étale extension of $\mathcal{O}(F_v)$, so that Ξ'_v is an isomorphism between the canonical models of G^* and G over $\mathcal{O}(F_v)$.

Unfortunately not all connected reductive groups can be realized as pure inner forms of quasi-split groups, due to the fact that $H^1(F, G^*) \to H^1(F, G^*_{\mathrm{ad}})$ can fail to be surjective. The simplest example is certainly the group of elements having reduced norm equal to 1 in a non-split quaternion algebra, an inner form of SL_2, considered in [LL79]. To circumvent this problem, Kaletha defined larger Galois cohomology groups in [Kal16] for the local case and in [Kal] for the global case. More precisely, he constructed central extensions (Galois gerbes bound by commutative groups in the terminology of [LR87])

$$1 \to P_v \to \mathcal{E}_v \to \mathrm{Gal}(\overline{F_v}/F_v) \to 1$$

in the local case, v any place of F, and

$$1 \to P \to \mathcal{E} \to \mathrm{Gal}(\overline{F}/F) \to 1$$

in the global case. Here P_v and P are inverse limits of finite commutative algebraic groups defined over F_v or F, and we have denoted by $P_v \to \mathcal{E}_v$ the extension denoted by $u \to W$ in [Kal16], to emphasize the analogy between the local and global cases. The central extensions are obtained from certain classes $\xi_v \in H^2(F_v, P_v)$, $\xi \in H^2(F, P)$. Using these central extensions Kaletha defined, for Z a finite central algebraic subgroup of G^*, certain sets of 1-cocycles

$$Z^1(P_v \to \mathcal{E}_v, Z(\overline{F_v}) \to G^*(\overline{F_v})) \supset Z^1(F_v, G^*_{F_v}),$$

$$\text{resp. } Z^1(P \to \mathcal{E}, Z(\overline{F}) \to G^*(\overline{F})) \supset Z^1(F, G^*)$$

which naturally map to $Z^1(F_v, G^*_{\mathrm{ad},F_v})$ (resp. $Z^1(F, G^*_{\mathrm{ad}})$), so that such cocycles give rise to inner forms of G^*. Kaletha also proposed precise formulations of the local Langlands conjecture and Arthur multiplicity formula, using rigidifying data $(G^*_v, \Xi_v, z_v, \mathfrak{w}_v)$ (resp. $(G^*, \Xi, z, \mathfrak{w})$) where now z_v (resp. z) belongs to this larger group of 1-cocycles. For Z large enough, for example if Z contains the center of the derived subgroup of G^*, the map between the resulting cohomology sets

$$H^1(P \to \mathcal{E}, Z(\overline{F}) \to G^*(\overline{F})) \to H^1(F, G^*_{\mathrm{ad}})$$

is surjective, and so any G can be endowed with such a rigidifying datum $(G^*, \Xi, z, \mathfrak{w})$. From such a global rigidifying datum, one obtains local rigidifying data by localization. Each localization $z_v = \mathrm{loc}_v(z)$ of z is defined by pulling back via a morphism of central extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & P_v & \longrightarrow & \mathcal{E}_v & \longrightarrow & \mathrm{Gal}(\overline{F_v}/F_v) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & P & \longrightarrow & \mathcal{E} & \longrightarrow & \mathrm{Gal}(\overline{F}/F) & \longrightarrow & 1 \end{array} \tag{1.0.1}$$

and extending coefficients from $G^*(\overline{F})$ to $G^*(\overline{F_v})$.

In this paper we give an explicit, bottom-up realization of the central extension

$$1 \to P \to \mathcal{E} \to \mathrm{Gal}(\overline{F}/F) \to 1$$

constructed in [Kal]. Here "bottom-up" means that our construction is naturally an inverse limit over $k \geq 0$ of central extensions

$$1 \to P_k \to \mathcal{E}_k \to \mathrm{Gal}(E'_k/F) \to 1,$$

where E'_k/F is finite Galois extension, P_k is a finite commutative algebraic group over F such that $P_k(E'_k) = P_k(\overline{F})$, and $P = \varprojlim_{k \geq 0} P_k$. We also give bottom-up realizations of localization morphisms (1.0.1) and generalized Tate-Nakayama morphisms for tori ([Kal, Theorem 3.7.3], which generalizes [Tat66]), as well as compatibilities between them. We also show (Proposition 5.5.2) that our construction recovers the "canonical class" defined abstractly in [Kal, §3.5]. Apart from giving alternative proofs of some results in [Kal], our construction has the benefit that it allows one to compute with global rigid inner forms "at finite level", that is using a *finite* Galois extension of the base field F. In particular, we deduce that global rigid inner forms are almost everywhere unramified (Proposition 6.1.1), a fact which is obvious for pure inner forms, but surprisingly not for rigid inner forms. In the future our construction could be used to prove further properties of Kaletha's canonical class.

Our direct construction is also useful for explicit applications using Arthur's formula for automorphic multiplicities. Computing spaces of automorphic forms, along with action of a Hecke algebra, is possible for definite reductive groups thanks to reduction theory. Unfortunately non-commutative definite reductive groups are not quasi-split. Once such spaces are computed, one would like to interpret Hecke eigenforms as being related to (ersatz) motives, and Arthur's multiplicity formula makes this relation precise (see [Taïa] for some cases for which rigid inner forms are needed). For this it is necessary to compute localizations of rigidifying data, more precisely to solve the following problem.

Problem. *Given a connected reductive group G over a number field F, find*

- *a global rigidifying datum $\mathcal{D} = (G^*, \Xi, z, \mathfrak{w})$,*
- *a finite set S of places of F containing all archimedean places and all non-archimedean places v such that G_{F_v} is ramified,*
- *a reductive model of $\underline{G}$ over the ring $\mathcal{O}_{F,S}$ of S-integers in F such that for any $v \notin S$, the localization $\mathcal{D}_v$ of $\mathcal{D}$ at v is unramified with respect to the integral model $\underline{G}_{\mathcal{O}_{F_v}}$ of G_{F_v},*

- *for each $v \in S$, an explicit description of the localization $\mathcal{D}_v$ of $\mathcal{D}$ at v.*

Above "unramified" means that $\mathrm{loc}_v(z) \in B^1(F_v, G)$, and that the resulting isomorphism $\Xi'_v : G^*_{F_v} \simeq G_{F_v}$, which is well-defined up to composing with conjugation by an element of $G(F_v)$, identifies the conjugacy class of $\mathfrak{w}_v$ with a Whittaker datum for G_{F_v} compatible with the integral model $\underline{G}_{\mathcal{O}(F_v)}$, in the sense of [CS80]. At almost all places this is implied by the fact that $\mathfrak{w}_v$ is compatible with the canonical model of G^* and the fact that $\mathrm{loc}_v(z) \in Z^1(F_v^{\mathrm{unr}}/F_v, G^*)$, but for applications it is desirable to keep S as small as possible. For $v \in S$, the meaning of "explicit description of $\mathcal{D}_v$" is somewhat vague. In the case where $\mathrm{loc}_v(z)$ is cohomologically trivial this simply means a Whittaker datum for G_{F_v}. In general it means describing the localization $\mathcal{D}_v$ in a purely local fashion, so that it could be compared to a reference rigidifying datum. We give detailed steps to solve this problem in section 7, reducing the computation of localizations at places in S to computations in local and global class field theory. We give an example in section 7.2 in a case where G is a definite inner form of SL_2 over $F = \mathbb{Q}(\sqrt{3})$ which is split at all finite places, and for S the set of archimedean places, that is in "level one". It can be generalized effortlessly, and without additional computations, to the analogous inner forms of Sp_{2n} over F, for arbitrary $n \geq 2$.

Let us explain why this problem does not appear to be directly solvable using constructions in [Kal], which might be surprising when one considers the case of pure inner forms, as it is straightforward to restrict a 1-cocycle to a decomposition group. For explicit computations one can only work with finite extensions of F, and finite Galois modules. Although the localization maps (1.0.1) are canonical, unfortunately they do not arise from *canonical* morphism of central extensions of Galois groups by *finite* Galois modules, because of the possible non-vanishing of $H^1(F_v, P_k)$, where $P = \varprojlim_k P_k$. Similarly, the possible non-vanishing of $H^1(F, P_k)$ means that inflation morphisms

$$\begin{tikzcd} 1 \arrow[r] & P_{k+1} \arrow[r] \arrow[d] & \mathcal{E}_{k+1} \arrow[r] \arrow[d] & \mathrm{Gal}(\overline{F}/F) \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & P_k \arrow[r] & \mathcal{E}_k \arrow[r] & \mathrm{Gal}(\overline{F}/F) \arrow[r] & 1 \end{tikzcd} \tag{1.0.2}$$

are not defined canonically, where $\mathcal{E}_k$ is the central extension obtained using a 2-cocycle in the cohomology class of the image of ξ in $H^2(F, P_k)$. For applications to generalized Tate-Nakayama isomorphisms, Kaletha shows that these ambiguities are innocuous using a clever indirect argument (Lemma 3.7.10 in [Kal]) in cohomology (but only in cohomology). Note that in the local case, Kaletha gave an explicit construction of the inflation maps analogous to (1.0.2): see [Kal16, §4.5].

Our construction is a global analogue. The main difficulty lies in formulating and proving the analogue of [Kal16, Lemma 4.4] (which draws on [Lan83, §VI.1]) in the global case. First we reinterpret [Kal16, Lemma 4.4] using a modification AW^2 of the Akizuki-Witt map on 2-cocycles ([AT09, Ch. XV]) occurring in the construction of Weil groups attached to class formations. We study this modification systematically in section 3.1, in particular we observe that it is more flexible while retaining the interpretation in terms of central extensions. It is not difficult to establish the analogue of [Kal16, Lemma 4.4] where local fundamental cocycles are replaced by global fundamental cocycles. However, in Tate-Nakayama isomorphisms these global fundamental cocycles control Galois cohomology groups such as $H^1(E/F, T(\mathbb{A}_E)/T(E))$, where T is a torus over F split by the finite Galois extension E/F, whereas we are interested in cohomology groups such as $H^1(E/F, T(E))$. These are controlled by *Tate cocycles* defined by Tate in [Tat66], essentially as a consequence of the compatibility between local and global fundamental 2-cocycles. Unfortunately these do not seem to have an interpretation using the Akizuki-Witt map, and this makes the global case more challenging. We give an ad hoc definition of a certain map AWES^2 in Definition 4.2.1, which is compatible with the corestriction map in Eckmann-Shapiro's lemma for modules which are *twice* induced. This definition is crucial for the main technical result of this article, Theorem 4.4.2, constructing a family of Tate cocycles compatible under AWES^2, as well as local-global compatibility with local fundamental cocycles. We give a second proof as preparation for the algorithm in section 7. Once this is proved, we construct Kaletha's generalized Tate-Nakayama morphisms at the level of cocycles in section 5, and prove compatibilities with respect to inflation and localization. In particular we obtain an explicit version of Kaletha's localization maps at finite level and for cocycles. Although these explicit localization maps are not canonical, as they depend on a number of choices detailed in the paper to form cocycles, they are compatible with inflation and so yield a localization map between towers of central extensions (see Proposition 5.4.5).

As mentioned above, a consequence is that global rigid inner forms are unramified away from a finite set (Proposition 6.1.1), which is not obvious from the definition using cohomology classes. After the first version of this paper was written, we found a short proof of this ramification property using only Kaletha's characterization of the canonical class in [Kal, §3.5]. This proof is included in section 6, along with an example of a "non-canonical" class, which does not satisfy this ramification property.

The author thanks Tasho Kaletha for numerous interesting discussions on rigid inner forms and for his encouragement, as well as two anonymous referees for their comments and suggestions.

2 Notation

Let F be a number field. We denote by $\mathbb{A}$ the ring of adèles for F. Let $\overline{F}$ be an algebraic closure of F. All algebraic extensions of F considered will be subextensions of $\overline{F}$. If E is an algebraic extension of F, let $\mathcal{O}(E)$ be its ring of integers, $\mathbb{A}_E = E \otimes_F \mathbb{A}$, $I(E) = \mathbb{A}_E^\times$ the group of idèles and $C(E) = I(E)/E^\times$ the group of idèle classes. Let $\overline{\mathbb{A}} = \mathbb{A}_{\overline{F}}$. Let V be the set of all places of F. If $S \subset V$ and E is an algebraic extension of F, denote by S_E the set of places of E above S. If S is a set of places of F or E containing all archimedean places, let $I(E, S)$ be the subgroup of $I(E)$ consisting of idèles which are integral units away from S, and $\mathcal{O}(E, S)$ the ring of S-integral elements of E. For $S \subset V$ let $\overline{F}_S$ be the maximal subextension of $\overline{F}/F$ unramified outside S, and $\mathcal{O}_S = \mathcal{O}(\overline{F}_S, S)$. For E an algebraic extension of F and $u \in V_E$, we will denote by pr_u the projection $\mathbb{A}_E \to E_u$. For $v \in V$ we will denote by pr_v the projection $\mathbb{A}_{\overline{F}} \to \overline{F} \otimes_F F_v$.

As in [Kal] we fix a tower $(E_k)_{k\geq 0}$ of increasing finite Galois extensions of F, with $E_0 = F$ and $\bigcup_k E_k = \overline{F}$. Choose an increasing sequence $(S_k)_{k\geq 0}$ of finite subsets of V such that S_0 contains all archimedean places of F, S_k contains all non-archimedean places of F ramifying in E_k, and $I(E_k, S_k)$ maps onto $C(E_k)$. We also fix a set $\dot{V} \subset V_{\overline{F}}$ of representatives for the action of $\mathrm{Gal}(\overline{F}/F)$, that is $\dot{V}$ contains a place of $\overline{F}$ above every place of F. For E a Galois extension of F and $S' \subset V$ let $\dot{S}'_E$ be the set of places of E below $\dot{V}$ and above S', so that $\dot{S}'_E$ is a set of representatives for the action of $\mathrm{Gal}(E/F)$ on S'_E. We can assume that $\dot{V}$ is chosen so that for any finite Galois extension E/F and $\sigma \in \mathrm{Gal}(E/F)$, there exists $\dot{v} \in \dot{V}_E$ such that $\sigma \cdot \dot{v} = \dot{v}$. This follows from Chebotarev's density theorem by an inductive process as in [Kal, (3.8)]. For $v \in V$ and $k \geq 0$ we will denote by $\dot{v}_k$ the unique place in $\dot{V}_{E_k}$ above v. To avoid double subscripts we let $E_{k,\dot{v}} = E_{k,\dot{v}_k}$. For $v \in S$ let $\overline{F_v} = \varinjlim_k E_{k,\dot{v}_k}$, an algebraic closure of F_v, so that we have a well-defined inclusion $\mathrm{Gal}(\overline{F_v}/F_v) \subset \mathrm{Gal}(\overline{F}/F)$.

Remark 2.0.1. *The above hypotheses on $(S_k)_{k\geq 0}$ are weaker than Conditions 3.3.1 in [Kal]. For effective computations (see Section 7) it is useful to have S_k as small as possible, and so we have only imposed conditions on $(S_k)_{k\geq 0}$ that are necessary for constructions in the present article.*

The condition on the choice of $\dot{V}$ (corresponding to Condition 3.3.1.4 in [Kal]) will not be used for the main constructions in this article. However, the extension $P \to \mathcal{E} \to$ $\mathrm{Gal}(\overline{F}/F)$ and the morphism ι in Corollary 5.2.4 depend on the choice of $(E_k)_{k\geq 0}$ and $\dot{V}$, and so the above condition on $\dot{V}$ is necessary to obtain objects isomorphic to those in [Kal]. Note that Condition 3.3.1.4 in [Kal] is first used in [Kal, Lemma 3.3.2, 3], and so it is also used [Kal, Lemma 3.6.1] to obtain surjectivity of

$$H^1(P \to \mathcal{E}, Z \to G) \to H^1(F, G/Z)$$

for any connected reductive group G over F and finite central subgroup Z. This is crucial for applications to automorphic forms (see [Kal, §4.3]).

Condition 3.3.1.3 in [Kal], which we have not imposed, is used to prove that certain inflation maps are injective (Lemma 3.1.10, Lemma 3.2.7, Proposition 3.7.12).

If A is a commutative group, $A^\vee = \mathrm{Hom}(A, \mathbb{Q}/\mathbb{Z})$. If A is commutative group and $N \geq 1$ is an integer, $A[N]$ denotes the N-torsion subgroup of A. If A is a finite commutative group, $\exp(A)$ is the exponent of A, i.e. the smallest $N \geq 1$ such that $A[N] = A$. We will denote the group law of most abelian groups multiplicatively, except notably for groups of characters or cocharacters of tori. If G is a group and A a G-module, $A^G \subset A$ is the subgroup of G-invariants. If in addition $G = \mathrm{Gal}(E/F)$, we will write $N_{E/F}$ for the norm map, and $A^{N_{E/F}}$ for the subgroup of elements killed by $N_{E/F}$.

3 Preliminaries

3.1 A modification of the Akizuki-Witt map

Consider G a finite group, N a normal subgroup. If $s : G/N \to G$ is a section such that $s(1) = 1$ and A is a G-module, with group law written multiplicatively, then for $\alpha \in Z^2(G, A)$,

$$\widetilde{\mathrm{AW}}(\alpha) : (\sigma, \tau) \mapsto \prod_{n \in N} \frac{n\left(\alpha(s(\sigma), s(\tau))\right) \times \alpha(n, s(\sigma)s(\tau))}{\alpha(n, s(\sigma\tau))} \tag{3.1.1}$$

defines an element of $Z^2(G/N, A^N)$, the cohomology class of which only depends on that of α [AT09, Ch. XIII, §3], so that $\widetilde{\mathrm{AW}}$ descends to a map $H^2(G, A) \to H^2(G/N, A^N)$. We refer to [AT09, Ch. XIII, §3] for the natural interpretation of $\widetilde{\mathrm{AW}}$ in terms of central group extensions. Using the 2-cocycle relation for α at $(n, s(\sigma), s(\tau))$ we can express (3.1.1) as

$$\prod_{n \in N} \frac{\alpha(n, s(\sigma)) \times \alpha(ns(\sigma), s(\tau))}{\alpha(n, s(\sigma\tau))} = \prod_{n \in N} \frac{\alpha(n, s(\sigma)) \times \alpha(\tilde{\sigma}n, s(\tau))}{\alpha(n, s(\sigma\tau))}$$

where $\tilde{\sigma} \in G$ is any lift of σ, not necessarily equal to $s(\sigma)$. Using the 2-cocycle relation for α at $(\tilde{\sigma}, n, s(\tau))$ we can also rewrite this as

$$\widetilde{\mathrm{AW}}(\alpha)(\sigma, \tau) = \prod_{n \in N} \left(\frac{\alpha(n, s(\sigma)) \times \tilde{\sigma}\left(\alpha(n, s(\tau))\right)}{\alpha(n, s(\sigma\tau))} \times \frac{\alpha(\tilde{\sigma}, ns(\tau))}{\alpha(\tilde{\sigma}, n)} \right). \tag{3.1.2}$$

The following shows that with an appropriate choice of α in its cohomology class, this expression simplifies.

Lemma 3.1.1. *In any cohomology class in $H^2(G, A)$, there is a 2-cocycle α such that for all $n \in N$ and $\sigma \in G/N$, $\alpha(n, s(\sigma)) = 1$.*

Proof. It is well-known that any cohomology class contains a 2-cocycle α such that for all $\sigma \in G$, $\alpha(\sigma, 1) = 1 = \alpha(1, \sigma)$. We choose such an α, and we will construct $\beta : G \to A$ such that $\alpha \mathrm{d}(\beta)$ satisfies the required property. Let $\beta(1) = 1$, and choose the values of β on $N \smallsetminus \{1\}$ and $s(G/N \smallsetminus \{1\})$ arbitrarily. For $n \in N$ and $\sigma \in G/N$,

$$\mathrm{d}\beta(n, s(\sigma)) = \frac{\beta(n) \times n(\beta(s(\sigma)))}{\beta(ns(\sigma))},$$

and we are led to define $\beta(ns(\sigma)) = \alpha(n, s(\sigma)) \times \beta(n) \times n\left(\beta(s(\sigma))\right)$ for $n \in N \smallsetminus \{1\}$ and $\sigma \in G/N \smallsetminus \{1\}$. Note that this equality also holds when $n = 1$ or $\sigma = 1$. □

This motivates to the following modification AW^2 of the Akizuki-Witt map $\widetilde{\mathrm{AW}}$.

Definition 3.1.2. *Let Γ be an extension of G, i.e. Γ is a group endowed with a surjective morphism $\Gamma \to G$. Let A be a commutative group, with group law written multiplicatively. For $\alpha : \Gamma \times G \to A$, define $\mathrm{AW}^2(\alpha) : \Gamma \times G/N \to A$ by*

$$\mathrm{AW}^2(\alpha)(\sigma, \tau) = \prod_{n \in N} \frac{\alpha(\sigma, n\tilde{\tau})}{\alpha(\sigma, n)}$$

where $\sigma \in \Gamma$, $\tau \in G/N$ and $\tilde{\tau} \in G$ is any lift of τ.

Although this coincides with the original Akizuki-Witt map a priori only for classes α as in Lemma 3.1.1 (for A a G-module and $\Gamma = G$), this definition has the advantage that it does not require a choice of section s, and will be more convenient for taking cup-products. Moreover it is defined in a slightly more general setting, since it does not involve an action of G on A. This property will make "extracting N-th roots" in section 5 almost harmless. The definition has the disadvantage that, even when A is a G-module, $\Gamma = G$ and $\alpha \in Z^2(G, A)$, it is not automatic that $\mathrm{AW}^2(\alpha)$ factors through $G/N \times G/N$ or takes values in A^N.

For Γ an extension of G and A a commutative group recall [Kal16, §4.3] for $i \geq j \geq 0$ the commutative group $C^{i,j}(\Gamma, G, A)$ of functions $\Gamma^{i-j} \times G^j \to A$, which is naturally a subgroup of $C^i(\Gamma, A)$. If A is a Γ-module, the differential d maps $C^{i,j}(\Gamma, G, A)$ to $C^{i+1,j}(\Gamma, G, A)$. Let $Z^{i,j}(\Gamma, G, A)$ be its kernel.

The following proposition is the first evidence that AW^2 behaves nicely under weaker conditions than the one imposed in Lemma 3.1.1, retaining the interpretation in terms of central extensions.

Proposition 3.1.3. *Let Γ be an extension of G.*

1. *For* $\alpha \in Z^{2,1}(\Gamma, G, A)$*, we have* $\mathrm{AW}^2(\alpha) \in Z^{2,1}(\Gamma, G/N, A)$.

2. *If* $\Gamma = G$ *then* $\sigma \mapsto \prod_{n\in N} \alpha(n,\sigma)$ *descends to a map* $G/N \to A/A^N$ *mapping* 1 *to* 1.

3. *If* $\Gamma = G$*, the following are equivalent:*

   (a) $\mathrm{AW}^2(\alpha)$ *factors through* $G/N \times G/N$*,*

   (b) *for all* $\sigma \in N$ *and* $\tau \in G/N$*,* $\mathrm{AW}^2(\alpha)(\sigma, \tau) = 1$*,*

   (c) *for all* $\sigma \in G$*,* $\prod_{n\in N} \alpha(n,\sigma) \in A^N$.

4. *If* $\Gamma = G$ *and the above conditions are satisfied, then* $\mathrm{AW}^2(\alpha) \in Z^2(G/N, A^N)$ *belongs to the same cohomology class as* $\widetilde{\mathrm{AW}}(\alpha)$ *and we have a morphism of central extensions*

$$\begin{aligned} A \underset{\alpha}{\boxtimes} G &\longrightarrow \qquad A^N \underset{\mathrm{AW}^2(\alpha)}{\boxtimes} G/N \\ x \boxtimes \sigma &\longmapsto \left(\prod_{n\in N} n(x)\alpha(n,\sigma) \right) \boxtimes \overline{\sigma}. \end{aligned} \tag{3.1.3}$$

We only sketch the proof, since this proposition is not logically necessary for the rest of the paper.

Proof. 1. This is an easy computation.

2. Suppose that $\Gamma = G$. Using the cocycle relation for α, for every $\tau, \gamma \in N$,

$$\tau \left(\prod_{n\in N} \alpha(n,\gamma) \right) = \prod_{n\in N} \alpha(\tau n, \gamma)\alpha(\tau, n)/\alpha(\tau, n\gamma) = \prod_{n\in N} \alpha(n,\gamma)$$

and so $\prod_{n\in N} \alpha(n,\gamma) \in A^N$ for any $\gamma \in N$. Now for $\gamma \in N$ and $\sigma \in G$, using the cocycle relation again,

$$\prod_{n\in N} \alpha(n, \gamma\sigma) = \prod_{n\in N} \alpha(n\gamma, \sigma)\alpha(n,\gamma)n\left(\alpha(\gamma,\sigma)\right) \equiv \prod_{n\in N} \alpha(n,\sigma) \mod A^N.$$

3. Using the cocycle relation we can write

$$\mathrm{AW}^2(\alpha)(\sigma,\tau) = \prod_{n\in N} \frac{\alpha(\sigma n, \widetilde{\tau})}{\sigma\left(\alpha(n,\widetilde{\tau})\right)}.$$

The numerator only depends on $\alpha \mod N$, and the equivalence between (a) and (c) follows easily. The equivalence between (b) and (c) is obtained by taking $\sigma \in N$.

4. The fact that $\mathrm{AW}^2(\alpha)$ is cohomologous to $\widetilde{\mathrm{AW}}^2(\alpha)$ follows from the expression (3.1.2) for $\widetilde{\mathrm{AW}}$ and condition (c). This give an isomorphism $A^N \underset{\mathrm{AW}^2(\alpha)}{\boxtimes} G/N \simeq A^N \underset{\widetilde{\mathrm{AW}}^2(\alpha)}{\boxtimes} G/N$. Since we have an explicit map $A \underset{\alpha}{\boxtimes} G \to A^N \underset{\widetilde{\mathrm{AW}}^2(\alpha)}{\boxtimes} G/N$ by construction in [AT09, Ch. XIII, §3], finding formula (3.1.3) is a simple computation. Alternatively, one can directly check that (3.1.3) is a morphism.

$\square$

In order to investigate the effect on $\mathrm{AW}^2(\alpha)$ of the choice of α in its cohomology class, let us define a second map AW^1 on 1-cochains.

Definition 3.1.4. *Let A be a commutative group. For $\beta : G \to A$, define $\mathrm{AW}^1(\beta) : G/N \to A$ by the formula $\mathrm{AW}^1(\beta)(\sigma) = \prod_{n\in N} \beta(n\tilde{\sigma})/\beta(n)$, where $\tilde{\sigma} \in G$ is any lift of $\sigma \in G/N$.*

Proposition 3.1.5. *Suppose Γ is an extension of G, and A is a Γ-module. For any $\beta : G \to A$, we have $\mathrm{d}(\mathrm{AW}^1(\beta)) = \mathrm{AW}^2(\mathrm{d}(\beta))$ in $Z^{2,1}(\Gamma, G/N, A)$.*

Proof. For $\sigma \in \Gamma$ and $\tau \in G/N$, denoting $\bar{\sigma}$ the image of σ in G, we have

$$\begin{aligned}\mathrm{d}(\mathrm{AW}^1(\beta))(\sigma,\tau) &= \prod_{n\in N} \frac{\beta(n\bar{\sigma})}{\beta(n)} \frac{\sigma\left(\beta(n\tilde{\tau})\right)}{\sigma\left(\beta(n)\right)} \frac{\beta(n)}{\beta(n\bar{\sigma}\tilde{\tau})} \\ &= \prod_{n\in N} \frac{\beta(n\bar{\sigma})\sigma\left(\beta(n\tilde{\tau})\right)}{\beta(n\bar{\sigma}\tilde{\tau})\sigma\left(\beta(n)\right)}\end{aligned}$$

and

$$\begin{aligned}\mathrm{AW}^2\left(\mathrm{d}(\beta)\right)(\sigma,\tau) &= \prod_{n\in N} \frac{\beta(\bar{\sigma})\sigma\left(\beta(n\tilde{\tau})\right)}{\beta(\bar{\sigma}n\tilde{\tau})} \frac{\beta(\bar{\sigma}n)}{\beta(\bar{\sigma})\sigma\left(\beta(n)\right)} \\ &= \prod_{n\in N} \frac{\sigma\left(\beta(n\tilde{\tau})\right)}{\beta(\bar{\sigma}n\tilde{\tau})} \frac{\beta(\bar{\sigma}n)}{\sigma\left(\beta(n)\right)}.\end{aligned}$$

$\square$

Lemma 3.1.6. *The maps*

$$\{\beta : G \to A \,|\, \beta(1) = 1\} \to \{\beta : G/N \to A \,|\, \beta(1) = 1\}$$

induced by AW^1 and

$$\{\alpha : \Gamma \times G \to A \,|\, \alpha(\sigma, 1) = 1 \text{ for all } \sigma \in \Gamma\} \to \{\alpha : \Gamma \times G/N \to A \,|\, \alpha(\sigma, 1) = 1 \text{ for all } \sigma \in \Gamma\}$$

induced by AW^2 are both surjective.

Proof. Let $s : G/N \to G$ be a section such that $s(1) = 1$. Restricting AW^1 to the set of $\beta : G \to A$ such that $\beta|_N = 1$ and $\beta(ns(\sigma)) = 1$ for $\sigma \in G/N \smallsetminus \{1\}$ and $n \in N \smallsetminus \{1\}$ yields a bijective map onto $\{\beta : G/N \to A \,|\, \beta(1) = 1\}$.

Similarly, restricting AW^2 to the set of $\alpha : \Gamma \times G \to A$ such that

- for all $\sigma \in \Gamma$ and $n \in N$, $\alpha(\sigma, n) = 1$,
- for all $\sigma \in \Gamma$, $n \in N \smallsetminus \{1\}$ and $\tau \in G/N \smallsetminus \{1\}$, $\alpha(\sigma, ns(\tau)) = 1$,

yields a bijective map onto $\{\alpha : \Gamma \times G/N \to A \,|\, \alpha(\sigma, 1) = 1 \text{ for all } \sigma \in \Gamma\}$. □

The following corollary is readily deduced from Lemmas 3.1.1 and 3.1.6 and Proposition 3.1.5.

Corollary 3.1.7. *Suppose that A is a G-module. Consider $c \in H^2(G, A)$, and let $\alpha_N \in Z^2(G/N, A^N)$ be in the cohomology class of the image of c under $\widetilde{\mathrm{AW}}$. Assume that $\alpha_N(1, 1) = 1$. Then there exists $\alpha \in c$ such that $\alpha(1, 1) = 1$ and $\mathrm{AW}^2(\alpha) = \alpha_N$.*

Note that we have not imposed that α should satisfy the property in Lemma 3.1.1, and indeed it can happen that no such α maps to α_N under AW^2. A simple computation shows that if we fix a section $s : G/N \to G$ as above, then for $\alpha, \alpha' \in c$ as in Lemma 3.1.1, $\mathrm{AW}^2(\alpha/\alpha') \in B^2(G/N, N_N(A))$ where

$$N_N(A) = \{\prod_{n \in N} n(x) \,|\, x \in A\}.$$

3.2 Explicit Eckmann-Shapiro

Let G be a finite group acting transitively on the left on a set X. Choose $x_0 \in X$ and let H be the stabilizer of x_0, so that we have an identification of G-sets $X \simeq G/H$ mapping x_0 to the trivial coset.

Let A be a left H-module. Define

$$\mathrm{ind}_H^G(A) = \{f : G \to A \,|\, \forall h \in H, g \in G,\ f(hg) = h \cdot f(g)\}.$$

It is naturally a left G-module by defining $(g_1 \cdot f)(g_2) = f(g_2 g_1)$. Evaluation at 1 defines a surjective morphism of H-modules $\pi : \mathrm{ind}_H^G(A) \to A$, which admits a natural splitting: we can identify A with the sub-H-module of $\mathrm{ind}_H^G(A)$ consisting of all functions whose support is contained in H. Choose R a set of representatives for G/H. Then $\mathrm{ind}_H^G(A) = \bigoplus_{r \in R} r \cdot A$. For simplicity we assume that $1 \in R$.

If A happens to be a G-module, then

$$f \mapsto (gH \mapsto g \cdot f(g^{-1})) \tag{3.2.1}$$

defines an isomorphism of G-modules ϕ_H^G between $\mathrm{ind}_H^G(A)$ and $\mathrm{Maps}(X, A)$. The sub-H-module A of $\mathrm{ind}_H^G(A)$ corresponds to functions supported on x_0 under this isomorphism.

The Eckmann-Shapiro lemma states that for any $i \geq 0$, the composite

$$H^i(G, \mathrm{ind}_H^G(A)) \to H^i(H, \mathrm{ind}_H^G(A)) \to H^i(H, A)$$

is an isomorphism, where the first map is restriction and the second map is induced by π. See e.g. [Ser94, Ch. I, §2.5]. It is well-known (for example [Tat66, p.713]) that the inverse is obtained as the composite

$$H^i(H, A) \to H^i(H, \mathrm{ind}_H^G(A)) \to H^i(G, \mathrm{ind}_H^G(A))$$

where the first map is induced by the embedding of H-modules $A \to \mathrm{ind}_H^G(A)$ mentioned above and the second map is corestriction. In this paper we will use explicit formulas for this inverse map, especially in degree 2.

Proposition-Definition 3.2.1. *As above, G is a finite group, H is a subgroup of G, R is a set of representatives for G/H containing 1, and A is a G-module.*

1. *For $i \geq 0$ and $c \in C^i(H, A)$, define $\mathrm{ES}_R^i(c) \in C^i(G, \mathrm{ind}_H^G(A))$ by*

$$\mathrm{ES}_R^i(c)(r_1 h_1 r_2^{-1}, r_2 h_2 r_3^{-1}, \ldots, r_i h_i r_{i+1}^{-1})(h_{i+1} r_1^{-1}) = h_{i+1}\left(c(h_1, h_2, \ldots, h_i)\right)$$

where $r_1, \ldots, r_{i+1} \in R$ and $h_1, \ldots, h_{i+1} \in H$. If A happens to be a G-module, then using the identification (3.2.1) we can write

$$\phi_H^G\left(\mathrm{ES}_R^i(c)(r_1 h_1 r_2^{-1}, r_2 h_2 r_3^{-1}, \ldots, r_i h_i r_{i+1}^{-1})\right)(r_1 \cdot x_0) = r_1\left(c(h_1, h_2, \ldots, h_i)\right). \tag{3.2.2}$$

2. *For $i \geq 0$ and $c \in C^i(H, A)$, $\mathrm{d}(\mathrm{ES}_R^i(c)) = \mathrm{ES}_R^{i+1}(\mathrm{d}(c))$. Thus ES_R^i induces a map $H^i(H, A) \to H^i(G, \mathrm{ind}_H^G)$, which is an isomorphism that we still denote by ES_R^i.*

Proof. The formula for $\mathrm{ES}_R^i(c)$ follows from the explicit formula for corestriction for homogeneous cochains found in [NSW08, Ch. I, §5.4. p.48] specialized to the case at hand where c takes values in $A \subset \mathrm{ind}_H^G(A)$. □

4 Construction of Tate cocycles in a tower

Let us recall from [Tat66] the construction of the Tate-Nakayama isomorphism, which gives a relatively simple description of Galois cohomology groups of tori over F. Consider E a finite Galois extension of F, and S a not necessarily finite set of places of F containing all Archimedean places and all non-Archimedean places that ramify in E, and such that

$I(E,S)$ surjects to $C(E)$. Tate introduced the $\mathrm{Gal}(E/F)$-module $\mathrm{Ta}(E,S)$ which consists of all morphisms from the short exact sequence

$$\mathbb{Z}[S_E]_0 \to \mathbb{Z}[S_E] \to \mathbb{Z}$$

to the short exact sequence

$$\mathcal{O}(E,S)^\times \to I(E,S) \to C(E).$$

Equivalently,

$$\mathrm{Ta}(E,S) = \mathrm{Hom}(\mathbb{Z}[S_E], I(E,S)) \underset{\mathrm{Hom}(\mathbb{Z}[S_E],C(E))}{\times} C(E) \subset \mathrm{Maps}(S_E, I(E,S)).$$

Tate constructed, using local and global fundamental classes and compatibility between them, a *Tate class* $\alpha \in H^2(E/F, \mathrm{Ta}(E,S))$. Consider a torus T over F which is split by E, let $Y = X_*(T)$ be the associated $\mathrm{Gal}(E/F)$-module of cocharacters. The main result of [Tat66] is that taking cup-product with α gives isomorphisms in every degree $i \in \mathbb{Z}$

$$\widehat{H}^i(E/F, Y[S_E]_0) \longrightarrow \widehat{H}^{i+2}(E/F, T(\mathcal{O}(E,S))) \tag{4.0.1a}$$

$$\widehat{H}^i(E/F, Y[S_E]) \longrightarrow \widehat{H}^{i+2}(E/F, T(\mathbb{A}_E, S)) \tag{4.0.1b}$$

$$\widehat{H}^i(E/F, Y) \longrightarrow \widehat{H}^{i+2}(E/F, T(\mathbb{A}_E)/T(E)) \tag{4.0.1c}$$

where

$$T(\mathbb{A}_E, S) = Y \otimes_{\mathbb{Z}} I(E,S) = \prod_{w \in S_E} T(E_w) \times \prod_{w \notin S_E} T(\mathcal{O}_{E_w}).$$

We shall see that varying S among the sets of places containing a fixed finite set S_0 satisfying the above conditions does not result in any difficulty. Varying E (for example in the tower E_k that is fixed in this paper), however, leads to the surprising phenomenon that it is not completely obvious that all three isomorphisms (4.0.1) are compatible with inflation of cohomology classes on the right hand side. See [Kal, Lemma 3.1.4] for a precise statement and a proof in cohomology.

Our first goal is to construct a *compatible* family of Tate *cocycles*

$$\alpha_k \in Z^2(E_k/F, \mathrm{Maps}(V_{E_k}, I(E_k)))$$

for the Galois extensions E_k/F. We will give a precise meaning to technical notion of "compatibility" in Theorem 4.4.2. For now we simply mention that this compatibility is a global analogue of [Kal16, Lemma 4.4].

Unwinding the definition, one can see that for a fixed k, a Tate cocycle α_k for E_k/F is obtained as follows.

1. Choose a representative $\overline{\alpha}_k \in Z^2(E_k/F, C(E_k))$ of the fundamental class for E_k/F.

2. For each place v of F, choose a representative $\alpha_{k,v} \in Z^2(E_{k,\dot{v}}/F_v, E_{k,\dot{v}}^\times)$ of the fundamental class for $E_{k,\dot{v}}/F_v$. Let $\alpha'_k \in Z^2(E_k/F, \mathrm{Maps}(V_{E_k}, I(E_k)))$ be such that for any $v \in V$, the 2-cocycle

$$\begin{aligned} \mathrm{Gal}(E_{k,\dot{v}}/F_v)^2 &\longrightarrow I(E_k) \\ (\sigma, \tau) &\longmapsto \alpha'_k(\sigma, \tau)(\dot{v}_k) \end{aligned}$$

is cohomologous to $\alpha_{k,v}$ composed with $j_{k,v} : E_{k,\dot{v}}^\times \hookrightarrow I(E_k)$. Explicitly, α'_k can be obtained from $(\alpha_{k,v})_{v\in V}$ using (3.2.2).

3. Choose $\overline{\beta}_k \in C^1(E_k/F, \mathrm{Maps}(V_{E_k}, C(E_k)))$ such that $\overline{\alpha}_k/\overline{\alpha'_k} = \mathrm{d}(\overline{\beta}_k)$, where $\overline{\alpha}_k$ is seen as taking values in the set of constant maps $V_{E_k} \to C(E_k)$ and $\overline{\alpha'_k}$ is the composition of α'_k with the natural map $\mathrm{Maps}(V_{E_k}, I(E_k)) \to \mathrm{Maps}(V_{E_k}, C(E_k))$.

4. Lift $\overline{\beta}_k$ to $\beta_k \in C^1(E_k/F, \mathrm{Maps}(V_{E_k}, I(E_k)))$ arbitrarily, and define $\alpha_k = \alpha'_k \times \mathrm{d}(\beta_k)$.

In this section we will show that each step can be done compatibly with Akizuki-Witt-like maps. For cocycles $\alpha_{k,v}$ this was done in [Kal16, Lemma 4.4], we will however give a slightly different construction, using Corollary 3.1.7. The case of $\overline{\alpha}_k$ is very similar. A key point of the construction will be the definition (see 4.2.1) of an "Akizuki-Witt-Eckmann-Shapiro" map relating the maps AW for local and global Galois groups, and formula (3.2.2) (see Lemma 4.2.2).

4.1 Global fundamental cocycles

For any $k \geq 0$, the image of the fundamental class in $H^2(E_{k+1}/F, C(E_{k+1}))$ under the Akizuki-Witt map (3.1.1) (for the normal subgroup $\mathrm{Gal}(E_{k+1}/E_k)$, and any choice of section) is the fundamental class in $H^2(E_k/F, C(E_k))$. This is a direct consequence of [AT09, Ch. XIII, Theorem 6]. For $i \in \{1, 2\}$ write AW_k^i for the maps AW^i defined in section 3.1, for the normal subgroup $\mathrm{Gal}(E_{k+1}/E_k)$ of $\mathrm{Gal}(E_{k+1}/F)$. Using Corollary 3.1.7 we see that there exists a family $(\overline{\alpha}_k)_{k\geq 0}$ where each $\overline{\alpha}_k \in Z^2(E_k/F, C(E_k))$ represents the fundamental class, and such that for all $k \geq 0$ we have $\overline{\alpha}_k = \mathrm{AW}_k^2(\overline{\alpha}_{k+1})$.

Remark 4.1.1. *Alternatively, one could construct such a family using a method similar to [Kal16, §4.4] (and so [Lan83, §VI.1]), that is by choosing sections* $\mathrm{Gal}(E_{k+1}/E_k) \to W_{E_k}$*, where* W_{E_k} *is the Weil group of* E_k*, and multiplying them to produce sections* $\mathrm{Gal}(E_k/F) \to W_{E_k/F}$*, yielding fundamental cocycles compatible with* AW_k^2.

A third way would be to use a compactness argument and Lemma 3.1.1, as in the proof of Theorem 4.4.2 (using 2*-cochains instead of* 1*-cochains). The details for this last alternative are left to the reader.*

4.2 Local and adèlic fundamental classes

Fix $v \in V$. For $i \in \{1,2\}$ write $\mathrm{AW}^i_{k,v}$ for the maps AW^i defined in section 3.1, for the normal subgroup $\mathrm{Gal}(E_{k+1,\dot{v}}/E_{k,\dot{v}})$ of $\mathrm{Gal}(E_{k+1,\dot{v}}/F_v)$. As in the global case we can use Corollary 3.1.7 inductively to produce a family $(\alpha_{k,v})_{k\geq 0}$ where $\alpha_{k,v} \in Z^2(E_{k,\dot{v}}/F_v, E^\times_{k,\dot{v}})$ represents the fundamental class and for all $k \geq 0$, we have $\alpha_{k,v} = \mathrm{AW}^2_{k,v}(\alpha_{k+1,v})$. Alternatively we could simply use [Kal16, Lemma 4.4]: see Remark 4.1.1.

For each $k \geq 1$, choose representatives for $\mathrm{Gal}(E_k/E_{k-1})/\mathrm{Gal}(E_{k,\dot{v}}/E_{k-1,\dot{v}})$, and choose lifts of these representatives in $\mathrm{Gal}(\overline{F}/E_{k-1})$ to obtain a finite set $R_{k,v} \subset \mathrm{Gal}(\overline{F}/E_{k-1})$. We can and do assume that $1 \in R_{k,v}$. For convenience we also define $R_{0,v} = \{1\} \subset \mathrm{Gal}(\overline{F}/F)$. For any $k \geq 0$, $R'_{k,v} := R_{0,v}R_{1,v}\dots R_{k,v} \subset \mathrm{Gal}(\overline{F}/F)$ projects to a set of representatives for $\mathrm{Gal}(E_k/F)/\mathrm{Gal}(E_{k,\dot{v}}/F_v)$. For $v \in V$ and $k \geq 0$ let $\zeta_{k,v} : \{v\}_{E_k} \to \{v\}_{E_{k+1}}$ be the section such that for all $r \in R'_{k,v}$, $\zeta_{k,v}(r \cdot \dot{v}_k) = r \cdot \dot{v}_{k+1}$. Define $j_{k,v} : E^\times_{k,\dot{v}} \hookrightarrow I(E_k)$ by $(j_{k,v}(x))_{\dot{v}_k} = x$ and $(j_{k,v}(x))_w = 1$ for $w \neq \dot{v}_k$. We have natural inclusions $E^\times_{k,\dot{v}} \subset E^\times_{k+1,\dot{v}}$ and for $x \in E^\times_{k,\dot{v}}$ we have

$$j_{k,v}(x) = \prod_{r \in R_{k+1,v}} r\left(j_{k+1,v}(x)\right). \tag{4.2.1}$$

Following Proposition 3.2.1 define, for all $k \geq 0$, $\alpha'_k \in Z^2(E_k/F, \mathrm{Maps}(V_{E_k}, I(E_k)))$ by

$$\alpha'_k(r_1\sigma r_2^{-1}, r_2\tau r_3^{-1})(r_1 \cdot \dot{v}_k) = r_1\left(j_{k,v}(\alpha_{k,v}(\sigma,\tau))\right) \tag{4.2.2}$$

for $v \in V$, $\sigma, \tau \in \mathrm{Gal}(E_{k,\dot{v}}/F_v)$ and $r_1, r_2, r_3 \in R'_{k,v}$. That is, α'_k is obtained by aggregating

$$\phi^{\mathrm{Gal}(E_k/F)}_{\mathrm{Gal}(E_{k,\dot{v}}/F_v)}(\mathrm{ES}^2_{R'_{k,v}}(j_{k,v}(\alpha_{k,v}))) \in Z^2(E_k/F, \mathrm{Maps}(\{v\}_{E_k}, I(E_k)))$$

for $v \in V$.

Definition 4.2.1. *Suppose that A is a commutative group. For $k \geq 0$ and $\alpha : \mathrm{Gal}(\overline{F}/F) \times \mathrm{Gal}(E_{k+1}/F) \to \mathrm{Maps}(V_{E_{k+1}}, A)$, define*

$$\mathrm{AWES}^2_k(\alpha) : \mathrm{Gal}(\overline{F}/F) \times \mathrm{Gal}(E_k/F) \to \mathrm{Maps}(V_{E_k}, A)$$

by

$$\mathrm{AWES}^2_k(\alpha)(\sigma,\tau)(\sigma_k\tau \cdot w) := \prod_{n \in \mathrm{Gal}(E_{k+1}/E_k)} \frac{\alpha(\sigma, n\tilde{\tau})(\sigma_{k+1}n\tilde{\tau} \cdot \zeta_{k,v}(w))}{\alpha(\sigma, n)(\sigma_{k+1}n \cdot \zeta_{k,v}(\tau \cdot w))}.$$

In this formula $\sigma \in \mathrm{Gal}(\overline{F}/F)$ has image σ_{k+1} in $\mathrm{Gal}(E_{k+1}/F)$ and σ_k in $\mathrm{Gal}(E_k/F)$, $\tau \in \mathrm{Gal}(E_k/F)$ and $\tilde{\tau} \in \mathrm{Gal}(E_{k+1}/F)$ is any lift of τ, $v \in V$ and $w \in \{v\}_{E_k}$.

Note that AWES^2_k depends on the choice of representatives $R'_{k,v}$ only via $\zeta_{k,v}$.

Lemma 4.2.2. *For all $k \geq 0$ we have* $\mathrm{AWES}^2_k(\alpha'_{k+1}) = \alpha'_k$.

Note that a priori the left hand side is only a map $\mathrm{Gal}(E_{k+1}/F) \times \mathrm{Gal}(E_k/F) \to \mathrm{Maps}(V_{E_k}, I(E_{k+1}))$. The lemma implies that is is inflated from a map $\mathrm{Gal}(E_k/F)^2 \to \mathrm{Maps}(V_{E_k}, I(E_k))$.

Proof. Fix $\sigma \in \mathrm{Gal}(E_{k+1}/F)$, $\tau \in \mathrm{Gal}(E_k/F)$ and $\gamma \in R'_{k,v}$. In $\mathrm{Gal}(E_k/F)$ write $\tau\gamma = r_2 g_2$ where $r_2 \in R'_{k,v}$ and $g_2 \in \mathrm{Gal}(E_{k,\dot{v}}/F_v)$. Let $\tilde{\tau} \in \mathrm{Gal}(E_{k+1}/F)$ be any lift of τ and let $\widetilde{g_2} \in \mathrm{Gal}(E_{k+1,\dot{v}}/F_v)$ be any lift of g_2. Note that

$$
\begin{aligned}
\{n\tilde{\tau} \,|\, n \in \mathrm{Gal}(E_{k+1}/E_k)\} =& \{r_2 u n_v \widetilde{g_2}\gamma^{-1} \,|\, u \in R_{k+1,v},\ n_v \in \mathrm{Gal}(E_{k+1,\dot{v}}/E_{k,\dot{v}})\},\\
\mathrm{Gal}(E_{k+1}/E_k) =& \{r_2 u n_v r_2^{-1} \,|\, u \in R_{k+1,v},\ n_v \in \mathrm{Gal}(E_{k+1,\dot{v}}/E_{k,\dot{v}})\}.
\end{aligned}
$$

In $\mathrm{Gal}(E_k/F)$ write $\sigma_k r_2 = r_1 g_1$ where $r_1 \in R'_{k,v}$ and $g_1 \in \mathrm{Gal}(E_{k,\dot{v}}/F_v)$. Choose $\widetilde{g_1} \in \mathrm{Gal}(E_{k+1,\dot{v}}/F_v)$ lifting g_1. For every $u \in R_{k+1,v}$ we can decompose $\sigma r_2 u \in \mathrm{Gal}(E_{k+1}/F)$ as follows: $\sigma r_2 u = r_1 u' \widetilde{g_1} x_v$ where $u' \in R_{k+1,v}$ and $x_v \in \mathrm{Gal}(E_{k+1,\dot{v}}/E_{k,\dot{v}})$ depend on u. Moreover $u \mapsto u'$ realizes a bijection from $R_{k+1,v}$ to itself: $r_1^{-1}\sigma r_2 \widetilde{g_1}^{-1} \in \mathrm{Gal}(E_{k+1}/E_k)$ induces a permutation of the set of places of E_{k+1} lying above $\dot{v}_k$.

$$
\begin{aligned}
\mathrm{AWES}^2_k(\alpha'_{k+1})(\sigma,\tau)(\sigma_k\tau\gamma\cdot\dot{v}_k) =& \prod_{n \in \mathrm{Gal}(E_{k+1}/E_k)} \frac{\alpha'_{k+1}(\sigma, n\tilde{\tau})(\sigma n\tilde{\tau}\gamma\cdot\dot{v}_{k+1})}{\alpha'_{k+1}(\sigma,n)(\sigma n r_2\cdot\dot{v}_{k+1})}\\
=& \prod_{u,n_v} \frac{\alpha'_{k+1}(r_1u'\widetilde{g_1}x_v(r_2u)^{-1}, r_2un_v\widetilde{g_2}\gamma^{-1})(r_1u'\cdot\dot{v}_{k+1})}{\alpha'_{k+1}(r_1u'\widetilde{g_1}x_v(r_2u)^{-1}, r_2un_vr_2^{-1})(r_1u'\cdot\dot{v}_{k+1})}
\end{aligned}
$$

using the above bijections. Now apply definition (4.2.2) of α'_{k+1} to the numerator (resp. denominator), with (r_1, r_2, r_3) replaced by (r_1u', r_2u, γ) (resp. (r_1u', r_2u, r_2)):

$$
\begin{aligned}
\mathrm{AWES}^2_k(\alpha'_{k+1})(\sigma,\tau)(\sigma_k\tau\gamma\cdot\dot{v}_k) =& \prod_u r_1u' \left(\prod_{n_v} \frac{j_{k+1,v}\left(\alpha_{k+1,v}(\widetilde{g_1}x_v, n_v\widetilde{g_2})\right)}{j_{k+1,v}\left(\alpha_{k+1,v}(\widetilde{g_1}x_v, n_v)\right)} \right)\\
=& \prod_u r_1u' \left(j_{k+1,v}\left(\alpha_{k,v}(g_1,g_2)\right)\right)\\
=& r_1 \left(j_{k,v}(\alpha_{k,v}(g_1,g_2))\right)\\
=& \alpha'_k(r_1g_1r_2^{-1}, r_2g_2\gamma^{-1})(r_1\cdot\dot{v}_k)\\
=& \alpha'_k(\sigma,\tau)(\sigma\tau\gamma\cdot\dot{v}_k).
\end{aligned}
$$

The second equality follows from $\alpha_{k,v} = \mathrm{AW}^2_{k,v}(\alpha_{k+1,v})$. The third equality is a consequence of (4.2.1). The third equality follows from the definition (4.2.2) of α'_k, and the last equality from the definition of r_1, r_2, g_1, g_2. □

Remark 4.2.3. *One could define* AWES^2 *axiomatically, as we did for* AW^2 *in Section 3.1, for general quadruples* $(G, N, H, R_{G/N}, R_N)$ *where* G *is a finite group,* N *a normal subgroup of* G*,* H *a subgroup of* G*,* $R_{G/N} \subset G$ *a set of representatives for* $G/HN = (G/N)/(HN/N)$ *such that* $1 \in R_{G/N}$*, and* $R_N \subset N$ *a set of representatives for* $N/(N \cap H)$ *such that* $1 \in R_N$*. One could also state the generalization of Lemma 4.2.2 in this context, with an identical proof. Note that it would apply to 2-cocycles* α' *taking values in a* twice *induced module, that is* $\mathbb{Z}[G/H] \otimes_{\mathbb{Z}} \mathrm{ind}_H^G(A)$ *for some* H*-module* A*. Indeed Definition 4.2.1 is essentially used with* $A = (E_k \otimes_F F_v)^\times = \prod_{w|v} E_w^\times$*, which is already induced with respect to the subgroup* $\mathrm{Gal}(E_{k,\dot{v}}/F_v)$ *of* $\mathrm{Gal}(E_k/F)$*. We will not need this generality, however.*

4.3 Properties of AWES_k^2

To establish the analogue of Proposition 3.1.5, we introduce variants of AWES_k^2 in degrees 0 and 1.

Definition 4.3.1. *Fix* $k \geq 0$*.*

1. *Suppose that* A *is a commutative group. For* $\beta : \mathrm{Gal}(E_{k+1}/F) \to \mathrm{Maps}(V_{E_{k+1}}, A)$*, define* $\mathrm{AWES}_k^1(\beta) : \mathrm{Gal}(E_k/F) \to \mathrm{Maps}(V_{E_k}, A)$ *by*

$$\mathrm{AWES}_k^1(\beta)(\sigma)(\sigma \cdot w) = \prod_{n \in \mathrm{Gal}(E_{k+1}/E_k)} \frac{\beta(n\tilde{\sigma})(n\tilde{\sigma} \cdot \zeta_{k,v}(w))}{\beta(n)(n \cdot \zeta_{k,v}(\sigma \cdot w))}$$

   for $\sigma \in \mathrm{Gal}(E_k/F)$ *and* $w \in \{v\}_{E_k}$*. In this formula* $\tilde{\sigma} \in \mathrm{Gal}(E_{k+1}/F)$ *is any lift of* σ*.*

2. *Suppose that* A *is a* $\mathrm{Gal}(E_{k+1}/E_k)$*-module. For* $\beta \in \mathrm{Maps}(V_{E_{k+1}}, A)$ *define* $\mathrm{AWES}_k^0(\beta) \in \mathrm{Maps}(V_{E_k}, A^{\mathrm{Gal}(E_{k+1}/E_k)})$ *by*

$$\mathrm{AWES}_k^0(\beta)(w) = N_{E_{k+1}/E_k}(\beta(\zeta_{k,v}(w)))$$

   for $w \in \{v\}_{E_k}$*.*

Lemma 4.3.2. *Fix* $k \geq 0$*.*

1. *Suppose that* A *is a* $\mathrm{Gal}(\overline{F}/F)$*-module. For* $\beta : \mathrm{Gal}(E_{k+1}/F) \to \mathrm{Maps}(V_{E_{k+1}}, A)$*, we have the equality of maps* $\mathrm{Gal}(\overline{F}/F) \times \mathrm{Gal}(E_k/F) \to \mathrm{Maps}(V_{E_k}, A)$

$$\mathrm{AWES}_k^2(\mathrm{d}(\beta)) = \mathrm{d}(\mathrm{AWES}_k^1(\beta)).$$

2. *Suppose that A is a $\mathrm{Gal}(E_{k+1}/F)$-module. For $\beta \in \mathrm{Maps}(V_{E_{k+1}}, A)$, we have the equality of maps $\mathrm{Gal}(E_{k+1}/F) \to \mathrm{Maps}(V_{E_k}, A)$*

$$\mathrm{AWES}_k^1(\mathrm{d}(\beta)) = \mathrm{d}(\mathrm{AWES}_k^0(\beta)).$$

The right hand side is a map $\mathrm{Gal}(E_k/F) \to \mathrm{Maps}(V_{E_k}, N_{E_{k+1}/E_k}(A))$.

Proof. 1. Let $v \in S$, $w \in \{w\}_k$, $\sigma \in \mathrm{Gal}(E_{k+1}/F)$ and $\tau \in \mathrm{Gal}(E_k/F)$. Let $\overline{\sigma}$ be the image of σ in $\mathrm{Gal}(E_k/F)$, and fix $\tilde{\tau} \in \mathrm{Gal}(E_{k+1}/F)$ lifting τ. We have

$$\begin{aligned}
\mathrm{d}(\mathrm{AWES}_k^1(\beta))(\sigma,\tau)(\overline{\sigma}\tau\cdot w) &= \frac{\mathrm{AWES}_k^1(\beta)(\overline{\sigma})(\overline{\sigma}\tau\cdot w)\sigma(\mathrm{AWES}_k^1(\beta)(\tau))(\overline{\sigma}\tau\cdot w)}{\mathrm{AWES}_k^1(\beta)(\overline{\sigma}\tau)(\overline{\sigma}\tau\cdot w)} \\
&= \prod_{n\in\mathrm{Gal}(E_{k+1}/E_k)} \frac{\beta(n\sigma)(n\sigma\cdot\zeta_{k,v}(\tau\cdot w))}{\beta(n)(n\cdot\zeta_{k,v}(\overline{\sigma}\tau\cdot w))} \\
&\qquad \times \sigma\left(\frac{\beta(n\tilde{\tau})(n\tilde{\tau}\cdot\zeta_{k,v}(w))}{\beta(n)(n\cdot\zeta_{k,v}(\tau\cdot w))}\right) \\
&\qquad \times \frac{\beta(n)(n\cdot\zeta_{k,v}(\overline{\sigma}\tau\cdot w))}{\beta(n\sigma\tilde{\tau})(n\sigma\tilde{\tau}\cdot\zeta_{k,v}(w))} \\
&= \prod_{n\in\mathrm{Gal}(E_{k+1}/E_k)} \frac{\sigma\left(\beta(n\tilde{\tau})(n\tilde{\tau}\cdot\zeta_{k,v}(w))\right)}{\beta(n\sigma\tilde{\tau})(n\sigma\tilde{\tau}\cdot\zeta_{k,v}(w))} \\
&\qquad \times \frac{\beta(n\sigma)(n\sigma\cdot\zeta_{k,v}(\tau\cdot w))}{\sigma\left(\beta(n)(n\cdot\zeta_{k,v}(\tau\cdot w))\right)} \\
&= \prod_{n\in\mathrm{Gal}(E_{k+1}/E_k)} \frac{\sigma\left(\beta(n\tilde{\tau})(n\tilde{\tau}\cdot\zeta_{k,v}(w))\right)}{\beta(\sigma n\tilde{\tau})(\sigma n\tilde{\tau}\cdot\zeta_{k,v}(w))} \\
&\qquad \times \frac{\beta(\sigma n)(\sigma n\cdot\zeta_{k,v}(\tau\cdot w))}{\sigma\left(\beta(n)(n\cdot\zeta_{k,v}(\tau\cdot w))\right)} \\
&= \prod_{n\in\mathrm{Gal}(E_{k+1}/E_k)} \frac{\mathrm{d}\beta(\sigma,n\tilde{\tau})(\sigma n\tilde{\tau}\cdot\zeta_{k,v}(w))}{\beta(\sigma)(\sigma n\tilde{\tau}\cdot\zeta_{k,v}(w))} \\
&\qquad \times \frac{\beta(\sigma)(\sigma n\cdot\zeta_{k,v}(\tau\cdot w)))}{\mathrm{d}\beta(\sigma,n)(\sigma n\cdot\zeta_{k,v}(\tau\cdot w))} \\
&= \prod_{n\in\mathrm{Gal}(E_{k+1}/E_k)} \frac{\mathrm{d}\beta(\sigma,n\tilde{\tau})(\sigma n\tilde{\tau}\cdot\zeta_{k,v}(w))}{\mathrm{d}\beta(\sigma,n)(\sigma n\cdot\zeta_{k,v}(\tau\cdot w))} \\
&= \mathrm{AWES}_k^2(d\beta)(\sigma,\tau)(\overline{\sigma}\tau\cdot w).
\end{aligned}$$

We have used the fact that for any $u \in \{v\}_{E_{k+1}}$,

$$\mathrm{card}\,\{n \in \mathrm{Gal}(E_{k+1}/E_k) \,|\, n\tilde{\tau}\cdot\zeta_{k,v}(w) = u\} = \mathrm{card}\,\{n \in \mathrm{Gal}(E_{k+1}/E_k) \,|\, n\cdot\zeta_{k,v}(\tau\cdot w)) = u\}$$

that implies

$$\prod_{n\in\mathrm{Gal}(E_{k+1}/E_k)} \beta(\sigma)(\sigma n\tilde{\tau}\cdot\zeta_{k,v}(w)) = \prod_{n\in\mathrm{Gal}(E_{k+1}/E_k)} \beta(\sigma)(\sigma n\cdot\zeta_{k,v}(\tau\cdot w))).$$

2. Let $v \in S$ and $w \in \{v\}_{E_k}$. Let $\sigma \in \mathrm{Gal}(E_k/F)$ and fix $\tilde{\sigma} \in \mathrm{Gal}(E_{k+1}/F)$ lifting σ.

$$\begin{aligned}
\mathrm{d}(\mathrm{AWES}_k^0(\beta))(\sigma)(\sigma \cdot w) &= \frac{\sigma(\mathrm{AWES}_k^0(\beta))(\sigma \cdot w)}{\mathrm{AWES}_k^0(\beta)(\sigma \cdot w)} \\
&= \prod_{n \in \mathrm{Gal}(E_{k+1}/E_k)} \frac{\tilde{\sigma} n(\beta(\zeta_{k,v}(w)))}{n(\beta(\zeta_{k,v}(\sigma \cdot w)))} \\
&= \prod_{n \in \mathrm{Gal}(E_{k+1}/E_k)} \frac{n\tilde{\sigma}(\beta(\zeta_{k,v}(w)))}{n(\beta(\zeta_{k,v}(\sigma \cdot w)))} \\
&= \prod_{n \in \mathrm{Gal}(E_{k+1}/E_k)} \frac{\mathrm{d}\beta(n\tilde{\sigma})(n\tilde{\sigma} \cdot \zeta_{k,v}(w)) \times \beta(n\tilde{\sigma} \cdot \zeta_{k,v}(w))}{\mathrm{d}\beta(n)(n \cdot \zeta_{k,v}(\sigma \cdot w)) \times \beta(n \cdot \zeta_{k,v}(\sigma \cdot w))} \\
&= \prod_{n \in \mathrm{Gal}(E_{k+1}/E_k)} \frac{\mathrm{d}\beta(n\tilde{\sigma})(n\tilde{\sigma} \cdot \zeta_{k,v}(w))}{\mathrm{d}\beta(n)(n \cdot \zeta_{k,v}(\sigma \cdot w))} \\
&= \mathrm{AWES}_k^1(\mathrm{d}\beta)(\sigma)(\sigma \cdot w).
\end{aligned}$$

Again we have used the fact that for any $u \in \{v\}_{E_{k+1}}$,

$$\mathrm{card}\,\{n \in \mathrm{Gal}(E_{k+1}/E_k) \,|\, n\tilde{\sigma} \cdot \zeta_{k,v}(w) = u\} = \mathrm{card}\,\{n \in \mathrm{Gal}(E_{k+1}/E_k) \,|\, n \cdot \zeta_{k,v}(\sigma \cdot w)) = u\}$$

that implies

$$\prod_{n \in \mathrm{Gal}(E_{k+1}/E_k)} \beta(n\tilde{\sigma} \cdot \zeta_{k,v}(w)) = \prod_{n \in \mathrm{Gal}(E_{k+1}/E_k)} \beta(n \cdot \zeta_{k,v}(\sigma \cdot w))).$$

□

Corollary 4.3.3. *Fix $k \geq 0$, and suppose that A is a $\mathrm{Gal}(E_{k+1}/F)$-module.*

1. *Let $\beta : \mathrm{Gal}(E_{k+1}/F) \to \mathrm{Maps}(V_{E_{k+1}}, A)$ be such that $\mathrm{AWES}_k^2(\mathrm{d}(\beta))$ factors through $\mathrm{Gal}(E_k/F)^2$. Then $\mathrm{AWES}_k^1(\beta)$ takes values in $\mathrm{Maps}\left(V_{E_k}, A^{\mathrm{Gal}(E_{k+1}/E_k)}\right)$.*
2. *If $\beta \in Z^1\left(\mathrm{Gal}(E_{k+1}/F), \mathrm{Maps}(V_{E_{k+1}}, A)\right)$ then*

$$\mathrm{AWES}_k^1(\beta) \in Z^1\left(\mathrm{Gal}(E_k/F), \mathrm{Maps}\left(V_{E_k}, A^{\mathrm{Gal}(E_{k+1}/E_k)}\right)\right).$$

Proof. 1. Recall that a priori $\mathrm{AWES}_k^1(\beta) : \mathrm{Gal}(E_k/F) \to \mathrm{Maps}(V_{E_k}, A)$. By the previous lemma, for all $w \in V_{E_k}$, $\sigma \in \mathrm{Gal}(E_{k+1}/F)$ and $\tau \in \mathrm{Gal}(E_k/F)$, the quotient

$$\frac{\mathrm{AWES}_k^1(\beta)(\overline{\sigma})(\overline{\sigma}\tau \cdot w) \times \sigma\left(\mathrm{AWES}_k^1(\beta)(\tau)(\tau \cdot w)\right)}{\mathrm{AWES}_k^1(\beta)(\overline{\sigma}\tau)(\overline{\sigma}\tau \cdot w)}$$

depends on σ only via its image $\overline{\sigma} \in \mathrm{Gal}(E_k/F)$. Taking $\sigma \in \mathrm{Gal}(E_{k+1}/E_k)$ shows that $\mathrm{AWES}_k^1(\beta)(\tau)(\tau \cdot w)$ is invariant under $\mathrm{Gal}(E_{k+1}/E_k)$.

2. This follows directly from the first point and a second application of the previous lemma.

$\square$

We now establish the analogue of Lemma 3.1.6 for AWES_k^1 and AWES_k^2.

Lemma 4.3.4. *Let $k \geq 0$. Suppose that A is a commutative group.*

1. *The map*

$$\begin{aligned}\big\{\beta : \mathrm{Gal}(E_{k+1}/F) \to \mathrm{Maps}(V_{E_{k+1}}, A) \,\big|\, \beta(1) = 1\big\} \\ \to \{\beta : \mathrm{Gal}(E_k/F) \to \mathrm{Maps}(V_{E_k}, A) \,|\, \beta(1) = 1\}\end{aligned}$$

induced by AWES_k^1 is surjective.

2. *Let $K \subset \overline{F}$ be a Galois extension of F containing E_{k+1}. The map*

$$\begin{aligned}\big\{\alpha : \mathrm{Gal}(K/F) \times \mathrm{Gal}(E_{k+1}/F) \to \mathrm{Maps}(V_{E_{k+1}}, A) \,\big|\, \forall \sigma \in \mathrm{Gal}(K/F),\, \alpha(\sigma, 1) = 1\big\} \\ \to \{\alpha : \mathrm{Gal}(K/F) \times \mathrm{Gal}(E_k/F) \to \mathrm{Maps}(V_{E_k}, A) \,|\, \forall \sigma \in \mathrm{Gal}(K/F),\, \alpha(\sigma, 1) = 1\}\end{aligned}$$

induced by AWES_k^2 is surjective.

Proof. As in the proof of Lemma 3.1.6, in each case we exhibit a subset of the source such that restricting to this subset yields a bijection. Choose a section $s : \mathrm{Gal}(E_k/F) \to \mathrm{Gal}(E_{k+1}/F)$ such that $s(1) = 1$.

1. Restrict to the set of $\beta : \mathrm{Gal}(E_{k+1}/F) \to \mathrm{Maps}(V_{E_{k+1}}, A)$ such that for $n \in \mathrm{Gal}(E_{k+1}/E_k)$, $\sigma \in \mathrm{Gal}(E_k/F)$, $v \in V$ and $u \in \{v\}_{E_{k+1}}$, $\beta(ns(\sigma))(ns(\sigma) \cdot u) = 1$ unless $n = 1$, $\sigma \neq 1$ and u belongs to the image of $\zeta_{k,v} : \{v\}_{E_k} \to \{v\}_{E_{k+1}}$.

2. Restrict to the set of $\alpha : \mathrm{Gal}(K/F) \times \mathrm{Gal}(E_{k+1}/F) \to \mathrm{Maps}(V_{E_{k+1}}, A)$ such that for $\sigma \in \mathrm{Gal}(K/F)$, $n \in \mathrm{Gal}(E_{k+1}/E_k)$, $\tau \in \mathrm{Gal}(E_k/F)$, $v \in V$ and $u \in \{v\}_{E_{k+1}}$, $\alpha(\sigma, ns(\tau))(\sigma ns(\tau) \cdot u) = 1$ unless $n = 1$, $\tau \neq 1$ and u belongs to the image of $\zeta_{k,v} : \{v\}_{E_k} \to \{v\}_{E_{k+1}}$.

$\square$

4.4 Tate cocycles

Recall that for every $k \geq 0$ the kernel $C(E_k)^1$ of the surjective norm map $\|\cdot\|_k : C(E_k) \to \mathbb{R}_{>0}$ is compact, and that these norm maps commute with the norm maps for the Galois action $N_{E_{k+1}/E_k} : C(E_{k+1}) \to C(E_k)$, that is $\|x\|_{k+1} = \|N_{E_{k+1}/E_k}(x)\|_k$ for all $x \in$

$C(E_{k+1})$. In this section we will see the fundamental cocycles $\overline{\alpha}_k \in \mathbb{Z}^2(E_k/F, C(E_k))$ defined in Section 4.1 as taking values in $\mathrm{Maps}(V_{E_k}, C(E_k))$, by seeing elements of $C(E_k)$ as constant functions $V_{E_k} \to C(E_k)$.

Lemma 4.4.1. *There exists a family* $\left(\overline{\beta}_k^{(0)}\right)_{k\geq 0}$*, where* $\overline{\beta}_k^{(0)} : \mathrm{Gal}(E_k/F) \to \mathrm{Maps}(V_{E_k}, C(E_k))$*, such that:*

1. *For any* $k \geq 0$ *we have* $\overline{\alpha}_k/\overline{\alpha'_k} = \mathrm{d}\left(\overline{\beta}_k^{(0)}\right)$*, where* $\overline{\alpha'_k} := \alpha'_k \mod E_k^\times$.
2. *For any* $k \geq 0$ *we have*
$$\mathrm{AWES}_k^1\left(\overline{\beta}_{k+1}^{(0)}\right) \in \mathrm{Maps}(\mathrm{Gal}(E_k/F), \mathrm{Maps}(V_{E_k}, C(E_k))).$$
3. *For any* $k \geq 0$ *we have* $\left\|\mathrm{AWES}_k^1\left(\overline{\beta}_{k+1}^{(0)}\right)\right\|_k = \left\|\overline{\beta}_k^{(0)}\right\|_k$*, as functions* $\mathrm{Gal}(E_k/F) \times V_{E_k} \to \mathbb{R}_{>0}$.

Proof. For a given k, the existence of $\overline{\beta}_k^{(0)}$ satisfying the first condition is a consequence of compatibility between local and global fundamental classes (see [Tat66]). Note that if $\overline{\beta}_{k+1}^{(0)}$ is such that $\overline{\alpha}_{k+1}/\overline{\alpha'_{k+1}} = \mathrm{d}\left(\overline{\beta}_{k+1}^{(0)}\right)$, then by Lemma 4.3.2

$$\mathrm{d}\left(\mathrm{AWES}_k^1\left(\overline{\beta}_{k+1}^{(0)}\right)\right) = \mathrm{AWES}_k^2\left(\mathrm{d}\left(\overline{\beta}_{k+1}^{(0)}\right)\right) = \mathrm{AWES}_k^2(\overline{\alpha}_{k+1})/\overline{\mathrm{AWES}_k^2(\alpha'_{k+1})} = \overline{\alpha}_k/\overline{\alpha'_k} \tag{4.4.1}$$

factors through $\mathrm{Gal}(E_k/F)^2$, and by Corollary 4.3.3 $\mathrm{AWES}_k^1\left(\overline{\beta}_{k+1}^{(0)}\right)$ takes values in $\mathrm{Maps}(V_{E_k}, C(E_k))$. So the second condition in the lemma is a consequence of the first one.

Let us start with a family $\left(\overline{\beta}_k^{(0)}\right)_{k\geq 0}$ satisfying the first condition, and show that we can inductively multiply $\overline{\beta}_k^{(0)}$, $k \geq 1$, by a 1-coboundary so that the third condition is also satisfied. By (4.4.1) we know that

$$\mathrm{AWES}_k^1\left(\overline{\beta}_{k+1}^{(0)}\right)/\overline{\beta}_k^{(0)} \in Z^1(\mathrm{Gal}(E_k/F), \mathrm{Maps}(V_{E_k}, C(E_k)))$$

and by vanishing of $H^1(\mathrm{Gal}(E_k/F), \mathrm{Maps}(V_{E_k}, C(E_k)))$ there exists $b_k : V_{E_k} \to C(E_k)$ such that $\mathrm{AWES}_k^1\left(\overline{\beta}_{k+1}^{(0)}\right)/\overline{\beta}_k^{(0)} = \mathrm{d}(b_k)$. Choose $\tilde{b}_k : V_{E_{k+1}} \to C(E_{k+1})$ such that for any $w \in \{v\}_{E_k}$, $\left\|\tilde{b}_k(\zeta_{k,v}(w))\right\|_{k+1} = \|b_k(\tau \cdot \dot{v}_k)\|_k$. Equivalently, $\left\|\mathrm{AWES}_k^0(\tilde{b}_k)\right\|_k = \|b_k\|_k$. Substituting $\overline{\beta}_{k+1}^{(0)}/\mathrm{d}(\tilde{b}_k)$ for $\overline{\beta}_{k+1}^{(0)}$, the third condition becomes satisfied. ☐

Theorem 4.4.2. *There exists a family* $(\beta_k)_{k\geq 0}$ *with* $\beta_k \in C^1(E_k/F, \mathrm{Maps}(V_{E_k}, I(E_k, S_k)))$ *such that*

1. *For any $k \geq 0$ we have $\overline{\alpha}_k/\overline{\alpha'_k} = \mathrm{d}\left(\overline{\beta_k}\right)$, where $\overline{\beta_k} \in C^1(E_k/F, \mathrm{Maps}(V_{E_k}, C(E_k)))$ is the projection of β_k.*

2. *For any $k \geq 0$ we have $\mathrm{AWES}^1_k(\beta_{k+1}) = \beta_k$.*

Therefore, the family $(\alpha_k)_{k\geq 0}$ defined by $\alpha_k = \alpha'_k \times \mathrm{d}(\beta_k)$ is a family of Tate cocycles, compatible in the sense that $\mathrm{AWES}^2_k(\alpha_{k+1}) = \alpha_k$ for all $k \geq 0$.

Proof. Let $\left(\overline{\beta}_k^{(0)}\right)_{k\geq 0}$ be a family as in the previous Lemma. The space

$$X_k := \left\{\overline{\beta}_k : \mathrm{Gal}(E_k/F) \to \mathrm{Maps}(V_{E_k}, C(E_k)) \,\middle|\, \|\overline{\beta}_k\|_k = \|\overline{\beta}_k^{(0)}\|_k \text{ and } \overline{\alpha}_k/\overline{\alpha'_k} = \mathrm{d}\left(\overline{\beta}_k\right)\right\}$$

is compact for the topology induced by the product topology on

$$\mathrm{Maps}(\mathrm{Gal}(E_k/F), \mathrm{Maps}(V_{E_k}, C(E_k))) = \prod_{\mathrm{Gal}(E_k/F)\times V_{E_k}} C(E_k).$$

Moreover $\overline{\beta}_k^{(0)} \in X_k$. The inverse system $\left((X_k)_{k\geq 0}, \left(\mathrm{AWES}^1_k : X_{k+1} \to X_k\right)_{k\geq 0}\right)$ consists of non-empty compact topological spaces and continuous maps between them, therefore $\varprojlim_{k\geq 0} X_k \neq \emptyset$. Choose $\left(\overline{\beta}_k\right)_k \in \varprojlim X_k$. Such a family satisfies the two conditions in the proposition, but note that $\overline{\beta}_k$ takes values in $C(E_k)$.

Let us inductively choose lifts β_k of $\overline{\beta}_k$ such that $\mathrm{AWES}^1_k(\beta_{k+1}) = \beta_k$. Note that this imposes $\beta_k(1) = 1$ for all k. Choose any β_0 lifting $\overline{\beta}_0$ such that $\beta_0(1) = 1$. Suppose that β_k is given. If β is any lift of $\overline{\beta}_{k+1}$ such that $\beta(1) = 1$, then $\beta_k/\mathrm{AWES}^1_k(\beta)$ is a mapping $\mathrm{Gal}(E_k/F) \to \mathrm{Maps}(V_{E_k}, \mathcal{O}(E_{k+1}, S_{k+1}))$. By Lemma 4.3.4, there exists $\nu : \mathrm{Gal}(E_{k+1}/F) \to \mathrm{Maps}(V_{E_{k+1}}, \mathcal{O}(E_{k+1}, S_{k+1}))$ such that $\nu(1) = 1$ and $\beta_k/\mathrm{AWES}^1_k(\beta) = \mathrm{AWES}^1_k(\nu)$, and we let $\beta_{k+1} = \beta \times \nu$. □

Remark 4.4.3. *This result solves two problems at once:*

1. *Constructing a family of Tate cocycles $(\alpha_k)_{k\geq 0}$ compatible with respect to AWES^2_k, which will be useful to compare (generalized) Tate-Nakayama isomorphisms in the tower $(E_k)_{k\geq 0}$, by taking cup-products (Lemma 5.2.1 and Proposition 5.2.3).*

2. *Constructing a family $(\beta_k)_{k\geq 0}$ compatible with respect to AWES^1_k and realizing local-global compatibility, which will be useful to compare local and global (generalized) Tate-Nakayama isomorphisms (Lemmas 5.4.1 and 5.4.4 and Propositions 5.4.3 and 5.4.5).*

The proof suggests that it is not possible to solve the first problem separately from the second. One can show that if families $(\alpha_{k,v})_{k\geq 0, v\in V}$, $(R_{k,v})_{k\geq 0, v\in V}$ and $(\overline{\alpha}_k)_{k\geq 0}$ as above *are fixed, then* $(\overline{\beta_k})_{k\geq 0}$ *is determined up to*

$$B^1\left(\mathrm{Gal}(\overline{F}/F),\ \varprojlim_{k\geq 0} C(E_k)^0\right)$$

where $C(E_k)^0$ *is the connected component of* 1 *in* $C(E_k)$*, i.e. the closure of* $(\mathbb{R}\otimes_{\mathbb{Q}} E_k)^{\times,0}$ *in* $C(E_k)$*, where* $(\mathbb{R}\otimes_{\mathbb{Q}} E_k)^{\times,0}$ *is the connected component of* 1 *in* $(\mathbb{R}\otimes_{\mathbb{Q}} E_k)^{\times}$*.*

Note that while $\alpha_{k,v}$, α_k and $R_{k,v}$ can simply be chosen sequentially as k grows, the existence of a family $(\beta_k)_{k\geq 0}$ in Theorem 4.4.2 follows from a compactness argument. Let us give an alternative, constructive but more intricate argument for the existence of $(\beta_k)_{k\geq 0}$. For simplicity we assume that for any $k\geq 0$, E_{k+1} contains the narrow Hilbert class field of E_k, i.e. $N_{E_{k+1}/E_k}(C(E_{k+1}))$ is contained in the image of $(\mathbb{R}\otimes_{\mathbb{Q}} E_k)^{\times,0}\times \widehat{\mathcal{O}(E_k)}^{\times}$ in $C(E_k)$. This can be achieved by discarding some of the E_k's. Choose $\overline{\beta}_1^{(0)}$ such that $\mathrm{d}\left(\overline{\beta}_1^{(0)}\right) = \overline{\alpha}_1/\overline{\alpha'_1}$. Note that $\overline{\beta}_0^{(1)} := \mathrm{AWES}_0^1(\overline{\beta}_1^{(0)}) = 1$. For good measure let $\beta_0^{(1)} = 1$ and $\alpha_0 = 1$. We now proceed to inductively construct $\overline{\beta}_{k+1}^{(0)}$, $\beta_k^{(1)}$ and ϵ_{k-1} for $k\geq 1$, satisfying the following properties.

1. $\overline{\beta}_{k+1}^{(0)} : \mathrm{Gal}(E_{k+1}/F) \to \mathrm{Maps}(V_{E_{k+1}}, C(E_{k+1}))$ is such that $\overline{\alpha'_{k+1}} \times \mathrm{d}\left(\overline{\beta}_{k+1}^{(0)}\right) = \overline{\alpha}_{k+1}$.
2. $\beta_k^{(1)} : \mathrm{Gal}(E_k/F) \to \mathrm{Maps}(V_{E_k}, I(E_k, S_k))$ is a lift of $\mathrm{AWES}_k^1\left(\overline{\beta}_{k+1}^{(0)}\right)$ such that $\beta_k^{(1)}(1) = 1$.
3. $\epsilon_k \in \mathrm{Maps}(V_{E_k}, \widehat{\mathcal{O}(E_k)}^{\times})$ is such that $\mathrm{AWES}_{k-1}^1(\beta_k^{(1)}) = \beta_{k-1}^{(1)}\mathrm{d}(\epsilon_{k-1})$.

Let $k\geq 0$, assume that $\overline{\beta}_{k+1}^{(0)}$ and $\beta_k^{(1)}$ are constructed. First choose any $\overline{\beta}_{k+2}^{(0)}$: $\mathrm{Gal}(E_{k+2}/F) \to \mathrm{Maps}(V_{E_{k+2}}, C(E_{k+2}))$ such that $\overline{\alpha'_{k+2}} \times \mathrm{d}\left(\overline{\beta}_{k+2}^{(0)}\right) = \overline{\alpha}_{k+2}$. As we saw in the proof of Lemma 4.4.1, there exists $\overline{z}_{k+1} \in \mathrm{Maps}(V_{E_{k+1}}, C(E_{k+1}))$ such that $\mathrm{AWES}_{k+1}^1(\overline{\beta}_{k+2}^{(0)}) = \overline{\beta}_{k+1}^{(0)} \times \mathrm{d}(\overline{z}_{i+1})$. Applying AWES_k^1, we get

$$\mathrm{AWES}_k^1 \circ \mathrm{AWES}_{k+1}^1\left(\overline{\beta}_{k+2}^{(0)}\right) = \mathrm{AWES}_k^1\left(\overline{\beta}_{k+1}^{(0)}\right) \times \mathrm{d}\left(\mathrm{AWES}_k^0\left(\overline{z}_{k+1}\right)\right)$$

and we would like to let $\epsilon_k \in \mathrm{Maps}(V_{E_k}, (\mathbb{R}\otimes_{\mathbb{Q}} E_k)^{\times,0}\times\widehat{\mathcal{O}(E_k)}^{\times})$ be a lift of $\mathrm{AWES}_k^0\left(\overline{z}_{k+1}\right)$, which exists thanks to the hypothesis that E_{k+1} contains the narrow Hilbert class field of E_k. This is not quite right, since we want $\epsilon_k \in \mathrm{Maps}(V_{E_k}, \widehat{\mathcal{O}(E_k)}^{\times})$. By surjectivity of

$$\mathrm{AWES}_k^0 \circ \mathrm{AWES}_{k+1}^0 : \mathrm{Maps}(V_{E_{k+2}}, (\mathbb{R}\otimes_{\mathbb{Q}} E_{k+2})^{\times,0}) \to \mathrm{Maps}(V_{E_k}, (\mathbb{R}\otimes_{\mathbb{Q}} E_k)^{\times,0})$$

we see that up to dividing $\overline{\beta}_{k+2}^{(0)}$ by an element of $B^1(\mathrm{Gal}(E_{k+2}/F), \mathrm{Maps}(V_{E_{k+2}}, (\mathbb{R} \otimes_{\mathbb{Q}} E_{k+2})^{\times,0}))$, we can find $\epsilon_k \in \mathrm{Maps}(V_{E_k}, \widehat{\mathcal{O}(E_k)}^{\times})$. Now let $\beta_k^{(2)} = \beta_k^{(1)} \times \mathrm{d}(\epsilon_k)$, and as we saw in the proof of Theorem 4.4.2, there exists $\beta_{k+1}^{(1)} : \mathrm{Gal}(E_{k+1}/F) \to \mathrm{Maps}(V_{E_{k+1}}, I(E_{k+1}, S_{k+1}))$ a lift of $\mathrm{AWES}_{k+1}^1 \left(\overline{\beta}_{k+2}^{(0)}\right)$ such that $\beta_{k+1}^{(1)}(1) = 1$ and $\mathrm{AWES}_k^1(\beta_{k+1}^{(1)}) = \beta_k^{(2)}$. This concludes the construction of $\left(\overline{\beta}_{k+2}^{(0)}, \beta_{k+1}^{(1)}, \epsilon_k\right)$.

Define inductively $\beta_k^{(i+1)} = \mathrm{AWES}_k^1\left(\beta_{k+1}^{(i)}\right)$ for $i \geq 0$. Then for all $i > k \geq 0$, we have

$$\beta_k^{(i+2-k)} = \beta_k^{(i+1-k)} \times \mathrm{d}\left(\mathrm{AWES}_k^0 \circ \cdots \circ \mathrm{AWES}_{i-1}^0(\epsilon_i)\right)$$

and since $\mathrm{AWES}_k^0 \circ \cdots \circ \mathrm{AWES}_{i-1}^0(\epsilon_i) \in \mathrm{Maps}\left(V_{E_k}, N_{E_i/E_k}\left(\widehat{\mathcal{O}(E_i)}^{\times}\right)\right)$, by the existence theorem in local class field theory and Krasner's lemma the sequences $(\beta_k^{(i)})_{i>0}$ converge and we can define $\beta_k = \lim_{i \to +\infty} \beta_k^{(i)}$.

5 Generalized Tate-Nakayama morphisms

In this section we will construct N-th roots of the cochains $(\alpha_{k,v})_{v \in V}$, α_k', β_k and α_k for all $N \geq 1$ and $k \geq 0$. This is necessary to establish the global analogue of [Kal16, §4.5], i.e. to make explicit the morphism $\iota_{\dot{V}}$ of [Kal, Theorem 3.7.3] for the tower $(E_k)_{k \geq 0}$, and to study the localization map [Kal, (3.19)].

5.1 Choice of N-th roots

Proposition 5.1.1. *For any $v \in V$, there exists a family $\left(\sqrt[N]{\alpha_{k,v}}\right)_{N \geq 1, k \geq 0}$ where $\sqrt[N]{\alpha_{k,v}} : \mathrm{Gal}(E_{k,\dot{v}}/F_v)^2 \to \overline{F_v}^{\times}$ such that*

1. *for all $k \geq 0$, $\sqrt[1]{\alpha_{k,v}} = \alpha_{k,v}$,*
2. *for all $k \geq 0$ and $N, N' \geq 1$ such that N divides N', $\sqrt[N']{\alpha_{k,v}}^{N'/N} = \sqrt[N]{\alpha_{k,v}}$,*
3. *for all $k \geq 0$ and $N \geq 1$, $\mathrm{AW}_{k,v}^2(\sqrt[N]{\alpha_{k+1,v}}) = \sqrt[N]{\alpha_{k,v}}$.*

Proof. Using Bézout identities, we see that it is enough to construct families $\left(\sqrt[\ell^m]{\alpha_{k,v}}\right)_{m \geq 0, k \geq 0}$ for all primes ℓ. So fix a prime number ℓ. For a fixed $k \geq 0$, there exists a family $\left(\sqrt[\ell^m]{\alpha_{k,v}}\right)_{m \geq 0}$ satisfying the first two conditions in the proposition, and such that for all $m \geq 0$ and $\sigma \in \mathrm{Gal}(E_{k,\dot{v}}/F_v)$, $\sqrt[\ell^m]{\alpha_{k,v}}(\sigma, 1) = 1$. If we choose two such families for k and $k+1$, the last condition might not be satisfied, i.e. for some $m \geq 1$ the obstruction

$$\frac{\mathrm{AW}_{k,v}^2(\sqrt[\ell^m]{\alpha_{k+1,v}})}{\sqrt[\ell^m]{\alpha_{k,v}}} : \mathrm{Gal}(E_{k+1,\dot{v}}/F_v) \times \mathrm{Gal}(E_{k,\dot{v}}/F_v) \to \mu_{\ell^m}$$

could be non-trivial. Note that the target is contained in μ_{ℓ^m} because $\mathrm{AW}^2_{k,v}(\alpha_{k+1,v}) = \alpha_{k,v}$. Recall that $\mathbb{Z}_\ell(1)$ is defined as $\varprojlim_{m\geq 0} \mu_{\ell^m}$. By the second condition these obstructions, as m varies, glue to give a mapping

$$\mathrm{Gal}(E_{k+1,\dot{v}}/F_v) \times \mathrm{Gal}(E_{k,\dot{v}}/F_v) \to \mathbb{Z}_\ell(1)$$

which maps any element of $\mathrm{Gal}(E_{k+1,\dot{v}}/F_v) \times \{1\}$ to 1. Applying Lemma 3.1.6 with $A = \mathbb{Z}_\ell(1)$, we obtain that $(\sqrt[\ell^m]{\alpha_{k+1,v}})_{m\geq 0}$ can be chosen so that $\mathrm{AW}^2_{k,v}(\sqrt[\ell^m]{\alpha_{k+1,v}}) = \sqrt[\ell^m]{\alpha_{k,v}}$ for all $m \geq 0$. $\square$

Fix such a family for each $v \in V$. Recall from Section 4.2 the embedding $j_{k,v} : E^\times_{k,\dot{v}} \hookrightarrow I(E_k)$. We now want to extend to $j_{k,v} : \overline{F_v}^\times \hookrightarrow I(\overline{F})$. For $x \in \overline{F_v}^\times$, there exists $i \geq 0$ such that $x \in E^\times_{k+i,\dot{v}}$. Define

$$j_{k,v}(x) = \prod_{r \in R_{k+1,v} \dots R_{k+i,v}} r(j_{k+i,v}(x))$$

which does not depend on the choice of a big enough i. These extended embeddings $j_{k,v}$ also satisfy a compatibility formula similar to (4.2.1): for any $x \in \overline{F_v}^\times$ we have

$$j_{k,v}(x) = \prod_{r \in R_{k+1,v}} r\left(j_{k+1,v}(x)\right). \tag{5.1.1}$$

For $N \geq 1$ define $\sqrt[N]{\alpha'_k} : \mathrm{Gal}(E_k/F)^2 \to \mathrm{Maps}(V_{E_k}, I(\overline{F}))$ by

$$\sqrt[N]{\alpha'_k}(r_1 \sigma r_2^{-1}, r_2 \tau r_3^{-1})(r_1 \cdot \dot{v}_k) = r_1\left(j_{k,v}(\sqrt[N]{\alpha_{k,v}}(\sigma,\tau))\right)$$

for $r_1, r_2, r_3 \in R'_{k,v}$ and $\sigma, \tau \in \mathrm{Gal}(E_{k,\dot{v}}/F_v)$. Obviously $\sqrt[1]{\alpha'_k} = \alpha'_k$ and whenever N divides N', $\sqrt[N']{\alpha'_k}^{N'/N} = \sqrt[N]{\alpha'_k}$. By the same proof as Lemma 4.2.2, thanks to (5.1.1), we have

$$\mathrm{AWES}^2_k\left(\sqrt[N]{\alpha'_{k+1}}\right) = \sqrt[N]{\alpha'_k}.$$

Note that for any $k \geq 0$ and $v \in V$, there exists $i \geq 0$ such that $\sqrt[N]{\alpha_{k,v}}$ takes values in $E^\times_{k+i,\dot{v}}$ and so for any $w \in \{v\}_{E_k}$, $\sqrt[N]{\alpha'_k}(-,-)(w)$ takes values in $\mathbb{A}^\times_{E_{k+i}}$.

We now want to construct N-th roots $\sqrt[N]{\alpha_k}$ of the Tate classes α_k constructed in Section 4.4. For this it is necessary to take N-th roots of idèles, which may not be idèles. For S' a finite subset of V, let $\mathcal{I}(F, S') \subset \prod_{v\in V}\left(\overline{F} \otimes_F F_v\right)^\times$ be the set of families $(x_v)_v$ such that for any $v \notin S'$, there exists a finite Galois extension K/F unramified above v such that $x_v \in (\mathcal{O}_K \otimes_{\mathcal{O}_F} \mathcal{O}_{F_v})^\times = \prod_{w|v} \mathcal{O}^\times_{K_w}$. Let $\mathcal{I}(F) = \varinjlim_{S'} \mathcal{I}(F, S')$. Recall (Theorem 4.4.2) that $\alpha_k : \mathrm{Gal}(E_k/F)^2 \to \mathrm{Maps}(V_{E_k}, I(E_k))$ has the following properties:

- for all $\sigma, \tau \in \mathrm{Gal}(E_k/F)$ and $w_1, w_2 \in V_{E_k}$, $\alpha_k(\sigma,\tau)(w_1)/\alpha_k(\sigma,\tau)(w_2) \in E_k^\times$,

- for all $\sigma, \tau \in \mathrm{Gal}(E_k/F)$, $v \in V$ and $w \in \{v\}_{E_k}$, $\alpha_k(\sigma,\tau)(w) \in I(E_k)$ is a unit away from $S_{k,E_k} \cup \{v\}_{E_k}$.

It is crucial for $\sqrt[N]{\alpha_k}$ to enjoy similar properties.

Proposition 5.1.2. *There exists a family* $\left(\sqrt[N]{\alpha_k}\right)_{N\geq 1, k\geq 0}$ *where* $\sqrt[N]{\alpha_k} : \mathrm{Gal}(E_k/F)^2 \to \mathcal{I}(F)$ *such that*

1. *for all* $k \geq 0$, $\sqrt[1]{\alpha_k} = \alpha_k$,
2. *for all* $k \geq 0$ *and* $N, N' \geq 1$ *such that* N *divides* N', $\sqrt[N']{\alpha_k}^{N'/N} = \sqrt[N]{\alpha_k}$,
3. *for all* $k \geq 0$ *and* $N \geq 1$, $\mathrm{AWES}_k^2(\sqrt[N]{\alpha_{k+1}}) = \sqrt[N]{\alpha_k}$,
4. *for all* $k \geq 0$, $N \geq 1$, $\sigma, \tau \in \mathrm{Gal}(E_k/F)$ *and* $w_1, w_2 \in V_{E_k}$, $\sqrt[N]{\alpha_k}(\sigma,\tau)(w_1)/\sqrt[N]{\alpha_k}(\sigma,\tau)(w_2) \in \overline{F}^\times$,
5. *for all* $k \geq 0$, $N \geq 1$, $\sigma, \tau \in \mathrm{Gal}(E_k/F)$, $v \in V$ *and* $w \in \{v\}_{E_k}$, $\sqrt[N]{\alpha_k}(\sigma,\tau)(w) \in \mathcal{I}(F, S_k \cup \{v\} \cup N)$.

Proof. It will be convenient to fix an archimedean place u of F, so that in particular $\dot{u}_k \in S_{k,E_k}$ for all $k \geq 0$. As in the proof of Proposition 5.1.1 it is enough to restrict to powers of a fixed prime ℓ.

First we show how to construct a family $\left(\sqrt[\ell^m]{\alpha_k}\right)_{m\geq 0}$ for a fixed $k \geq 0$. For $m \geq 0$ and $\sigma, \tau \in \mathrm{Gal}(E_k/F)$ choose roots $\sqrt[\ell^m]{\alpha_k}(\sigma,\tau)(\sigma\tau \cdot \dot{u}_k) \in \mathcal{I}(F, S_k \cup \ell)$ such that $\sqrt[\ell^{m+1}]{\alpha_k}(\sigma,\tau)(\sigma\tau \cdot \dot{u}_k)^\ell = \sqrt[\ell^m]{\alpha_k}(\sigma,\tau)(\sigma\tau \cdot \dot{u}_k)$. We can further impose that $\sqrt[\ell^m]{\alpha_k}(\sigma,1)(\sigma \cdot \dot{u}_k) = 1$ for all $\sigma \in \mathrm{Gal}(E_k/F)$. Then choose, for $\sigma, \tau \in \mathrm{Gal}(E_k/F)$, $v \in V$ and $w \in \{v\}_{E_k} \smallsetminus \{\sigma\tau \cdot \dot{u}_k\}$, ℓ^m-th roots of $\alpha_k(\sigma,\tau)(w)/\alpha_k(\sigma,\tau)(\sigma\tau \cdot \dot{u}_k)$ in $(\overline{F}_{S_k \cup \{v\} \cup \ell})^\times$, and define $\sqrt[\ell^m]{\alpha_k}(\sigma,\tau)(w)$ as the products of these ℓ^m-th roots with $\sqrt[\ell^m]{\alpha_k}(\sigma,\tau)(\sigma\tau \cdot \dot{u}_k)$. This can be done compatibly as m varies. Again we can impose $\sqrt[\ell^m]{\alpha_k}(\sigma,1)(w) = 1$ for all $\sigma \in \mathrm{Gal}(E_k/F)$. We obtain a family $\left(\sqrt[\ell^m]{\alpha_k}\right)_{m\geq 0}$ satisfying all conditions in the proposition except for the third one.

The fact that these choices can be made compatibly as k varies, i.e. in such a way that the third condition is also satisfied, can be proved as in Proposition 5.1.1, using the fact that $\mathrm{AWES}_k^2(\alpha_{k+1}) = \alpha_k$ and Lemma 4.3.4 instead of Lemma 3.1.6. □

Fix a family $\left(\sqrt[N]{\alpha_k}\right)_{N\geq 1, k\geq 0}$ as in the proposition. We want to compare $\sqrt[N]{\alpha_k'}$ and $\sqrt[N]{\alpha_k}$. Recall (Theorem 4.4.2) that $\alpha_k = \alpha_k' \mathrm{d}(\beta_k)$, where $\beta_k : \mathrm{Gal}(E_k/F) \to \mathrm{Maps}(V_{E_k}, I(E_k, S_k))$.

Proposition 5.1.3. *There exists a family* $\left(\sqrt[N]{\beta_k}\right)_{N\geq 1, k\geq 0}$ *where* $\sqrt[N]{\beta_k} : \mathrm{Gal}(E_k/F) \to \mathrm{Maps}\left(V_{E_k}, \mathcal{I}(F, S_k \cup N)\right)$ *such that*

1. *for all* $k \geq 0$, $\sqrt[1]{\beta_k} = \beta_k$,

2. *for all* $k \geq 0$ *and* $N, N' \geq 1$ *such that* N *divides* N', $\sqrt[N']{\beta_k}^{N'/N} = \sqrt[N]{\beta_k}$,

3. *for all* $k \geq 0$ *and* $N \geq 1$, $\mathrm{AWES}_k^1(\sqrt[N]{\beta_{k+1}}) = \sqrt[N]{\beta_k}$.

Proof. Only the third condition is non-trivial, and the proof proceeds as in Propositions 5.1.1 and 5.1.2. □

Fix a family $\left(\sqrt[N]{\beta_k}\right)_{N\geq 1, k\geq 0}$ as in the proposition. Note that $\mathrm{d}\left(\sqrt[N]{\beta_k}\right) : \mathrm{Gal}(\overline{F}_{S_k\cup N}/F) \times \mathrm{Gal}(E_k/F) \to \mathrm{Maps}\left(V_{E_k}, \mathcal{I}(F, S_k \cup N)\right)$.

Definition 5.1.4. *For* $k \geq 0$ *and* $N \geq 1$, *let*

$$\delta_k(N) = \frac{\sqrt[N]{\alpha_k}}{\sqrt[N]{\alpha'_k}\ \mathrm{d}\left(\sqrt[N]{\beta_k}\right)} : \mathrm{Gal}(\overline{F}_{S_k\cup N}/F) \times \mathrm{Gal}(E_k/F) \to \mathrm{Maps}\left(V_{E_k}, \mathcal{I}(F)[N]\right)$$

where $\mathcal{I}(F)[N]$ *is the subgroup of* N*-torsion elements in* $\mathcal{I}(F)$.

By construction, we have:

- For all $k \geq 0$, $N \geq 1$ and $w \in V_{E_k}$, there exists a finite Galois extension K of F containing E_k such that $\delta_k(N)(w)$ factors through $\mathrm{Gal}(K/F) \times \mathrm{Gal}(E_k/F)$.

- For all $k \geq 0$, $N \geq 1$, $\sigma \in \mathrm{Gal}(\overline{F}_{S_k\cup N}/F)$, $\tau \in \mathrm{Gal}(E_k/F)$, $v \in V$ and $w \in \{v\}_{E_k}$,
$$\delta_k(N)(\sigma, \tau)(w) \in \mathcal{I}(F, S_k \cup \{v\} \cup N)[N].$$

- For all $k \geq 0$ and $N, N' \geq 1$ such that N divides N', we have $\delta_k(N')^{N'/N} = \delta_k(N)$.

- For all $k \geq 0$ and $N \geq 1$, $\mathrm{AWES}_k^2(\delta_{k+1}(N)) = \delta_k(N)$.

5.2 Generalized Tate-Nakayama morphism for the global tower

Using the compatible families of cochains constructed in the previous section, we now want to recast several of Kaletha's constructions in cohomology, but for actual cochains. First we describe the extension $P_{\dot{V}} \to \mathcal{E}_{\dot{V}} \to \mathrm{Gal}(\overline{F}/F)$ explicitly as a projective limit of extensions $P(E_k, \dot{S}'_{E_k}, N) \to \mathcal{E}_k(S', N) \to \mathrm{Gal}(\overline{F}_{S'\cup N}/F)$ constructed using $\sqrt[N]{\alpha_k}$, for varying k, S', N. This is the global analogue of [Kal16, §4.5]. Then we make explicit the morphism $\iota_{\dot{V}}$ of [Kal, Theorem 3.7.3] using this projective limit. To avoid repeating similar calculations we deduce these two constructions from Lemma 5.2.1 below.

Let us recall notation from [Kal, Lemma 3.1.7]. Suppose that $S' \subset V$. If M is an abelian group, define $!_k : M[S'_{E_k}] \to M[S'_{E_{k+1}}]$ by $!_k(\Lambda)(\zeta_{k,v}(w)) = \Lambda(w)$ for $v \in S'$ and $w \in \{v\}_{E_k}$, and $!_k(\Lambda)(u) = 0$ if $u \notin \{\zeta_{k,v}(w) \,|\, v \in S',\ w \in \{v\}_{E_k}\}$. Here $\zeta_{k,v}$ is the section of the natural projection $\{v\}_{E_{k+1}} \to \{v\}_{E_k}$ defined in Section 4.2.

Recall also the notion of unbalanced cup-product $\sqcup$ from [Kal16, §4.3].

Lemma 5.2.1. *Let T be a torus defined over F. Denote $Y = X_*(T)$. Let k be big enough so that E_k splits T. Let $N \geq 1$ be an integer. Let S' be a finite subset of V containing S_{k+1}. Let $\Lambda \in Y[S'_{E_k}]_0^{N_{E_k/F}} = \widehat{Z}^{-1}(\mathrm{Gal}(E_k/F), Y[S'_{E_k}]_0)$. Then we have an equality of maps $\mathrm{Gal}(\overline{F}_{S' \cup N}/F) \to T(\mathcal{O}_{S' \cup N})$:*

$$\sqrt[N]{\alpha_k} \underset{E_k/F}{\sqcup} \Lambda = \sqrt[N]{\alpha_{k+1}} \underset{E_{k+1}/F}{\sqcup} !_k(\Lambda).$$

Note that if $S_k \subset S'' \subset S'$ and the support of Λ is contained in S''_{E_k}, then the left hand side is inflated from a map $\mathrm{Gal}(\overline{F}_{S'' \cup N}/F) \to T(\mathcal{O}_{S'' \cup N})$.

Proof. For $\sigma \in \mathrm{Gal}(\overline{F}_{S' \cup N}/F)$ we have

$$\begin{aligned}\left(\sqrt[N]{\alpha_k} \sqcup_{E_k/F} \Lambda\right)(\sigma) &= \prod_{\tau \in \mathrm{Gal}(E_k/F)} \sqrt[N]{\alpha_k}(\sigma, \tau) \otimes \sigma\tau(\Lambda) \\ &= \prod_{\tau \in \mathrm{Gal}(E_k/F)} \prod_{w \in S'_{E_k}} \sqrt[N]{\alpha_k}(\sigma, \tau)(w) \otimes \sigma\tau(\Lambda)(w).\end{aligned}$$

Note that in this last expression, the tensor products land in $\mathcal{I}(F, S' \cup N) \otimes_{\mathbb{Z}} Y$, but the product over S'_{E_k} belongs to $\mathcal{O}^\times_{S' \cup N} \otimes_{\mathbb{Z}} Y = T(\mathcal{O}_{S' \cup N})$ because $\sum_{w \in S'_{E_k}} \Lambda(w) = 0$, using the third condition in Proposition 5.1.2. Compare with the pairing [Kal, (3.24)]. Recall that $\sqrt[N]{\alpha_k} = \mathrm{AWES}^2_k(\sqrt[N]{\alpha_{k+1}})$ by construction in Theorem 4.4.2, so that

$$\begin{aligned}&\left(\sqrt[N]{\alpha_k} \underset{E_{k+1}/F}{\sqcup} \Lambda\right)(\sigma) \\ &= \prod_{\tau \in \mathrm{Gal}(E_k/F)} \prod_{v \in S'} \prod_{w \in \{v\}_{E_k}} \prod_{n \in \mathrm{Gal}(E_{k+1}/E_k)} \frac{\alpha(\sigma, n\widetilde{\tau})(\sigma_{k+1} n\widetilde{\tau} \cdot \zeta_{k,v}(w))}{\alpha(\sigma, n)(\sigma_{k+1} n \cdot \zeta_{k,v}(\tau \cdot w))} \otimes \sigma\tau\left(\Lambda(w)\right)\end{aligned}$$

where σ_{k+1} is the image of σ in $\mathrm{Gal}(E_{k+1}/F)$. We recognize $\left(\sqrt[N]{\alpha_{k+1}} \underset{E_{k+1}/F}{\sqcup} !_k(\Lambda)\right)$ at the numerator, by writing the product over $\tau \in \mathrm{Gal}(E_k/F)$ and $n \in \mathrm{Gal}(E_{k+1}/E_k)$ as a product over $\tau' \in \mathrm{Gal}(E_{k+1}/F)$ with $\tau' = n\widetilde{\tau}$. We obtain

$$\begin{aligned}&\left(\sqrt[N]{\alpha_{k+1}} \underset{E_{k+1}/F}{\sqcup} !_k(\Lambda)\right)(\sigma) / \left(\sqrt[N]{\alpha_k} \underset{E_k/F}{\sqcup} \Lambda\right)(\sigma) = \\ &\prod_{\tau \in \mathrm{Gal}(E_k/F)} \prod_{v \in S'} \prod_{w \in \{v\}_{E_k}} \prod_{n \in \mathrm{Gal}(E_{k+1}/E_k)} \sqrt[N]{\alpha_{k+1}}(\sigma, n)(\sigma n \cdot \zeta_{k,v}(\tau \cdot w)) \otimes \sigma\tau\left(\Lambda(w)\right).\end{aligned}$$

To simplify this expression we use the change of variable $u = \tau \cdot w$ to get

$$\prod_{\substack{v \in S' \\ n \in \mathrm{Gal}(E_{k+1}/E_k)}} \prod_{u \in \{v\}_{E_k}} \sqrt[N]{\alpha_{k+1}}(\sigma, n)(\sigma n \cdot \zeta_{k,v}(u)) \otimes \sigma \left(\sum_{\tau \in \mathrm{Gal}(E_k/F)} \tau\left(\Lambda(\tau^{-1} \cdot u)\right)\right)$$

and the sum over τ vanishes since $N_{E_k/F}(\Lambda) = 0$ by assumption. $\square$

Let $k \geq 0$ and $N \geq 1$, and let S' be a finite subset of V containing S_k. Recall the finite sub-$\mathrm{Gal}(E_k/F)$-module $M(E_k, \dot{S}'_{E_k}, N)$ of $\mathrm{Maps}(\mathrm{Gal}(E_k/F) \times S'_{E_k}, \frac{1}{N}\mathbb{Z}/\mathbb{Z})$ defined in [Kal, §3.3], and the finite commutative algebraic group $P(E_k, \dot{S}'_{E_k}, N)$ such that $X^*(P(E_k, \dot{S}'_{E_k}, N)) = M(E_k, \dot{S}'_{E_k}, N)$. For any finite commutative algebraic group Z over F such that $\exp(Z)|N$ and the Galois action on $A := X^*(Z)$ factors through $\mathrm{Gal}(E_k/F)$, we have an identification $\Psi(E_k, S', N) : \mathrm{Hom}(P(E_k, \dot{S}'_{E_k}, N), Z) \simeq A^\vee[\dot{S}'_{E_k}]_0^{N_{E_k/F}}$ (see [Kal, Lemma 3.3.2]). Recall also the 2-cocycle $\xi_k \in Z^2(\mathrm{Gal}(\overline{F}_{S' \cup N}/F), P(E_k, \dot{S}'_{E_k}, N))$ from [Kal, (3.5)], defined using an unbalanced cup-product:

$$\xi_k(S', N) = \mathrm{d}\,(\sqrt[N]{\alpha_k}) \underset{E_k/F}{\sqcup} c_{\mathrm{univ}}(E_k, S', N) \tag{5.2.1}$$

where $c_{\mathrm{univ}}(E_k, S', N) \in M(E_k, \dot{S}'_{E_k}, N)^\vee[\dot{S}'_{E_k}]_0^{N_{E_k/F}}$ is the image of $\mathrm{Id}_{P(E_k, \dot{S}'_{E_k}, N)}$ under $\Psi(E_k, S', N)$. Explicitly, for $w \in S'_{E_k}$ and $f \in M(E_k, \dot{S}'_{E_k}, N)$, $c_{\mathrm{univ}}(E_k, S', N)(w)(f) = f(1, w)$. The restriction of $\mathrm{d}\left(\sqrt[N]{\alpha_k}\right)$ to S'_{E_k} is a 3-cocycle

$$\mathrm{Gal}(\overline{F}_{S' \cup N}/F) \times \mathrm{Gal}(E_k/F)^2 \to \mathrm{Maps}(S'_{E_k}, \mathcal{I}(F, S' \cup N)[N])$$

such that

$$\frac{\mathrm{d}\left(\sqrt[N]{\alpha_k}\right)(\sigma_1, \sigma_2, \sigma_3)(w_1)}{\mathrm{d}\left(\sqrt[N]{\alpha_k}\right)(\sigma_1, \sigma_2, \sigma_3)(w_2)} \in \mu_N(\overline{F}) \subset \mathcal{I}(F, S' \cup N)[N].$$

This property allows to pair $\mathrm{d}\left(\sqrt[N]{\alpha_k}\right)(\sigma_1, \sigma_2, \sigma_3)$ with an element of $M(E_k, \dot{S}'_{E_k}, N)^\vee[\dot{S}'_{E_k}]_0$ to get an element of $P(E_k, \dot{S}'_{E_k}, N)$, as in [Kal, Fact 3.2.3]. This is the pairing used in the definition of $\xi_k(S', N)$ (5.2.1). The 2-cocycle $\xi_k(S', N)$ is universal in the sense that for any morphism of algebraic groups $f : P(E_k, \dot{S}'_{E_k}, N) \to Z$ over F we have

$$f_*(\xi_k(S', N)) = \mathrm{d}\,(\sqrt[N]{\alpha_k}) \underset{E_k/F}{\sqcup} \Psi(E_k, S', N)(f). \tag{5.2.2}$$

Definition 5.2.2. *Let $k \geq 0$ and $N \geq 1$, and let S' be a finite subset of V containing S_k. Define $\mathcal{E}_k(S', N)$ as the central extension $P(E_k, \dot{S}'_{E_k}, N) \underset{\xi_k(S', N)}{\boxtimes} \mathrm{Gal}(\overline{F}_{S' \cup N}/F)$.*

Recall that set-theoretically this is $P(E_k, \dot{S}'_{E_k}, N) \times \mathrm{Gal}(\overline{F}_{S' \cup N}/F)$, with group law

$$(x \boxtimes \sigma)(y \boxtimes \tau) = x\sigma(y)\xi_k(S', N)(\sigma, \tau) \boxtimes \sigma\tau.$$

Suppose $Z \hookrightarrow T$ is an injective morphism of commutative algebraic groups over F with Z finite, $\exp(Z)|N$ and T a torus split by E_k. Denote $A = X^*(Z)$, $Y = X_*(T)$ and $\overline{Y} = X_*(T/Z)$, so that we have a short exact sequence $0 \to Y \to \overline{Y} \to A^\vee \to 0$. Recall from [Kal, §3.7] the subgroup $\overline{Y}[S'_{E_k}, \dot{S}'_{E_k}]$ of $\overline{Y}[S'_{E_k}]$ consisting of all elements whose image in $A^\vee[S'_{E_k}]$ is supported on $\dot{S}'_{E_k}$. Also let $\overline{Y}[S'_{E_k}, \dot{S}'_{E_k}]_0 = \overline{Y}[S'_{E_k}, \dot{S}'_{E_k}] \cap \overline{Y}[S'_{E_k}]_0$

and $\overline{Y}[S'_{E_k}, \dot{S}'_{E_k}]_0^{N_{E_k/F}} = \overline{Y}[S'_{E_k}, \dot{S}'_{E_k}] \cap \overline{Y}[S'_{E_k}]_0^{N_{E_k/F}}$. As shown in [Kal, Proposition 3.7.8], we have a morphism

$$\begin{aligned} \iota_k(S', N) : \overline{Y}[S'_{E_k}, \dot{S}'_{E_k}]_0^{N_{E_k/F}} &\longrightarrow Z^1(P(E_k, \dot{S}'_{E_k}, N) \to \mathcal{E}_k(S', N), Z \to T(\mathcal{O}_{S' \cup N})) \\ \Lambda &\longmapsto \left(x \boxtimes \sigma \mapsto \Psi(E_k, S', N)^{-1}([\Lambda])(x) \times \left(\sqrt[N]{\alpha_k} \underset{E_k/F}{\sqcup} N\Lambda \right)(\sigma) \right) \end{aligned}$$

where $[\Lambda]$ is the image of Λ in $A^\vee[\dot{S}'_{E_k}]_0^{N_{E_k/F}}$. As explained in the proof of [Kal, Proposition 3.7.8], the fact that $\iota_k(S', N)(\Lambda)$ is a 1-cocycle is essentially equivalent to

$$\mathrm{d}\,(\sqrt[N]{\alpha_k}) \underset{E_k/F}{\sqcup} N\Lambda = \mathrm{d}\,(\sqrt[N]{\alpha_k}) \underset{E_k/F}{\sqcup} [\Lambda]. \tag{5.2.3}$$

Note that different pairings are used to form cup-products in this equality: [Kal, (3.24)] on the left, [Kal, (3.3)] on the right. To be rigorous we should point out that [Kal, Proposition 3.7.8] is stated with additional assumptions on S', but it is easy to check that the first point in this proposition does not use these assumptions.

As N and S' vary, there are natural morphisms between the extensions $\mathcal{E}_k(S', N)$, compatible with $\iota_k(S', N)$. Verifying this is purely formal, so we omit this verification.

The more challenging and interesting compatibility is when k varies. This is the main goal of this paper, and we can finally harvest the fruit of our labour. Assume that S' also contains S_{k+1}. Recall ([Kal, (3.7)]) the natural injection $M(E_k, \dot{S}'_{E_k}, N) \hookrightarrow M(E_{k+1}, \dot{S}'_{E_{k+1}}, N)$ mapping f to

$$(\sigma, w) \mapsto \begin{cases} f(\overline{\sigma}, \overline{w}) & \text{if } \sigma^{-1} \cdot w \in \dot{V}_{E_{k+1}}, \\ 0 & \text{otherwise.} \end{cases}$$

where $\overline{\sigma}$ (resp. $\overline{w}$) is the image of σ in $\mathrm{Gal}(E_k/F)$ (resp. V_{E_k}), and the dual surjective morphism $\rho_k(S', N) : P(E_{k+1}, \dot{S}'_{E_{k+1}}, N) \to P(E_k, \dot{S}'_{E_k}, N)$.

It is formal to check that for any finite commutative algebraic group Z over F such that $\exp(Z)|N$ and the Galois action on $A := X^*(Z)$ factors through $\mathrm{Gal}(E_k/F)$ and any finite $s' \subset V$, the following diagram is commutative.

$$\begin{tikzcd} \mathrm{Hom}(P(E_k, \dot{S}'_{E_k}, N), Z) \arrow[r, "\Psi(E_k, S', N)"] \arrow[d, "\rho_k(S', N)^*"] & A^\vee[\dot{S}'_{E_k}]_0^{N_{E_k/F}} \arrow[d, "!_k"] \\ \mathrm{Hom}(P(E_{k+1}, \dot{S}'_{E_{k+1}}, N), Z) \arrow[r, "\Psi(E_{k+1}, S', N)"] & A^\vee[\dot{S}'_{E_{k+1}}]_0^{N_{E_{k+1}/F}} \end{tikzcd} \tag{5.2.4}$$

Proposition 5.2.3. *Let $k \geq 0$ and $N \geq 1$, and let S' be a finite subset of V containing S_{k+1}.*

1. *Composition with $\rho_k(S',N)$ maps $\xi_{k+1}(S',N)$ to $\xi_k(S',N)$. In particular, we have a natural surjective morphism of extensions*

$$\begin{aligned} \mathcal{E}_{k+1}(S',N) &\longrightarrow \mathcal{E}_k(S',N) \\ x \boxtimes \sigma &\longmapsto \rho_k(S',N)(x) \boxtimes \sigma \end{aligned}$$

2. *Let $Z \hookrightarrow T$ be an injective morphism of commutative algebraic groups over F with Z finite and T a torus split by E_k. Assume that $\exp(Z)|N$. Let $Y = X_*(T)$ and $\overline{Y} = X_*(T/Z)$. Then the following diagram commutes*

$$\begin{tikzcd} \overline{Y}[S'_{E_k}, \dot{S}'_{E_k}]_0^{N_{E_k/F}} \arrow[r, "\iota_k(S',N)"] \arrow[d, "!_k"] & Z^1(P(E_k, \dot{S}'_{E_k}, N) \to \mathcal{E}_k(S',N), Z \to T(\mathcal{O}_{S' \cup N})) \arrow[d] \\ \overline{Y}[S'_{E_{k+1}}, \dot{S}'_{E_{k+1}}]_0^{N_{E_{k+1}/F}} \arrow[r, "\iota_{k+1}(S',N)"] & Z^1(P(E_{k+1}, \dot{S}'_{E_{k+1}}, N) \to \mathcal{E}_{k+1}(S',N), Z \to T(\mathcal{O}_{S' \cup N})) \end{tikzcd}$$

where the right vertical map is the inflation map induced by the morphism of extensions defined above.

Proof. 1. We use an argument similar to the proof of [Kal, Lemma 3.2.8]. We will apply Lemma 5.2.1. This way we avoid explicit computations with 3-cocycles $\mathrm{d}\left(\sqrt[N]{\alpha_k}\right)$. Denote $Z = P(E_k, \dot{S}'_{E_k}, N)$ and $A = X^*(Z)$. Fix a surjective morphism $X \to A$ where X is a free $\mathbb{Z}[\mathrm{Gal}(E_k/F)]$-module, and let $\overline{X}$ be the kernel. Associated to $X, \overline{X}$ are tori $T, \overline{T}$ and a short exact sequence $1 \to Z \to T \to \overline{T} \to 1$. Let $Y = X_*(T) = \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{Z})$ and $\overline{Y} = X_*(\overline{T}) = \mathrm{Hom}_{\mathbb{Z}}(\overline{X}, \mathbb{Z})$. We have a short exact sequence $0 \to Y[S'_{E_k}]_0 \to \overline{Y}[S'_{E_k}]_0 \to A^\vee[S'_{E_k}]_0 \to 0$, where $A = \mathrm{Hom}(X^*(Z), \mathbb{Q}/\mathbb{Z})$. The $\mathrm{Gal}(E_k/F)$-modules Y and $Y[S'_{E_k}]$ are cohomologically trivial (for Tate cohomology) and we have a short exact sequence $0 \to Y[S'_{E_k}]_0 \to Y[S'_{E_k}] \to Y \to 0$, therefore $Y[S'_{E_k}]_0$ is also cohomologically trivial. This implies in particular that there exists $\Lambda \in \overline{Y}[S'_{E_k}]_0^{N_{E_k/F}}$ mapping to the class of $c_{\mathrm{univ}}(E_k, S', N)$ in $A^\vee[S'_{E_k}]_0^{N_{E_k/F}} / I_{E_k/F}\left(A^\vee[S'_{E_k}]_0\right)$. Since $I_{E_k/F}\left(\overline{Y}[S'_{E_k}]_0\right)$ surjects to $I_{E_k/F}\left(A^\vee[S'_{E_k}]_0\right)$, we can even assume that the image $[\Lambda]$ of Λ in $A^\vee[S'_{E_k}]_0^{N_{E_k/F}}$ equals $c_{\mathrm{univ}}(E_k, S', N)$. Then $\Lambda \in \overline{Y}[S'_{E_k}, \dot{S}'_{E_k}]_0^{N_{E_k/F}}$, and applying Lemma 5.2.1 to $N\Lambda \in Y[S'_{E_k}]_0^{N_{E_k/F}}$ and taking the coboundary, we obtain the identity between 2-cocycles taking values in Z

$$\mathrm{d}\left(\sqrt[N]{\alpha_k}\right) \underset{E_k/F}{\sqcup} N\Lambda = \mathrm{d}\left(\sqrt[N]{\alpha_{k+1}}\right) \underset{E_{k+1}/F}{\sqcup} !_k(N\Lambda).$$

Using identity (5.2.3) on both sides, we obtain

$$\xi_k(S',N) = \mathrm{d}\left(\sqrt[N]{\alpha_{k+1}}\right) \underset{E_{k+1}/F}{\sqcup} [!_k(\Lambda)].$$

Moreover $[!_k(\Lambda)] = !_k([\Lambda]) = !_k(c_{\mathrm{univ}}(E_k, S', N)) = !_k\left(\Psi(E_k, S', N)\left(\mathrm{Id}_{P(E_k, \dot{S}'_{E_k}, N)}\right)\right)$ equals $\Psi(E_{k+1}, S', N)\left(\rho_k(S', N)\right)$ by commutativity of diagram (5.2.4). Therefore

$$\begin{aligned}\xi_k(S', N) =& \mathrm{d}\left(\sqrt[N]{\alpha_{k+1}}\right) \underset{E_{k+1}/F}{\sqcup} \Psi(E_{k+1}, S', N)(\rho_k(S', N)) \\ =& \rho_k(S', N)_*(\xi_{k+1}(S', N)).\end{aligned}$$

2. Let $\Lambda \in \overline{Y}[S'_{E_k}, \dot{S}'_{E_k}]_0^{N_{E_k/F}}$. The inflation of $\iota_k(S', N)(\Lambda)$ is the element of

$$Z^1(P(E_{k+1}, \dot{S}'_{E_{k+1}}, N) \to \mathcal{E}_{k+1}(S', N), Z \to T(\mathcal{O}_{S' \cup N}))$$

mapping $x \boxtimes \sigma \in \mathcal{E}_{k+1}(S', N)$ to

$$\Psi(E_k, S', N)^{-1}([\Lambda])(\rho_k(S', N)(x)) \times \left(\sqrt[N]{\alpha_k} \underset{E_k/F}{\sqcup} N\Lambda\right)(\sigma).$$

By (5.2.4) we have $\Psi(E_k, S', N)^{-1}([\Lambda]) \circ \rho_k(S', N) = \Psi(E_{k+1}, S', N)(!_k([\Lambda]))$ and moreover $!_k([\Lambda]) = [!_k(\Lambda)]$. The conclusion then follows from Lemma 5.2.1 applied to $N\Lambda$.

□

Thanks to the first part of Proposition 5.2.3 and obvious compatibilities with respect to enlarging S' and replacing N by a multiple, we can now define the extension $P \to \mathcal{E}$ of $\mathrm{Gal}(\overline{F}/F)$ as the projective limit of the extensions $P(E_k, \dot{S}'_{E_k}, N) \to \mathcal{E}_k(S', N)$ over triples (k, N, S') such that $S' \supset S_k$.

Let $Z \hookrightarrow T$ be an injective morphism of commutative algebraic groups over F with Z finite and T a torus. Let $Y = X_*(T)$ and $\overline{Y} = X_*(T/Z)$, and denote

$$\overline{Y}[V_{\overline{F}}, \dot{V}]_0^{N/F} = \varinjlim_{k, S'} \overline{Y}[S'_{E_k}, \dot{S}'_{E_k}]_0^{N_{E_k/F}}$$

where the limit is over pairs k, S' such that E_k splits T and $S' \supset S_k$.

Corollary 5.2.4. *Let $Z \hookrightarrow T$ be an injective morphism of commutative algebraic groups over F with Z finite and T a torus. Let $\overline{T} = T/Z$ and let $Y = X_*(T)$, $\overline{Y} = X_*(\overline{T})$. Then the morphisms $(\iota_k(S', N))_{k, S', N}$, for k, S', N such that E_k splits T, $\exp(Z) | N$ and $S' \supset S_k$, splice into a morphism*

$$\iota : \overline{Y}[V_{\overline{F}}, \dot{V}]_0^{N/F} \to Z^1(P \to \mathcal{E}, Z \to T(\overline{F})). \tag{5.2.5}$$

In Section 5.5 we will check that the class of the extension $P \to \mathcal{E}$ coincides with Kaletha's "canonical class" from [Kal]. Granting this, it is clear that ι in (5.2.5) lifts the cohomological isomorphism $\iota_{\dot{V}}$ of [Kal, Theorem 3.7.3].

5.3 Generalized Tate-Nakayama morphism for the local towers

In this section we fix $v \in V$. We want to study the relation of the map ι defined in Corollary 5.2.4 with the localization map loc_v defined in [Kal, §3.6]. This will necessitate defining loc_v (for varying k, S', N) for cochains rather than in cohomology. The first step is to recall several constructions from [Kal16]. We choose notation similar to the global case instead of notation used in [Kal16]. For $k \geq 0$ and $N \geq 1$, we have a central extension

$$P(E_{k,\dot{v}}, N) \to \mathcal{E}_{k,v}(N) \to \mathrm{Gal}(\overline{F_v}/F_v)$$

where $P(E_{k,\dot{v}}, N) := \mathrm{Res}_{E_{k,\dot{v}}/F_v}(\mu_N)/\mu_N$. In particular $M(E_{k,\dot{v}}, N) := X^*(P(E_{k,\dot{v}}, N))$ can be identified with $\mathbb{Z}/N\mathbb{Z}[\mathrm{Gal}(E_{k,\dot{v}}/F_v)]_0$. The central extension

$$\mathcal{E}_{k,v}(N) := P(E_{k,\dot{v}}, N) \underset{\xi_{k,v}(N)}{\boxtimes} \mathrm{Gal}(\overline{F_v}/F_v)$$

is defined using the 2-cocycle

$$\xi_{k,v}(N) := \mathrm{d}\left(\sqrt[N]{\alpha_{k,v}}\right) \underset{E_{k,\dot{v}}/F_v}{\sqcup} c_{\mathrm{univ}}(E_{k,\dot{v}}, N)$$

where $c_{\mathrm{univ}}(E_{k,\dot{v}}, N) \in X^*(P(E_{k,\dot{v}}, N))^\vee$ is killed by $N_{E_{k,\dot{v}}/F_v}$, and is defined as $f \mapsto f(1)$.

Suppose $Z \hookrightarrow T$ is an injective morphism of commutative algebraic groups over F_v with Z finite, $\exp(Z)|N$ and T a torus split by $E_{k,\dot{v}}$. Denote $Y = X_*(T)$ and $\overline{Y} = X_*(T/Z)$. We have a morphism

$$\begin{aligned}\iota_{k,v}(N) : \overline{Y}^{N_{E_{k,\dot{v}}/F_v}} &\longrightarrow Z^1(P(E_{k,\dot{v}}, N) \to \mathcal{E}_{k,v}(N), Z \to T(\overline{F_v}))\\ \Lambda &\longmapsto \left(x \boxtimes \sigma \mapsto \Psi(E_{k,\dot{v}}, N)^{-1}([\Lambda])(x) \times \left(\sqrt[N]{\alpha_{k,v}} \underset{E_{k,\dot{v}}/F_v}{\sqcup} N\Lambda\right)(\sigma)\right)\end{aligned}$$

The following lemma and proposition, using a formulation analogous to Lemma 5.2.1 and Proposition 5.2.3, are essentially proved in [Kal16, Lemma 4.5 and Lemma 4.7]. Note that we have arranged for the 1-cochain denoted α_k in [Kal16, Lemma 4.5] to be trivial. This slightly simplifies formulae. Then Kaletha's proof becomes a simpler analogue of that of Lemma 5.2.1, using $\mathrm{AW}^2_k(\sqrt[N]{\alpha_{k+1,v}}) = \sqrt[N]{\alpha_{k,v}}$ instead of $\mathrm{AWES}^2_k(\sqrt[N]{\alpha_{k+1}}) = \sqrt[N]{\alpha_k}$.

Lemma 5.3.1. *Let T be a torus defined over F_v. Denote $Y = X_*(T)$. Let k be big enough so that $E_{k,\dot{v}}$ splits T. Let $N \geq 1$ be an integer. Let $\Lambda \in Y^{N_{E_{k,\dot{v}}/F_v}}$. Then we have an equality of maps $\mathrm{Gal}(\overline{F_v}/F_v) \to T(\overline{F_v})$:*

$$\sqrt[N]{\alpha_{k,v}} \underset{E_{k,\dot{v}}/F_v}{\sqcup} \Lambda = \sqrt[N]{\alpha_{k+1,v}} \underset{E_{k+1,\dot{v}}/F_v}{\sqcup} \Lambda.$$

As in the global case, there are natural morphisms $\rho_{k,v}(N) : P(E_{k+1,\dot{v}}, N) \to P(E_{k,\dot{v}}, N)$, denoted p in [Kal16, (3.2)]. There are also natural morphisms as N varies, which we do not bother to name. As in the global case (5.2.4), for any finite commutative algebraic group Z over F_v such that $\exp(Z)|N$ and the Galois action on $A := X^*(Z)$ factors through $\mathrm{Gal}(E_{k,\dot{v}}/F_v)$, we have a commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}(P(E_{k,\dot{v}}, N), Z) \arrow[r, "\Psi(E_{k,\dot{v}},N)"] \arrow[d, "\rho_{k,v}(N)^*"] & (A^\vee)^{N_{E_{k,\dot{v}}/F_v}} \arrow[d, hook] \\
\mathrm{Hom}(P(E_{k+1,\dot{v}}, N), Z) \arrow[r, "\Psi(E_{k+1,\dot{v}},N)"] & (A^\vee)^{N_{E_{k+1,\dot{v}}/F_v}}
\end{tikzcd} \tag{5.3.1}$$

Proposition 5.3.2. *Let $k \geq 0$ and $N \geq 1$.*

1. *Composition with $\rho_{k,v}(N)$ maps $\xi_{k+1,v}(N)$ to $\xi_{k,v}(N)$. In particular, we have a natural morphism of extensions*

$$\begin{aligned}
\mathcal{E}_{k+1,v}(N) &\longrightarrow \mathcal{E}_{k,v}(N) \\
x \boxtimes \sigma &\longmapsto \rho_{k,v}(N)(x) \boxtimes \sigma.
\end{aligned}$$

2. *Let $Z \hookrightarrow T$ be an injective morphism of commutative algebraic groups over F_v with Z finite and T a torus split by $E_{k,\dot{v}}$. Assume that $\exp(Z)|N$. Let $Y = X_*(T)$ and $\overline{Y} = X_*(T/Z)$. Then the following diagram commutes*

$$\begin{tikzcd}
\overline{Y}^{N_{E_{k,\dot{v}}/F_v}} \arrow[r, "\iota_{k,v}(N)"] \arrow[d, equal] & Z^1(P(E_{k,\dot{v}}, N) \to \mathcal{E}_{k,v}(N), Z \to T(\overline{F_v})) \arrow[d] \\
\overline{Y}^{N_{E_{k+1,\dot{v}}/F_v}} \arrow[r, "\iota_{k+1,v}(N)"] & Z^1(P(E_{k+1,v}, N) \to \mathcal{E}_{k+1,v}(N), Z \to T(\overline{F_v}))
\end{tikzcd}$$

   where the right vertical map is inflation for the morphism of extensions defined above.

Proof. The proof is similar to that of Proposition 5.2.3, in fact slightly easier, so we omit it. ☐

Let $Z \hookrightarrow T$ be an injective morphism of commutative algebraic groups over F_v with Z finite and T a torus. Let $Y = X_*(T)$ and $\overline{Y} = X_*(T/Z)$. Denote $\overline{Y}^{N/F_v} = \overline{Y}^{N_{E_{k,\dot{v}}/F_v}}$ for any k such that $E_{k,\dot{v}}$ splits T.

Corollary 5.3.3. *Let $Z \hookrightarrow T$ be an injective morphism of commutative algebraic groups over F_v with Z finite and T a torus. Let $Y = X_*(T)$ and $\overline{Y} = X_*(T/Z)$. Then the* *morphisms* $(\iota_{k,v}(N))_{k,N}$*, for* k, N *such that* $E_{k,\dot{v}}$ *splits* T *and* $\exp(Z)|N$*, splice into a morphism*

$$\iota_v : \overline{Y}^{N/F_v} \to Z^1(P_v \to \mathcal{E}_v, Z \to T(\overline{F_v}))$$

lifting the morphism in cohomology of [Kal16, Theorem 4.8].

5.4 Localization

In this section $v \in V$ is fixed. We want to study the relationship between ι (Corollary 5.2.4), ι_v (Corollary 5.3.3) and loc_v ([Kal, §3.6]). We study it for fixed $k \geq 0$ first.

Recall ([Kal, (3.11)]) the morphisms $\mathrm{loc}_{k,v}(S', N) : P(E_{k,\dot{v}}, N) \to P(E_k, \dot{S}'_{E_k}, N)$. If $v \in S'$ it is dual to $f \mapsto (\sigma \mapsto f(\sigma, \dot{v}))$. We define it to be trivial if $v \notin S'$. It is $\mathrm{Gal}(E_{k,\dot{v}}/F_v)$-equivariant, and there are obvious commuting diagrams as S' and N vary.

For M a $\mathrm{Gal}(E_k/F)$-module, recall the morphism $l_{k,v} : M[S'_{E_k}]^{N_{E_k/F}} \to M^{N_{E_{k,\dot{v}}/F_v}}$ (denoted l_v^k in [Kal, Lemma 3.7.2]) defined by

$$l_{k,v}(\Lambda) = \sum_{r \in R'_{k,v}} r^{-1} \left(\Lambda(r \cdot \dot{v}_k)\right)$$

if $v \in S'$, and zero otherwise.

Lemma 5.4.1. *Let* T *be a torus defined over* F*. Denote* $Y = X_*(T)$*. Let* k *be big enough so that* E_k *splits* T*. Let* $N \geq 1$ *be an integer. Let* S' *be a finite subset of* V *containing* S_k*. Let* $\Lambda \in Y[S'_{E_k}]_0^{N_{E_k/F}}$*.*

Let $i \geq 0$ *be big enough so that* $\sqrt[N]{\alpha_{k,v}}$ *takes values in* $E^\times_{k+i,\dot{v}}$*. Then we have an equality of maps* $\mathrm{Gal}(\overline{F}/F) \to T(\overline{F} \otimes_F F_v)$*:*

$$\mathrm{pr}_v\left(\sqrt[N]{\alpha_k} \underset{E_k/F}{\sqcup} \Lambda\right) = \mathrm{ES}^1_{R'_{k+i,v}}\left(\sqrt[N]{\alpha_{k,v}} \underset{E_{k,\dot{v}}/F_v}{\sqcup} l_{k,v}(\Lambda)\right) \times \mathrm{d}\left(\mathrm{pr}_v(\sqrt[N]{\beta_k}) \underset{E_k/F}{\sqcup} \Lambda\right) \times \left(\mathrm{pr}_v(\delta_k(N)) \underset{E_k/F}{\sqcup} \Lambda\right).$$

In particular, upon restriction to $\mathrm{Gal}(\overline{F_v}/F_v)$ *and projection to* $T(\overline{F_v})$*:*

$$\mathrm{pr}_{\dot{v}}\left(\sqrt[N]{\alpha_k} \underset{E_k/F}{\sqcup} \Lambda\right) = \left(\sqrt[N]{\alpha_{k,v}} \underset{E_{k,\dot{v}}/F_v}{\sqcup} l_{k,v}(\Lambda)\right) \times \mathrm{d}\left(\mathrm{pr}_{\dot{v}}(\sqrt[N]{\beta_k}) \underset{E_k/F}{\sqcup} \Lambda\right) \times \left(\mathrm{pr}_{\dot{v}}(\delta_k(N)) \underset{E_k/F}{\sqcup} \Lambda\right).$$

Note that the first equality implicitly uses the identification

$$\mathrm{ind}_{\mathrm{Gal}(E_{k+i,\dot{v}}/F_v)}^{\mathrm{Gal}(E_{k+i}/F)}(E^\times_{k+i,\dot{v}}) \xrightarrow{\sim} (E_{k+i} \otimes_F F_v)^\times$$

$$f \longmapsto \prod_{g \in \mathrm{Gal}(E_{k+i,\dot{v}}/F_v))\backslash \mathrm{Gal}(E_{k+i}/F)} g^{-1}(f(g))$$

to see $\mathrm{ES}^1_{R'_{k+i,v}}\left(\sqrt[N]{\alpha_{k,v}} \underset{E_{k,\dot{v}}/F_v}{\sqcup} l_{k,v}(\Lambda) \right)$ as a map $\mathrm{Gal}(E_{k+i}/F) \to T(E_{k+i} \otimes_F F_v)$.

Proof. Recall that by definition of $\delta_k(N)$, we have $\sqrt[N]{\alpha_k} = \sqrt[N]{\alpha'_k}\mathrm{d}(\sqrt[N]{\beta_k})\delta_k(N)$, and we compute unbalanced cup-products with these three terms separately. In the case of $\delta_k(N)$ there is nothing to prove, so we first consider $\mathrm{d}(\sqrt[N]{\beta_k})$. By [Kal16, Fact 4.3] we have

$$\mathrm{d}(\sqrt[N]{\beta_k}) \underset{E_k/F}{\sqcup} \Lambda = \mathrm{d}\left(\sqrt[N]{\beta_k} \underset{E_k/F}{\sqcup} \Lambda \right)$$

and thus upon restriction to $\mathrm{Gal}(\overline{F_v}/F_v)$,

$$\mathrm{pr}_{\dot{v}}\left(\mathrm{d}(\sqrt[N]{\beta_k}) \underset{E_k/F}{\sqcup} \Lambda \right) = \mathrm{d}\left(\mathrm{pr}_{\dot{v}}\left(\sqrt[N]{\beta_k} \underset{E_k/F}{\sqcup} \Lambda \right) \right).$$

Let us now consider $\sqrt[N]{\alpha'_k}$. For $\sigma \in \mathrm{Gal}(E_k/F)$ we have

$$\mathrm{pr}_v\left(\left(\sqrt[N]{\alpha'_k} \sqcup_{E_k/F} \Lambda \right)(\sigma) \right) = \prod_{\gamma \in R'_{k,v}} \prod_{\tau \in \mathrm{Gal}(E_k/F)} \sqrt[N]{\alpha'_k}(\sigma, \tau)(\sigma\tau\gamma \cdot \dot{v}_k) \otimes \sigma\tau\left(\Lambda(\gamma \cdot \dot{v}_k)\right).$$

Write $\tau\gamma = r\tau'$ and $\sigma r = r'\sigma'$ where $r, r' \in R'_{k,v}$ and $\tau', \sigma' \in \mathrm{Gal}(E_{k,\dot{v}}/F_v)$ are functions of (σ, γ, τ). For σ and γ fixed the map $\tau \mapsto (r, \tau')$ is bijective onto $R'_{k,v} \times \mathrm{Gal}(E_{k,\dot{v}}/F_v)$. We obtain

$$\begin{aligned} &\mathrm{pr}_v\left(\left(\sqrt[N]{\alpha'_k} \sqcup_{E_k/F} \Lambda \right)(\sigma) \right) \\ &= \prod_{\gamma \in R'_{k,v}} \prod_{r \in R'_{k,v}} \prod_{\tau' \in \mathrm{Gal}(E_{k,\dot{v}}/F_v)} \sqrt[N]{\alpha'_k}(r'\sigma' r^{-1}, r\tau'\gamma^{-1})(r' \cdot \dot{v}_k) \otimes r'\sigma'\tau'\gamma^{-1}\left(\Lambda(\gamma \cdot \dot{v}_k)\right) \end{aligned}$$

where $r'\sigma' = \sigma r$, $r' \in R'_{k,v}$ and $\sigma' \in \mathrm{Gal}(E_{k,\dot{v}}/F_v)$ being functions of r. Recall that by definition,

$$\sqrt[N]{\alpha'_k}(r'\sigma' r^{-1}, r\tau'\gamma^{-1})(r' \cdot \dot{v}_k) = r'\left(j_{k,v}\left(\sqrt[N]{\alpha_{k,v}}(\sigma', \tau') \right) \right).$$

Therefore

$$\begin{aligned} &\mathrm{pr}_v\left(\left(\sqrt[N]{\alpha'_k} \underset{E_k/F}{\sqcup} \Lambda \right)(\sigma) \right) \\ &= \prod_{\gamma \in R'_{k,v}} \prod_{r \in R'_{k,v}} \prod_{\tau' \in \mathrm{Gal}(E_{k,\dot{v}}/F_v)} r'\left(j_{k,v}\left(\sqrt[N]{\alpha_{k,v}}(\sigma', \tau') \right) \otimes \sigma'\tau'\gamma^{-1}\left(\Lambda(\gamma \cdot \dot{v}_k)\right) \right) \end{aligned}$$

$$= \prod_{r\in R'_{k,v}} r' \left(\prod_{\tau'\in \mathrm{Gal}(E_{k,\dot{v}}/F_v)} j_{k,v}\left(\sqrt[N]{\alpha_{k,v}}(\sigma',\tau')\right) \otimes \sigma'\tau'\left(l_{k,v}(\Lambda)\right)\right).$$

The map $r \mapsto r'$ from $R'_{k,v}$ to itself is bijective, so we can write this as

$$\prod_{r'\in R'_{k,v}} r' \left(\prod_{\tau'\in \mathrm{Gal}(E_{k,\dot{v}}/F_v)} j_{k,v}\left(\sqrt[N]{\alpha_{k,v}}(\sigma',\tau')\right) \otimes \sigma'\tau'\left(l_{k,v}(\Lambda)\right)\right)$$

where σ' depends on r' and is the unique element of $\mathrm{Gal}(E_{k,\dot{v}}/F_v)$ such that $\sigma^{-1}r'\sigma' \in R'_{k,v}$. Choose $i \geq 0$ such that for any $\tau' \in \mathrm{Gal}(E_{k,\dot{v}}/F_v)$, $\sqrt[N]{\alpha_{k,v}}(\sigma',\tau') \in E^\times_{k+i,\dot{v}}$. Using (5.1.1) we obtain

$$\mathrm{pr}_v\left(\left(\sqrt[N]{\alpha'_k} \underset{E_k/F}{\sqcup} \Lambda\right)(\sigma)\right) = \prod_{r'\in R'_{k+i,v}} r' \left(\prod_{\tau'\in \mathrm{Gal}(E_{k,\dot{v}}/F_v)} j_{k+i,v}\left(\sqrt[N]{\alpha_{k,v}}(\sigma',\tau')\right) \otimes \sigma'\tau'\left(l_{k,v}(\Lambda)\right)\right)$$

and it is easy to check that this is equal to $\mathrm{ES}^1_{R'_{k+i,v}}\left(\sqrt[N]{\alpha_{k,v}} \underset{E_k/F}{\sqcup} l_{k,v}(\Lambda)\right)(\sigma)$. □

It is formal to check that for any finite commutative algebraic group Z over F such that $\exp(Z)|N$ and the Galois action on $A := X^*(Z)$ factors through $\mathrm{Gal}(E_k/F)$, and any finite set of places S' of F such that $S' \supset S_k$, the following diagram is commutative.

$$\begin{tikzcd}
\mathrm{Hom}(P(E_k,\dot{S}'_{E_k},N),Z) \arrow[r, "\Psi(E_k,S',N)"] \arrow[d, "(\mathrm{loc}_{k,v}(S',N))^*"'] & A^\vee[\dot{S}'_{E_k}]_0^{N_{E_k/F}} \arrow[d, "l_{k,v}"] \\
\mathrm{Hom}(P(E_{k,\dot{v}},N),Z) \arrow[r, "\Psi(E_{k,\dot{v}},S',N)"] & (A^\vee)^{N_{E_{k,\dot{v}}/F_v}}
\end{tikzcd} \tag{5.4.1}$$

Definition 5.4.2. *For $k \geq 0$, $N \geq 1$ and S' a finite subset of V containing S_k, let $\eta_{k,v}(S',N) : \mathrm{Gal}(\overline{F_v}/F_v) \to P(E_k,\dot{S}'_{E_k},N)$ be the restriction of $\mathrm{pr}_{\dot{v}}(\delta_k(N)) \underset{E_k/F}{\sqcup} c_{\mathrm{univ}}(E_k,S',N)$ to $\mathrm{Gal}(\overline{F_v}/F_v)$.*

Proposition 5.4.3. *Let $k \geq 0$, $N \geq 1$ and S' a finite subset of V containing S_k.*

1. *The restriction of the 2-cocycle $\xi_k(S',N)$ to $\mathrm{Gal}(\overline{F_v}/F_v)$ equals*
$$\left(\mathrm{loc}_{k,v}(S',N)\right)_*(\xi_{k,v}(N)) \times \mathrm{d}\left(\eta_{k,v}(S',N)\right)$$
and so the morphism $\mathrm{loc}_{k,v}(S',N) : P(E_{k,\dot{v}},N) \to P(E_k,\dot{S}'_{E_k},N)$ can be extended to a morphism of extensions
$$\begin{aligned}\mathrm{loc}_{k,v}(S',N) : \mathcal{E}_{k,v}(N) &\longrightarrow \mathcal{E}_k(S',N)\\ x \boxtimes \sigma &\longmapsto \frac{\mathrm{loc}_{k,v}(S',N)(x)}{\eta_{k,v}(S',N)(\sigma)} \boxtimes \sigma.\end{aligned}$$

2. *Let $Z \hookrightarrow T$ be an injective morphism of commutative algebraic groups over F with Z finite and T a torus split by E_k. Assume that* $\exp(Z)|N$. *Let $Y = X_*(T)$ and $\overline{Y} = X_*(T/Z)$. Then for any $\Lambda \in \overline{Y}[S'_{E_k}, \dot{S}'_{E_k}]_0^{N_{E_k/F}}$, the following identity holds in $Z^1(P(E_{k,\dot{v}}, N) \to \mathcal{E}_{k,v}(N), Z \to T(\overline{F_v}))$:*

$$\mathrm{pr}_{\dot{v}}\left(\iota_k(S', N)(\Lambda) \circ \mathrm{loc}_{k,v}(S', N)\right) = \iota_{k,v}(N)(l_{k,v}(\Lambda)) \times \mathrm{d}\left(\mathrm{pr}_{\dot{v}}(\sqrt[N]{\beta_k}) \underset{E_k/F}{\sqcup} N\Lambda\right). \tag{5.4.2}$$

Proof. The proof is similar to that of Proposition 5.2.3, and we will be more concise.

1. Let $Z = P(E_k, \dot{S}'_{E_k}, N)$ and $A = X^*(Z)$. As in the proof of Proposition 5.2.3 we can find an embedding $Z \hookrightarrow T$ where T is a torus over F, split over E_k and such that $Y := X_*(T)$ is a free $\mathbb{Z}[\mathrm{Gal}(E_k/F)]$-module. Let $\overline{Y} = X_*(T/Z)$. There exists $\Lambda \in \overline{Y}[S'_{E_k}, \dot{S}'_{E_k}]_0^{N_{E_k/F}}$ such that its image $[\Lambda]$ in $A^\vee[\dot{S}'_{E_k}]_0^{N_{E_k/F}}$ equals $c_{\mathrm{univ}}(E_k, S', N)$. Applying Lemma 5.4.1 to $N\Lambda \in Y$ and taking the coboundary, we obtain the identity between 2-cocycles $\mathrm{Gal}(\overline{F_v}/F_v)^2 \to T(\overline{F_v})$

$$\mathrm{d}\left(\sqrt[N]{\alpha_k}\right) \underset{E_k/F}{\sqcup} N\Lambda = \left(\mathrm{d}\left(\sqrt[N]{\alpha_{k,v}}\right) \underset{E_{k,\dot{v}}/F_v}{\sqcup} Nl_{k,v}(\Lambda)\right) \times \mathrm{d}\left(\mathrm{pr}_{\dot{v}}(\delta_k(N)) \underset{E_k/F}{\sqcup} N\Lambda\right).$$

   Since $\mathrm{d}\left(\sqrt[N]{\alpha_k}\right)^N = 1$, $\mathrm{d}\left(\sqrt[N]{\alpha_{k,v}}\right)^N = 1$ and $\delta_k(N)^N = 1$ all three terms take values in $Z \subset T(\overline{F_v})$ and the equality can be written

$$\mathrm{d}\left(\sqrt[N]{\alpha_k}\right) \underset{E_k/F}{\sqcup} [\Lambda] = \left(\mathrm{d}\left(\sqrt[N]{\alpha_{k,v}}\right) \underset{E_{k,\dot{v}}/F_v}{\sqcup} l_{k,v}([\Lambda])\right) \times \mathrm{d}\left(\mathrm{pr}_{\dot{v}}(\delta_k(N)) \underset{E_k/F}{\sqcup} [\Lambda]\right)$$

   using the pairing $\mu_N \times A^\vee \to Z$. Using the fact that

$$l_{k,v}(c_{\mathrm{univ}(E_k, S', N)}) = \Psi(E_{k,\dot{v},S',N})(\mathrm{loc}_{k,v}(S', N))$$

   thanks to (5.4.1), we obtain the desired equality.

2. This is a direct consequence of Lemma 5.4.1 applied to $N\Lambda$, using also the commutative diagram (5.4.1) with $[\Lambda]$ in the top right corner.

□

Lemma 5.4.4. *Let T be a torus defined over F. Denote $Y = X_*(T)$. Let k be big enough so that E_k splits T. Let $N \geq 1$ be an integer. Let S' be a finite subset of V containing S_{k+1}. Let $\Lambda \in Y[S'_{E_k}]_0^{N_{E_k/F}}$. Then we have an equality of maps $\mathrm{Gal}(\overline{F}_{S' \cup N}/F) \to Y \otimes_{\mathbb{Z}} \mathcal{I}(F, S' \cup N)[N]$:*

$$\delta_k(N) \underset{E_k/F}{\sqcup} \Lambda = \delta_{k+1}(N) \underset{E_{k+1}/F}{\sqcup} !_k(\Lambda) \tag{5.4.3}$$

and an equality in $Y \otimes_{\mathbb{Z}} \mathcal{I}(F, S' \cup N)$:

$$\sqrt[N]{\beta_k} \underset{E_k/F}{\sqcup} \Lambda = \sqrt[N]{\beta_{k+1}} \underset{E_{k+1}/F}{\sqcup} !_k(\Lambda). \tag{5.4.4}$$

Note that in (5.4.4) the left hand side belongs to $Y \otimes_{\mathbb{Z}} \mathcal{I}(F, S_k \cup N)$.

Proof. For (5.4.3) the proof is identical to that of Lemma 5.2.1. For (5.4.4) the proof is similar and easier, so we omit it. □

The localization maps $l_{k,v}$ are compatible with increasing k, i.e. $l_{k+1,v} \circ !_k = l_{k,v}$. This is proved in [Kal, Lemma 3.7.2]. Thus for any embedding $Z \hookrightarrow T$ of commutative algebraic groups over F with Z finite and T a torus, they splice into

$$l_v : \overline{Y}[V_{\overline{F}}, \dot{V}]_0^{N/F} \to \overline{Y}^{N/F_v}$$

where $\overline{Y} = X_*(T/Z)$.

The localization morphisms $\mathrm{loc}_{k,v}(S', N) : P(E_{k,\dot{v}}, N) \to P(E_k, \dot{S}'_{E_k}, N)$ are also compatible with varying k. We formulate this compatibility, together with (5.2.4), (5.3.1) and (5.4.1), using a commutative cubic diagram below. For any finite commutative algebraic group Z over F such that $\exp(Z)|N$ and the Galois action on $A := X^*(Z)$ factors through $\mathrm{Gal}(E_k/F)$, and any finite set of places S' of F such that $S' \supset S_{k+1}$, the following cubic diagram is commutative.

$$\begin{tikzcd}
& \mathrm{Hom}(P(E_k, \dot{S}'_{E_k}, N), Z) \arrow[rr, "\Psi(E_k,S',N)"] \arrow[dl, "\mathrm{loc}_{k,v}(S',N)^*"'] \arrow[dd, "\rho_k(S',N)^*"] & & A^\vee[\dot{S}'_{E_k}]_0^{N_{E_k/F}} \arrow[dl, "l_{k,v}"'] \arrow[dd, "!_k"] \\
\mathrm{Hom}(P(E_{k,\dot{v}}, N), Z) \arrow[rr, "\Psi(E_{k,\dot{v}},N)"' near start] \arrow[dd, "\rho_{k,v}(N)^*"'] & & (A^\vee)^{N_{E_{k,\dot{v}}/F_v}} \arrow[dd] & \\
& \mathrm{Hom}(P(E_{k+1}, \dot{S}'_{E_{k+1}}, N), Z) \arrow[rr, "\Psi(E_{k+1},S',N)" near start] \arrow[dl, "\mathrm{loc}_{k+1,v}(S',N)^*"'] & & A^\vee[\dot{S}'_{E_{k+1}}]_0^{N_{E_{k+1}/F}} \arrow[dl, "l_{k+1,v}"'] \\
\mathrm{Hom}(P(E_{k+1,\dot{v}}, N), Z) \arrow[rr, "\Psi(E_{k+1,\dot{v}},N)"] & & (A^\vee)^{N_{E_{k+1,\dot{v}}/F_v}} &
\end{tikzcd} \tag{5.4.5}$$

In fact the commutativity of the left face follows from the commutativity of the other faces and the fact that the morphisms Ψ are isomorphisms.

Proposition 5.4.5. *1. For any $k \geq 0$, $N \geq 1$ and S' a finite subset of V containing S_{k+1} we have $\eta_{k,v}(S', N) = \rho_k(S', N)_* (\eta_{k+1,v}(S', N))$, and a commutative diagram* *of central extensions*

$$\begin{array}{ccc} \mathcal{E}_{k+1,v}(N) & \xrightarrow{\mathrm{loc}_{k+1,v}(S',N)} & \mathcal{E}_{k+1}(S',N) \\ \downarrow & & \downarrow \\ \mathcal{E}_{k,v}(N) & \xrightarrow{\mathrm{loc}_{k,v}(S',N)} & \mathcal{E}_{k}(S',N) \end{array} \tag{5.4.6}$$

Therefore as k, S', N *vary, the morphisms* $\mathrm{loc}_{k,v}(S',N)$ *yield* $\mathrm{loc}_v : \mathcal{E}_v \to \mathcal{E}$.

2. *Let* $Z \hookrightarrow T$ *be an injective morphism of commutative algebraic groups over* F *with* Z *finite and* T *a torus. Let* $Y = X_*(T)$ *and* $\overline{Y} = X_*(T/Z)$. *Let* $\Lambda \in \overline{Y}[V_{\overline{F}}, \dot{V}]_0^{N/F}$. *For* k, S', N *such that* E_k *splits* T, $N \geq 1$ *is divisible by* $\exp(Z)$, S' *contains* S_k *and* Λ *comes from an element* $\Lambda_k \in \overline{Y}[S'_k, \dot{S}'_{E_k}]_0^{N_{E_k/F}}$, *let* $\kappa_v(\Lambda) = \mathrm{pr}_{\dot{v}}(\sqrt[N]{\beta_k}) \underset{E_k/F}{\sqcup} N\Lambda_k \in T(\overline{F_v})$. *As the notation suggests, it does not depend on the choice of* k, S', N. *Then the following identity holds in* $Z^1(P_v \to \mathcal{E}_v, Z \to T(\overline{F_v}))$:

$$\mathrm{pr}_{\dot{v}}\left(\iota(\Lambda) \circ \mathrm{loc}_v\right) = \iota_v(l_v(\Lambda)) \times \mathrm{d}\left(\kappa_v(\Lambda)\right). \tag{5.4.7}$$

Proof. 1. The equality $\eta_{k,v}(S',N) = \rho_k(S',N)_*\left(\eta_{k+1,v}(S',N)\right)$ follows from (5.4.3) in Lemma 5.4.4, using the same argument as in the proof of Proposition 5.2.3. Commutativity of diagram (5.4.6) follows from this equality and the equality $\mathrm{loc}_{k,v}(S',N) \circ \rho_{k,v}(N) = \rho_k(S',N) \circ \mathrm{loc}_{k+1,v}(S',N)$, which is equivalent to commutativity of the left face of (5.4.5) for $Z = P(E_k, \dot{S}'_{E_k}, N)$.

2. The fact that $\kappa_v(\Lambda)$ does not depend on the choice of k, S', N follows from (5.4.4) in Lemma 5.4.4, and (5.4.7) is (5.4.2) in Proposition 5.4.3.

□

5.5 Comparison with Kaletha's canonical class

We follow the convention in [Kal] and define, for a projective system $(Q_k)_{k\geq 0}, (Q_{k+1} \to Q_k)_{k\geq 0}$ of commutative algebraic groups over F and R a F-algebra, $\left(\varprojlim_k Q_k\right)(R) = \varprojlim_k Q_k(R)$. In particular

$$\varinjlim_{E/F \text{ finite}} \left((\varprojlim_k Q_k)(E) \right) \longrightarrow \left(\varprojlim_k Q_k \right)(\overline{F})$$

is not surjective in general. For $\mathrm{Gal}(\overline{F}/F)$- or $\mathrm{Gal}(\overline{F_v}/F_v)$-modules which arise naturally as projective limits (such as $Q(\overline{F})$, $Q(\overline{F_v})$ or $Q(\mathbb{A})$ for $Q = \varprojlim_k Q_k$ as above), we will only consider *continuous* cochains, for the topology on projective limits induced by the discrete topology on each term.

As in [Kal] we let $P = \varprojlim_{k,S',N} P(E_k, \dot{S}'_{E_k}, N)$. Each term $P(E_k, \dot{S}'_{E_k}, N)$ is finite, so that we can also simply consider the profinite $\mathrm{Gal}(\overline{F}/F)$-module $P(\overline{F})$, which equals $P(\overline{F_v})$ for any $v \in V$.

The 2-cocycles $\xi_k(S', N)$ are compatible by Proposition 5.2.3, and so we obtain a 2-cocycle $\xi \in Z^2(F, P)$ which corresponds to the extension $P \to \mathcal{E}$ of $\mathrm{Gal}(\overline{F}/F)$ introduced at the end of Section 5.2. The goal of this section is to check that ξ represents the canonical class in $H^2(\mathrm{Gal}(\overline{F}/F), P)$ defined in [Kal, §3.5], so that our $P \to \mathcal{E}$ is isomorphic to Kaletha's, canonically by [Kal, Proposition 3.4.6].

As in [Kal, §3.3], fix a cofinal sequence $(N_k)_{k\geq 0}$ in $\mathbb{Z}_{>0}$ (for the partial order defined by divisibility) with $N_0 = 1$ and such that for any $k \geq 0$, S_k contains all places dividing N_k (this is possible up to enlarging the finite sets S_k). To simplify notation we write $P_k = P(E_k, \dot{S}_{k,E_k}, N_k)$, $M_k = M(E_k, \dot{S}_{k,E_k}, N_k) = X^*(P_k)$, $\rho_k : P_{k+1} \twoheadrightarrow P_k$ and $c_{\mathrm{univ},k} = c_{\mathrm{univ}}(E_k, S_k, N_k)$.

First we need to go back to the construction of a resolution of P by pro-tori in [Kal, Lemma 3.5.1].

Lemma 5.5.1. *There exists a family of resolutions, for $k \geq 0$,*

$$1 \to P_k \to T_k \to \overline{T}_k \to 1$$

of P_k by tori $T_k, \overline{T}_k$ defined over F and split by E_k, and morphisms $r_k : T_{k+1} \to T_k$ and $\overline{r}_k : \overline{T}_{k+1} \to \overline{T}_k$, such that

1. *For all $k \geq 0$, the diagram*

$$\begin{tikzcd} P_{k+1} \arrow[r] \arrow[d, "\rho_k"] & T_{k+1} \arrow[r] \arrow[d, "r_k"] & \overline{T}_{k+1} \arrow[d, "\overline{r}_k"] \\ P_k \arrow[r] & T_k \arrow[r] & \overline{T}_k \end{tikzcd} \tag{5.5.1}$$

   is commutative and $r_k, \overline{r}_k$ are surjective with connected kernels.

2. *Letting $Y_k = X_*(T_k)$ and $\overline{Y}_k = X_*(\overline{T}_k)$, there exists a family $(\Lambda_k)_{k\geq 0}$ where $\Lambda_k \in \overline{Y}_k[S_{k,E_k}, \dot{S}_{k,E_k}]_0^{N_{E_k/F}}$ maps to $c_{\mathrm{univ},k} \in M_k^\vee[\dot{S}_{k,E_k}]_0^{N_{E_k/F}}$ and $!_k(\Lambda_k) = \overline{r}_k(\Lambda_{k+1})$ in $\overline{Y}_k[S_{k+1,E_{k+1}}, \dot{S}_{k+1,E_{k+1}}]_0^{N_{E_{k+1}/F}}$.*

Proof. For $k \geq 0$ let $X'_k = \mathbb{Z}[\mathrm{Gal(E_k/F)}][M_k]$, so that there is a canonical surjective map of $\mathbb{Z}[\mathrm{Gal}(E_k/F)]$-modules $X'_k \to M_k$. Let $X_0 = X'_0$, and for $k \geq 0$ let $X_{k+1} = X_k \oplus X'_{k+1}$. We have a natural surjective morphism $X_k \to M_k$, which for $k > 0$ is obtained as the sum of $X_{k-1} \to M_{k-1} \hookrightarrow M_k$ and $X'_k \to M_k$. Let T_k be the torus over F such that $X^*(T_k) = X_k$, and let $U_k = T_k/P_k$. Compared to the construction in [Kal, Lemma

3.5.1], the only difference is that X'_{k+1} is free with basis M_{k+1} instead of $M_{k+1} \smallsetminus M_k$. Let $Y_k = X_*(T_k)$ and $\overline{Y}_k = X_*(U_k)$, so that we have an exact sequence

$$0 \to Y_k \to \overline{Y}_k \to M_k^\vee \to 0.$$

Let $\overline{X}'_k = \ker(X'_k \to M_k)$, $Y'_k = \mathrm{Hom}_{\mathbb{Z}}(X'_k, \mathbb{Z})$ and $\overline{Y}'_k = \mathrm{Hom}_{\mathbb{Z}}(\overline{X}'_k, \mathbb{Z})$ Since X'_k is a free $\mathbb{Z}[\mathrm{Gal}(E_k/F]$-module, using the same argument as in Proposition 5.2.3 we can find $\Upsilon_k \in \overline{Y}'_k[S_{k,E_k}, \dot{S}_{k,E_k}]_0^{N_{E_k/F}}$ mapping to $c_{\mathrm{univ},k}$. For all $k \geq 0$ we can identify $\overline{Y}_{k+1}$ with the group of $f \oplus g \in \overline{Y}_k \oplus \overline{Y}'_{k+1}$ such that $[f] = [g]$ in $M_k^\vee$. We use these identifications to construct Λ_k inductively from Υ_k. Let $\Lambda_0 = \Upsilon_0$, and for $k \geq 0$ let $\Lambda_{k+1} = !_k(\Lambda_k) \oplus \Upsilon_{k+1} \in (\overline{Y}_k \oplus \overline{Y}'_{k+1})[S_{k+1,E_{k+1}}, \dot{S}_{k+1,E_{k+1}}]_0^{N_{E_{k+1}/F}}$. Thanks to the equality $!_k(c_{\mathrm{univ},k}) = \rho_k(c_{\mathrm{univ},k+1})$, we have that $\Lambda_{k+1} \in \overline{Y}_{k+1}[S_{k+1,E_{k+1}}, \dot{S}_{k+1,E_{k+1}}]_0^{N_{E_{k+1}/F}}$. $\square$

Let us now recall how Kaletha pins down the canonical class ξ in [Kal, Proposition 3.5.2]. For $v \in V$, let $k_{0,v}$ be the minimal $k \geq 0$ such that $v \in S_k$. For $k \geq k_{0,v}$ let $P_{k,v} = P(E_{k,\dot{v}}, N_k)$. As in the global case $(\xi_{k,v})_{k \geq k_{0,v}}$ induce a continuous 2-cocycle $\xi_v \in Z^2(\mathrm{Gal}(\overline{F_v}/F_v, P_v)$ where $P_v = \varprojlim_k P_{k,v}$. Note that unlike in the global case, the cohomology class of ξ_v is simply characterized by the property that its image in $H^2(\mathrm{Gal}(\overline{F_v}/F_v), P_{k,v})$ is that of $\xi_{k,v}$ for every $k \geq k_{0,v}$. Uniqueness follows from vanishing of $\varinjlim_k^1 H^1(\mathrm{Gal}(\overline{F_v}/F_v), P_{k,v})$.

For $v \in V$ denote $R'_v = (R'_{k,v})_{k \geq 0}$. Consider a projective system $(Q_k)_{k \geq 0}, (Q_{k+1} \to Q_k)_{k \geq 0}$ of commutative algebraic groups over F, and let $Q = \varprojlim_k Q_k$. The Eckmann-Shapiro maps, for $k, i, j \geq 0$,

$$\mathrm{ES}^j_{R'_{k+i,v}} : C^j(\mathrm{Gal}(E_{k+i,\dot{v}}/F_v), Q_k(E_{k+i,\dot{v}})) \to C^j(\mathrm{Gal}(E_{k+i}/F), Q_k(E_{k+i} \otimes_F F_v))$$

are compatible (for k fixed and varying i, and then also for varying k) and yield a pro-Eckmann-Shapiro map

$$\mathrm{ES}^j_{R'_v} : C^j(F_v, Q(\overline{F_v})) \to C^j(F, Q(\overline{F} \otimes_F F_v)).$$

This is explained in [Kal, Appendix B], although notations differ: our set of *right* coset representatives $R'_{k,v}$ corresponds to the image of the composition in [Kal, Lemma B.1, 1.], by mapping $r \in R'_{k,v}$ to r^{-1}.

Define $x_k \in Z^2(\mathrm{Gal}(\overline{F}/F), P_k(\overline{\mathbb{A}}))$ by

$$x_k = \prod_{v \in S_k} \mathrm{ES}^2_{R'_v}(\mathrm{loc}_{k,v}(\xi_{k,v})) \in Z^2(\mathbb{A}, P_k).$$

The family $(x_k)_{k \geq 0}$ is easily seen to be compatible and so it defines a continus 2-cocycle $x \in Z^2(\mathrm{Gal}(\overline{F}/F), P(\overline{\mathbb{A}}))$. Kaletha checks that the class of x in $H^2(\mathrm{Gal}(\overline{F}/F), P(\overline{\mathbb{A}}))$

does not depend on the choice of sets of representatives $R_{k,v}$, nor does it depend on the choice of ξ_v in its cohomology class.

Kaletha shows [Kal, Proposition 3.5.2] that there is a unique class $\mathrm{cl}(\xi_{\mathrm{can}}) \in H^2(\mathrm{Gal}(\overline{F}/F), P(\overline{F}))$ such that.

1. For any $k \geq 0$, the image of $\mathrm{cl}(\xi_{\mathrm{can}})$ in $H^2(F, P_k)$ is $\mathrm{cl}(\xi_k)$.
2. The image of $\mathrm{cl}(\xi_{\mathrm{can}})$ in $H^2(\mathbb{A}, T \to \overline{T})$ coincides with the image of $\mathrm{cl}(x)$.

Adèlic cohomology groups of complexes of tori were defined and studied in [KS99, Appendix C], see [Kal, §3.5] for the case of projective systems of complexes of tori satisfying a Mittag-Leffler condition. The class $\mathrm{cl}(\xi_{\mathrm{can}})$ does not depend on the choice of a suitable pro-resolution $T \to \overline{T}$ of P by pro-tori, but for the following proposition it will be convenient to use the pro-resolution introduced in Lemma 5.5.1.

Proposition 5.5.2. *The 2-cocycle ξ belongs to the canonical class $\mathrm{cl}(\xi_{\mathrm{can}}) \in H^2(F, P)$ defined in [Kal, Definition 3.5.4].*

Proof. The first property above is obviously satisfied. The second property is equivalent to the existence of a compatible family $(a_k, b_k)_{k \geq 0}$ where $a_k \in C^1(F, T_k)$ and $b_k \in \overline{T}_k(\mathbb{A}_{\overline{F}})$ are such that $\overline{a_k} = \mathrm{d}(b_k)$ in $C^1(\mathbb{A}, \overline{T}_k)$ and

$$\xi_k = \prod_{v \in S_k} \mathrm{ES}^2_{R'_{k+i,v}}(\mathrm{loc}_{k,v}(\xi_{k,v})) \times \mathrm{d}(a_k)$$

in $Z^2(\mathbb{A}, T_k)$, for $i \geq 0$ large enough.

By Lemma 5.4.1 and thanks to the fact that Λ_k has support in the finite set S_{k,E_k}, for $i \geq 0$ big enough we have

$$\sqrt[N_k]{\alpha'_k} \underset{E_k/F}{\sqcup} N_k \Lambda_k = \prod_{v \in S_k} \mathrm{ES}^1_{R'_{k+i,v}} \left(\sqrt[N_k]{\alpha_{k,v}} \underset{E_{k,\dot{v}}/F_v}{\sqcup} N_k l_{k,v}(\Lambda_k) \right)$$

as maps $\mathrm{Gal}(E_k/F) \to T_k(\mathbb{A}_{E_{k+i}})$. Using an argument similar to the proof of Proposition 5.4.3, we deduce

$$\begin{aligned} \mathrm{d}\left(\sqrt[N_k]{\alpha'_k} \underset{E_k/F}{\sqcup} N_k \Lambda_k \right) &= \prod_{v \in S_k} \mathrm{ES}^2_{R'_{k+i,v}} \left(\mathrm{d}\left(\sqrt[N_k]{\alpha_{k,v}} \underset{E_{k,\dot{v}}/F_v}{\sqcup} N_k l_{k,v}(\Lambda_k) \right) \right) \\ &= \prod_{v \in S_k} \mathrm{ES}^2_{R'_{k+i,v}} (\mathrm{loc}_{k,v}(\xi_{k,v})) \end{aligned}$$

in $Z^2(\mathrm{Gal}(\overline{F}/F), \ker(T_k(\mathbb{A}_{\overline{F}}) \to \overline{T}_k(\mathbb{A}_{\overline{F}})))$. This leads us to define

$$a_k = \frac{\sqrt[N_k]{\alpha_k}}{\sqrt[N_k]{\alpha'_k}} \underset{E_k/F}{\sqcup} N_k \Lambda_k \in C^1(\mathrm{Gal}(E_k/F), T_k(\mathbb{A}_{E_{k+i}})).$$

Then

$$\overline{a_k} = \frac{\alpha_k}{\alpha'_k} \underset{E_k/F}{\sqcup} \Lambda_k = \mathrm{d}(b_k)$$

where $b_k = \beta_k \underset{E_k/F}{\sqcup} \Lambda_k \in \overline{T}(\mathbb{A}_{E_k})$.

The fact that $\overline{r}_k(b_{k+1}) = b_k$ for all $k \geq 0$ follows directly from (5.4.4) in Lemma 5.4.4. Using $\sqrt[N_{k+1}]{\alpha_k}^{N_{k+1}/N_k} = \sqrt[N_k]{\alpha_k}$ and Lemma 5.2.1 we find

$$\sqrt[N_k]{\alpha_k} \underset{E_k/F}{\sqcup} N_k \Lambda_k = \sqrt[N_{k+1}]{\alpha_k} \underset{E_k/F}{\sqcup} N_{k+1} \Lambda_k = \sqrt[N_{k+1}]{\alpha_{k+1}} \underset{E_{k+1}/F}{\sqcup} N_{k+1} !_k(\Lambda_k).$$

Lemma 5.2.1 also holds with $\sqrt[N]{\alpha_?}$ replaced by $\sqrt[N]{\alpha'_?}$ because this family also satisfies $\mathrm{AWES}^2_k(\sqrt[N]{\alpha'_{k+1}}) = \sqrt[N]{\alpha'_k}$, and so we similarly find

$$\sqrt[N_k]{\alpha'_k} \underset{E_k/F}{\sqcup} N_k \Lambda_k = \sqrt[N_{k+1}]{\alpha'_{k+1}} \underset{E_{k+1}/F}{\sqcup} N_{k+1} !_k(\Lambda_k).$$

The fact that $r_k(a_{k+1}) = a_k$ for all $k \geq 0$ follows from these two equalities and $\overline{r}_k(\Lambda_{k+1}) = !_k(\Lambda_k)$ (Lemma 5.5.1). $\square$

6 On ramification

6.1 A ramification property

We deduce a ramification property for Kaletha's generalized Galois cocycles from our explicit construction. Such a property is important to state Arthur's multiplicity formula in [Kal, §4.5], namely to guarantee that the global adèlic packets Π_φ are well-defined: see [Kal, Lemma 4.5.1].

Proposition 6.1.1. *Let G be a connected reductive group over F, and Z a finite central subgroup defined over F. For any $z \in Z^1(P \to \mathcal{E}, Z \to G)$, there exists a finite subset S' of V containing all archimedean places such that for any $v \in V \smallsetminus S'$, $\mathrm{pr}_{\dot{v}}(z \circ \mathrm{loc}_v)$ is unramified, i.e. inflated from an element of $Z^1(\mathrm{Gal}(K(v)/F_v), G(\mathcal{O}(K(v))))$ for some finite unramified extension $K(v)/F_v$.*

Proof. Let us first check that for $z' \in Z^1(P \to \mathcal{E}, Z \to G)$ in the same class as z, this ramification property holds for z if and only if it holds for z' (in general for distinct finite sets of places). There exists $g \in G(\overline{F})$ such that for any $w \in \mathcal{E}$, $z'(w) = g^{-1} z(w) w(g)$. Note that the action of $\mathcal{E}$ on $G(\overline{F})$ factors through $\mathrm{Gal}(\overline{F}/F)$. There exists a finite set $S'' \subset V$ containing all Archimedean places and a finite Galois extension E/F unramified away from S'' such that $g \in G(\mathcal{O}(E, S''))$. Thus if z satisfies the ramification property for S', z' satisfies it for $S' \cup S''$.

Thanks to [Kal, Lemma 3.6.2] it is enough to prove the statement in the case where G is a torus T. We remark that this reduction could force us to enlarge S'. As usual let $\overline{Y} = X_*(T/Z)$. Let $N = \exp(Z)$. There exists $k \geq 0$ such that E_k splits T and a finite $S' \subset V$ containing all places dividing N and S_k such that z is inflated from a unique element of $Z^1(P(E_k, \dot{S}'_{E_k}, N) \to \mathcal{E}_k(S', N), Z \to T(\mathcal{O}_{S'}))$, which we also denote by z. By [Kal, Proposition 3.7.8, 3.], up to replacing z with a cohomologous cocycle we can assume that $z = \iota_k(S', N)(\Lambda)$ for some $\Lambda \in \overline{Y}[S'_{E_k}, \dot{S}'_{E_k}]_0^{N_{E_k/F}}$, up to enlarging S' so that Conditions 3.3.1 in [Kal] are satisfied.

For $v \in V \smallsetminus S'$, the morphism $\mathrm{loc}_{k,v}(S', N) : \mathcal{E}_{k,v}(N) \to \mathcal{E}_k(S', N)$ is trivial on $P(E_{k,\dot{v}}, N)$ and so it factors through $\mathrm{Gal}(\overline{F_v}/F_v)$. Thanks to ramification properties of $\delta_k(N)$ (see Definition 5.1.4) and by definition of $\eta_{k,v}(S', N)$ (see Definition 5.4.2), $\eta_{k,v}(S', N) : \mathrm{Gal}(\overline{F_v}/F_v) \to P(E_k, \dot{S}'_{E_k}, N)$ factors through $\mathrm{Gal}(F_v^{\mathrm{nr}}/F_v)$. By construction in Proposition 5.1.3, $\sqrt[N]{\beta_k}$ takes values in $\mathcal{I}(F, S_k \cup N)$. Thus by definition of $\kappa_v(\Lambda)$ in Proposition 5.4.5, $\kappa_v(\Lambda) \in T(\mathcal{O}(F_v^{\mathrm{nr}}))$. The equality (5.4.7) in Proposition 5.4.5, which is inflated from (5.4.2) in Proposition 5.4.3, shows that $\mathrm{pr}_{\dot{v}}(z \circ \mathrm{loc}_v)$ is unramified. □

Note that it does not seem possible to choose $K(v) = K_v$ for some finite extension K/F.

6.2 Alternative proof

As announced in the introduction to this paper, we now give an alternative proof of Proposition 6.1.1, which relies solely on Kaletha's definition of the canonical class, and not on constructions in the present paper.

Alternative proof of Proposition 6.1.1. For $v \in V$ temporarily let $\xi_v \in Z^2(\mathrm{Gal}(\overline{F_v}/F_v), P_v)$ be *any* element of $Z^2(\mathrm{Gal}(\overline{F_v}/F_v), P_v)$ representing the class defined in [Kal16]. Choose a tower of resolutions $(1 \to P_k \to T_k \to U_k \to 1)_{k \geq 0}$ as in [Kal, Lemma 3.5.1], and as before write $T(\overline{\mathbb{A}}) = \varprojlim_k T_k(\overline{\mathbb{A}})$ and $U(\overline{\mathbb{A}}) = \varprojlim_k U_k(\overline{\mathbb{A}})$. Temporarily let ξ be *any* element of $Z^2(\mathrm{Gal}(\overline{F}/F), P)$ representing the canonical class defined in [Kal, §3.5]. Of course the 2-cocycles constructed in this paper are examples of elements of these cohomology classes, but we want to emphasize that the present proof does not require constructions in previous sections.

By definition of the canonical class there exists $a \in C^1(\mathbb{A}, T)$ and $b \in U(\overline{\mathbb{A}})$ such that

$$\xi = \prod_{v \in V} \mathrm{ES}^2_{R'_v}(\mathrm{loc}_v(\xi_v)) \times \mathrm{d}(a)$$

in $Z^2(\mathbb{A}, T)$ and $\overline{a} = \mathrm{d}(b)$ in $C^1(\mathbb{A}, U)$. In particular for any $v \in V$ we have

$$\mathrm{res}_v(\xi) = \mathrm{loc}_v(\xi_v) \times \mathrm{d}(a_v)$$

where res_v denotes restriction to $\mathrm{Gal}(\overline{F_v}/F_v)$ and $a_v = \mathrm{pr}_{\dot{v}}(\mathrm{res}_v(a))$. This equality holds in $Z^2(F_v, T)$, but ξ and $\mathrm{loc}_v(\xi_v)$ both take values in P. Let $b_v = \mathrm{pr}_{\dot{v}}(b)$, and choose a lift $\tilde{b}_v$ of b_v in $T(\overline{F_v})$. This is possible thanks to the surjectivity of all maps $P_{k+1} \to P_k$, by a simple diagram chasing argument (or more conceptually using vanishing of $\varprojlim_k^1 P_k$). Let $a'_v = a_v/\mathrm{d}(\tilde{b}_v)$. Then $a'_v \in C^1(F_v, P)$, and we have the equality

$$\mathrm{res}_v(\xi) = \mathrm{loc}_v(\xi_v) \times \mathrm{d}(a'_v)$$

in $Z^2(F_v, P)$.

Fix $k \geq 0$. For $v \in V$ denote by $a_{k,v}$ (resp. $b_{k,v}$, $\tilde{b}_{k,v}$, $a'_{k,v}$) the image of a_v (resp. b_v, $\tilde{b}_v$, a'_v) in $C^1(F_v, T_k)$ (resp. $U_k(\overline{F_v})$, $T_k(\overline{F_v})$, $C^1(F_v, P_k)$). Let us check that there is a finite set S' of places of F such that for all $v \notin S'$, $a'_{k,v} \in C^1(F_v, P_k)$ is unramified. There exists a finite set $S' \supset S_k$ and a finite Galois extension K of F containing E_k, splitting T_k and unramified away from S' such that $a_k \in C^1(K/F, T_k(\mathbb{A}_K)_{S'})$ and $b_k \in U_k(\mathbb{A}_K)_{S'}$ where $T_k(\mathbb{A}_K)_{S'}$ is defined as $X_*(T_k) \otimes_{\mathbb{Z}} I(K, S')$. So for $v \notin S'$, $a_{k,v} \in C^1(K_{\dot{v}}/F_v, T_k(\mathcal{O}(K_{\dot{v}})))$ is unramified. The group $P_k = \ker(T_k \to U_k)$ is killed by N_k, and so there is a unique morphism $U_k \to T_k$ such that the composition $U_k \to T_k \to U_k$ is the N_k-power map. Thus for any $v \notin S'$, $\tilde{b}_{k,v} \in T_k(\mathcal{O}(K_{\dot{v}})^{(N_k)})$ where $\mathcal{O}(K_{\dot{v}})^{(N_k)}$ is the finite étale extension of $\mathcal{O}(K_{\dot{v}})$ obtained by adjoining all N_k-th roots of elements in $\mathcal{O}(K_{\dot{v}})^\times$. We conclude that for $v \notin S'$, $a'_{k,v} \in C^1(\mathrm{Gal}(\mathcal{O}(K_{\dot{v}})^{(N_k)}/\mathcal{O}(F_v)), P_k)$ and

$$\mathrm{res}_v(\xi_k) = \mathrm{d}(a'_{k,v})$$

in $Z^2(F_v, P_k)$, where ξ_k is ξ composed with the surjection $P \to P_k$. This easily implies Proposition 6.1.1. $\square$

Note that the fact that for a fixed k, $\mathrm{res}_v(\xi_k)$ is the coboundary of an unramified 1-cochain for almost all $v \in V$ is straighforward from the definition. What the proof above shows is that the cochain $a'_{k,v}$ coming from "infinite level", which is unique up multiplication by a 1-coboundary, is unramified for almost all $v \in V$.

6.3 A non-canonical class failing the ramification property

Proposition 6.3.1. *Assume that $N_1 = 2$ and that S_1 is big enough so that P_1 is non-trivial. Then there exists $\xi^{\mathrm{bad}} \in Z^2(F, P)$ which coincides with the canonical class in $\varprojlim_k H^2(F, P_k)$ and such that for infinitely many places v of F, the 1-cochain $a_v \in C^1(F_v, P)$ such that $\mathrm{res}_v(\xi^{\mathrm{bad}}) = \mathrm{loc}_v(\xi_v)\mathrm{d}(a_v)$ is such that its image $a_{1,v} \in C^1(F_v, P_1)$ is ramified.*

Note that a_v is unique up to a 1-coboundary by [Kal, Proposition 3.4.5], and so the property "$a_{1,v}$ is unramified" is well-defined at all places $v \in V \smallsetminus S_1$.

Proof. Fix a tower of resolutions $(T_k \to U_k)_{k\geq 0}$ of P_k by tori as in [Kal, §3.5], and denote by π_k the morphism $(T_{k+1} \to U_{k+1}) \to (T_k \to U_k)$. Recall (discussion before Proposition 3.5.2 in [Kal] and [Wei94, Theorem 3.5.8]) that for any $j \geq 0$ the following short sequences are exact:

$$1 \to \varprojlim_k{}^1 H^j(F, P_k) \to H^{j+1}(F, P) \to \varprojlim_k H^{j+1}(F, P_k) \to 1 \tag{6.3.1}$$

$$1 \to \varprojlim_k{}^1 H^j(\mathbb{A}, T_k \to U_k) \to H^{j+1}(\mathbb{A}, T \to U) \to \varprojlim_k H^{j+1}(\mathbb{A}, T_k \to U_k) \to 1.$$

For any $k \geq 0$ and $j \geq 0$ the natural map $H^j(F, P_k) \to H^j(F, T_k \to U_k)$ is an isomorphism because

$$1 \to P_k(\overline{F}) \to T_k(\overline{F}) \to U_k(\overline{F}) \to 1$$

is exact (whereas $T_k(\overline{\mathbb{A}}) \to U_k(\overline{\mathbb{A}})$ is not surjective in general). By the five lemma this implies that the first short exact sequence (6.3.1) is isomorphic to

$$1 \to \varprojlim_k{}^1 H^j(F, T_k \to U_k) \to H^{j+1}(F, T \to U) \to \varprojlim_k H^{j+1}(F, T_k \to U_k) \to 1.$$

One could also check that $H^j(F, P) \to H^j(F, T \to U)$ is an isomorphism more directly by manipulating cocycles.

By [Kal, Lemma 3.5.3] the natural morphism

$$\varprojlim_k{}^1 H^1(F, P_k) \to \varprojlim_k{}^1 H^1(\mathbb{A}, T_k \to U_k) \tag{6.3.2}$$

is an isomorphism. So let us first define a non-trivial element of $\varprojlim_k{}^1 H^1(\mathbb{A}, T_k \to U_k)$. Choose, for any $k \geq 1$, a place $v_k \in V \smallsetminus S_1$ such that E_k/F is split above v_k and the v_k's are distinct. For any $k \geq 1$, the tori T_k, U_k, T_1 and U_1 are split over F_{v_k}, and the surjective morphism of tori $U_k \to U_1$ splits over F_{v_k} since it has connected kernel. Therefore

$$H^1(F_{v_k}, P_k) = H^1(F_{v_k}, T_k \to U_k) \simeq U_k(F_{v_k})/T_k(F_{v_k})$$

maps onto

$$H^1(F_{v_k}, P_1) = H^1(F_{v_k}, T_1 \to U_1) \simeq U_1(F_{v_k})/T_1(F_{v_k}).$$

Since we have assumed $N_1 = 2$, over F_{v_k} the multiplicative group P_1 is isomorphic to μ_2^r for some $r > 1$. For each $k \geq 1$ let $c_{k,v_k} \in Z^1(F_{v_k}, P_k) \subset Z^1(F_{v_k}, T_k \to U_k)$ be such that its image in $H^1(F_{v_k}, P_1)$ is ramified. Recall that $H^1(\mathbb{A}, T_k \to U_k)$ decomposes as a restricted direct product over places in V [KS99, Lemma C.1.B]. Define $c_k \in Z^1(\mathbb{A}, T_k \to U_k)$ by

$$\mathrm{pr}_v(c_k) = \begin{cases} 1 & \text{if } v \neq v_k \\ \mathrm{ES}^1_{R'_v}(c_{k,v_k}) & \text{if } v = v_k. \end{cases}$$

If $\tilde{c}_k \in C^1(\mathbb{A}, T \to U)$ lifts c_k, then $\mathrm{d}(\tilde{c}_k) \in Z^2(\mathbb{A}, T \to U)$ has trivial image in $Z^2(\mathbb{A}, T_k \to U_k)$. The family $(c_k)_{k \geq 1}$ defines an element of $\varprojlim^1_k H^1(\mathbb{A}, T_k \to U_k)$, whose image in $H^2(\mathbb{A}, T \to U)$ is the class of the convergent product $\prod_{k \geq 1} \mathrm{d}(\tilde{c}_k)$, for any choice of lifts $(\tilde{c}_k)_{k \geq 1}$. For simplicity we choose a lift $\tilde{c}_{k,v_k} \in C^1(F_v, T \to U)$ of c_{k,v_k} and define $\tilde{c}_k$ by

$$\mathrm{pr}_v(\tilde{c}_k) = \begin{cases} 1 & \text{if } v \neq v_k \\ \mathrm{ES}^1_{R'_v}(\tilde{c}_{k,v_k}) & \text{if } v = v_k. \end{cases}$$

By surjectivity of (6.3.2), there exists a family $(b_k)_{k \geq 1}$ with $b_k \in Z^1(\mathbb{A}, T_k \to U_k)$ such that for every $k \geq 1$, the class of $c_k b_k / \pi_k(b_{k+1})$ belongs to the image of $H^1(F, T_k \to U_k) \to H^1(\mathbb{A}, T_k \to U_k)$. This means that there exists $e_k \in C^0(\mathbb{A}, T_k \to U_k) = T_k(\overline{\mathbb{A}})$ such that for every $k \geq 0$,

$$f_k := c_k \frac{b_k}{\pi_k(b_{k+1})} \mathrm{d}(e_k) \in Z^1(F, T_k \to U_k).$$

Choose lifts $\tilde{b}_k \in C^1(\mathbb{A}, T \to U)$ of b_k, $\tilde{e}_k \in C^0(\mathbb{A}, T \to U) = T(\overline{\mathbb{A}})$ of e_k, and $\tilde{f}_k \in C^1(F, T \to U)$ of f_k. Then

$$g_k := \tilde{c}_k \frac{\tilde{b}_k}{\tilde{b}_{k+1}} \mathrm{d}(\tilde{e}_k) \tilde{f}_k^{-1} \in C^1(\mathbb{A}, T \to U)$$

takes values in the complex

$$\ker\left(T(\overline{\mathbb{A}}) \to T_k(\overline{\mathbb{A}})\right) \longrightarrow \ker\left(U(\overline{\mathbb{A}}) \to U_k(\overline{\mathbb{A}})\right)$$

and so $\prod_{k \geq 1} g_k$ is convergent in $C^1(\mathbb{A}, T \to U)$. Let $q = \prod_{k \geq 1} \mathrm{d}(\tilde{f}_k) \in Z^2(F, T \to U)$, which converges because f_k is a cocycle. In $Z^2(\mathbb{A}, T \to U)$ we have a factorization

$$q = \mathrm{d}(\tilde{b}_1) \times \left(\prod_{k \geq 1} \mathrm{d}(\tilde{c}_k) \right) \times \mathrm{d} \left(\prod_{k \geq 1} g_k^{-1} \right).$$

Moreover q defines a class in $H^2(F, T \to U) = H^2(F, P)$. Choose $a^{(1)} \in C^1(F, T \to U)$ such that $q \times \mathrm{d}(a^{(1)}) \in Z^2(F, P)$.

Let $\xi_{\mathrm{bad}} = \xi \times q \times \mathrm{d}(a^{(1)})$ in $Z^2(F, P)$, where $\xi \in Z^2(F, P)$ belongs to the canonical class. For any $v \in V$, by vanishing of $\varprojlim^1_k H^1(F_v, P) = \varprojlim^1_k H^1(F_v, T \to U)$ we know a priori that $\mathrm{res}_v(q)$ is the trivial class in $H^2(F_v, P)$. The point of the diagonal construction above is that we can write $\mathrm{res}_v(q)$ more explicitly as a coboundary. Let $a^{(2)} = \tilde{b}_1 \prod_{k \geq 1} g_k^{-1} \in C^1(\mathbb{A}, T \to U)$. Then for any place v, letting $a^{(2)}_v = \mathrm{pr}_{\dot{v}}(\mathrm{res}_v(a^{(2)}))$,

$$\mathrm{res}_v(q) = \begin{cases} \mathrm{d}(a^{(2)}_v) & \text{if } v \notin \{v_k | k \geq 1\}, \\ \mathrm{d}(a^{(2)}_v \times c^{(v)}_k) & \text{if } v = v_k. \end{cases}$$

Since ξ belongs to the canonical class, as in the alternative proof in section 6.2 there exists $a^{(3)} \in C^1(\mathbb{A}, T \to U)$ such that for any place v, $\mathrm{res}_v(\xi) = \mathrm{loc}_v(\xi_v) \times \mathrm{d}(a_v^{(3)})$. Let $a = a^{(1)}a^{(2)}a^{(3)} \in C^1(\mathbb{A}, T \to U)$. Then for every place v, letting $a_v = \mathrm{pr}_{\dot{v}}(\mathrm{res}_v(a))$,

$$\mathrm{res}_v(\xi_{\mathrm{bad}})/\mathrm{loc}_v(\xi_v) = \begin{cases} \mathrm{d}(a_v) & \text{if } v \notin \{v_k | k \geq 1\} \\ \mathrm{d}(a_v \times c_k^{(v)}) & \text{if } v = v_k. \end{cases}$$

By the same argument as in section 6.2, in this equality we can replace $a_v \in C^1(F_v, T \to U)$ by $a'_v \in C^1(F_v, P)$, and for almost all places v the image $a'_{1,v}$ of a'_v in $C^1(F_v, P_1)$ is unramified. We conclude that for almost all $k \geq 1$, $\mathrm{res}_{v_k}(\xi_{\mathrm{bad}})/\mathrm{loc}_{v_k}(\xi_{v_k})$ is the coboundary of an element of $C^1(F_{v_k}, P)$ whose image in $C^1(F_{v_k}, P_1)$ is ramified. □

This example shows that for [Kal, Lemma 4.5.1], it is important to use the canonical class and not an arbitrary lift in $H^2(F, P)$ of the canonical element of $\varprojlim_k H^2(F, P_k)$. More precisely, suppose that we form an extension $\mathcal{E}^{\mathrm{bad}}$ of $\mathrm{Gal}(\overline{F}/F)$ by P using a non-canonical class ξ^{bad} as above. Suppose that G is a reductive group that is an inner form of a quasi-split reductive group G^* over F. Realize G as a global rigid inner form (Ξ, z) of G^* with $z \in Z^1(P \to \mathcal{E}^{\mathrm{bad}}, Z \to G^*)$ for some finite central subgroup Z of G^*. Let $k \geq 0$ be big enough so that

1. G^* and G admit reductive models over $\mathcal{O}(F, S_k)$, that we fix,
2. G^* admits a global Whittaker datum $\mathfrak{w}$ compatible with this model at all $v \notin S_k$ in the sense of [CS80],
3. the restriction of z to P factors through a morphism $P(E_k, \dot{S}'_{E_k}, N_k) \to Z$, and for any $v \notin S_k$ the localization $z_v \in Z^1(F_v, G^*)$ is cohomologically trivial.

It can happen that the set V^{bad} of finite places $v \notin S_k$ such that the conjugacy classes of hyperspecial maximal compact subgroups $G(\mathcal{O}_{F_v})$ and $G^*(\mathcal{O}_{F_v})$ are *not* conjugate under the trivialization of (Ξ_v, z_v) is infinite. Using Proposition 6.3.1 one can easily give such examples with $G^* = \mathrm{Sp}_{2n}$ for any $n \geq 1$. Suppose for simplicity that G^* is split and that for a finite place v of F there are exactly two conjugacy classes of hyperspecial maximal compact subgroups in $G^*(F_v)$, as is the case for $G^* = \mathrm{Sp}_{2n}$. Suppose that φ is a global discrete Langlands parameter for G and that for every place v of F, φ_v is relevant for G_{F_v} i.e. that the local L-packet Π_{φ_v} is non-empty. Let V_φ^{bad} be the set of $v \in V^{\mathrm{bad}}$ such that the local parameter φ_v is unramified and endoscopic, i.e. the centralizer of $\varphi(\mathrm{Frob}_v)$ in $\widehat{G}$ is not connected. For every such v, Π_{φ_v} has two elements and the base point of this set for the rigidifying datum $(G^*_{F_v}, \Xi_v, z_v, \mathfrak{w}_v)$ is *not* $G(\mathcal{O}_{F_v})$-spherical. If V_φ^{bad} is infinite, no element of the adelic L-packet considered in [Kal, §4.5] is admissible, which is a problem to formulate a multiplicity formula for automorphic representations. In Example 6.3.2 below we point out that by [Elk87] there are infinitely many examples of (unconditional substitutes for) global Langlands parameters φ such that φ_v is endoscopic for infinitely many v. We do not know if there are examples with V_φ^{bad} infinite, but Proposition 6.3.1 and Example 6.3.2 certainly justify caution.

Example 6.3.2. *Consider first a prime number p and the group $\mathrm{SL}_2(\mathbb{Q}_p)$. There are two conjugacy classes of hyperspecial maximal compact subgroups of $\mathrm{SL}_2(\mathbb{Q}_p)$, represented by $K_1 = \mathrm{SL}_2(\mathbb{Z}_p)$ and its conjugate K_2 under $\mathrm{diag}(p,1) \in \mathrm{GL}_2(\mathbb{Q}_p)$. Therefore, for any Satake parameter $c = \mathrm{cl}(\mathrm{diag}(x,1))$, a semisimple conjugacy class in $\mathrm{PGL}_2(\mathbb{C})$,* a priori *there are two associated unramified representations of $\mathrm{SL}_2(\mathbb{Q}_p)$, say $\pi_{1,x}, \pi_{2,x}$ such that $\dim_{\mathbb{C}} \pi_{i,x}^{K_i} = 1$. Let $T = \{\mathrm{diag}(t,t^{-1}) \,|\, t \in \mathbb{Q}_p^\times\}$, a maximal torus in $\mathrm{SL}_2(\mathbb{Q}_p)$, and χ_x the unramified character $\mathrm{diag}(t,t^{-1}) \mapsto x^{v_p(t)}$ of T, where v_p is the p-adic valuation such that $v_p(p) = 1$. Let B be a Borel subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$ containing T. Then $\mathrm{Ind}_B^{\mathrm{SL}_2(\mathbb{Q}_p)}(\chi_x)$ is irreducible and isomorphic to $\pi_{1,x} \simeq \pi_{2,x}$ if $x \notin \{-1, p, p^{-1}\}$, whereas $\mathrm{Ind}_B^{\mathrm{SL}_2(\mathbb{Q}_p)}(\chi_{-1}) \simeq \pi_{1,-1} \oplus \pi_{2,-1}$ with $\pi_{1,-1} \not\simeq \pi_{2,-1}$. This is related to the fact that $\mathrm{diag}(-1,1)$ is, up to conjugation, the only semisimple element of $\mathrm{PGL}_2(\mathbb{C})$ whose centralizer is not connected (it has two connected components).*

Now let E be an elliptic curve over $\mathbb{Q}$. Let $f = \sum_{n \geq 1} a_n q^n$ be the associated [BCDT01] newform. By [Elk87] there are infinitely many primes p such that $a_p = 0$. In terms of the cuspidal automorphic representation $\pi = \otimes'_v \pi_v$ corresponding to f, this means that for infinitely many primes p, the Satake parameter of the unramified representation π_p of $\mathrm{GL}_2(\mathbb{Q}_p)$ (a semisimple conjugacy class in $\mathrm{GL}_2(\mathbb{C})$) has trace zero. Equivalently, its image in $\mathrm{PGL}_2(\mathbb{C})$ is $\mathrm{cl}(\mathrm{diag}(-1,1))$. Consider the conjectural *associated Langlands parameter $\varphi_E : L_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$ of π, where $L_{\mathbb{Q}}$ is the hypothetical Langlands group of $\mathbb{Q}$. Then its projection $\overline{\varphi_E}$ to $\mathrm{PGL}_2(\mathbb{C})$ is such that for infinitely many unramified primes p, $\overline{\varphi_E}(\mathrm{Frob}_p)$ is conjugated to $\mathrm{diag}(-1,1)$.*

This phenomenon has the following unconditional *consequence. Let $\widetilde{G}$ be an inner form of $\mathrm{GL}_2/\mathbb{Q}$, i.e. the group of invertible elements of a central simple algebra of degree 2 over $\mathbb{Q}$. Assume that E is relevant for $\widetilde{G}$, i.e. that for any prime p such that $\widetilde{G}_{\mathbb{Q}_p}$ is not split, π_p is a twist of the Steinberg representation or a supercuspidal representation of $\mathrm{GL}_2(\mathbb{Q}_p)$. By the Jacquet-Langlands correspondence [JL70], there is a unique automorphic cuspidal representation π' for $\widetilde{G}$ corresponding to π. Let G be the derived subgroup of $\widetilde{G}$, an inner form of $\mathrm{SL}_2/\mathbb{Q}$. By [LL79] and [Ram00], the restriction of π' to $G(\mathbb{A})$ (at the real place, one should consider $(\mathfrak{g}, K)$-modules) embeds in the space of cuspidal automorphic forms for G. This restriction is admissible but has infinite length: for any prime $p > 3$ such that $G_{\mathbb{Q}_p}$ is split and E has good supersingular reduction, $\pi'_p|_{G(\mathbb{Q}_p)}$ has length 2.*

Interestingly, the algorithm in [Elk87] uses primes which do not *split in certain quadratic extensions of* $\mathbb{Q}$*, while the counter-example in 6.3.1 is constructed using primes split in arbitrarily large extensions of the base field.*

7 Effective localization

We conclude by explaining how the constructive proof of the existence of a family of "local-global compatibility" cochains $(\beta_k)_{k\geq 0}$ at the end of section 4.4 allows one to explicitly compute all localizations of a global rigidifying datum, as promised in the introduction to this article.

7.1 A general procedure

Let G^* be a quasi-split connected reductive group over F. Fix a global Whittaker datum $\mathfrak{w}$ of G^*, i.e. choose a Borel subgroup B^* of G^* defined over F, let U be the unipotent radical of B^*, let χ be a generic unitary character of $U(\mathbb{A})/U(F)$, and let $\mathfrak{w}$ be the $G^*(F)$-conjugacy class of (B^*, χ). Let T a maximal torus of G^* defined over F, and E a finite Galois extension of F splitting T. Let S be a finite set of places of F such that

1. S contains all archimedean places of F and all places of F which ramify in E, and the (always injective) morphism $I(E,S)/\mathcal{O}(E,S)^\times \to C(E)$ is surjective (i.e. $\mathrm{Pic}(\mathcal{O}(E,S)) = 1$).

2. G^* admits a reductive model $\underline{G^*}$ over $\mathcal{O}(F,S)$ in the sense of [SGA70b, Exposé XIX, Définition 2.7] such that the schematic closure $\underline{T}$ of T in $\underline{G^*}$, which is a flat group scheme over $\mathcal{O}(F,S)$ since this ring is Dedekind, is a torus in the sense of [SGA70a, Exposé IX, Définition 1.3].

3. For any $v \notin S$, the Whittaker datum $\mathfrak{w}$ is compatible with the $G^*(F_v)$-conjugacy class of the hyperspecial maximal compact subgroup $\underline{G}^*(\mathcal{O}(F_v))$, in the sense of [CS80].

Let Z be a finite central subgroup of G, $N = \exp(Z)$ and $\overline{T} = T/Z$. Let $\underline{Z}$ be the schematic closure of Z in $\underline{T}$ (or $\underline{G}$), then $\underline{Z}$ is a group scheme of multiplicative type over $\mathcal{O}(F,S)$. Moreover $\overline{\underline{T}} := \underline{T}/\underline{Z}$ is a maximal torus of the reductive group scheme $\underline{G^*}/\underline{Z}$ (see [SGA70b, Exposé XXII, Corollaire 4.3.2]). Let $\dot{S}_E$ be a set of representatives for the action of $\mathrm{Gal}(E/F)$ on S_E. Finally, choose $\Lambda \in \overline{Y}[S_E, \dot{S}_E]_0^{N_{E/F}}$. If

$$\alpha_{E/F} \in Z^2\left(\mathrm{Gal}(E/F), \mathrm{Hom}(\mathbb{Z}[S_E]_0, \mathcal{O}(E,S))^\times\right)$$

is any Tate cocycle (as in [Tat66]), then taking the cup-product of $\alpha_{E/F}$ with Λ yields

$$\overline{z} \in Z^1(\mathrm{Gal}(\mathcal{O}(E,S)/\mathcal{O}(F,S)), \underline{\overline{T}}(\mathcal{O}(E,S))) \tag{7.1.1}$$

i.e. a Čech cocycle for the étale sheaf $\underline{\overline{T}}$ and the covering $\mathrm{Spec}(\mathcal{O}(E,S)) \to \mathrm{Spec}(\mathcal{O}(F,S))$. In particular we obtain a reductive group $\underline{G}$ over $\mathcal{O}(F,S)$ by twisting $\underline{G^*}$ with the image $\overline{\overline{z}}$ of $\overline{z}$ in

$$Z^1\left(\mathrm{Gal}(\mathcal{O}(E,S)/\mathcal{O}(F,S)), \underline{G^*}_{\mathrm{ad}}(\mathcal{O}(E,S))\right).$$

This realizes the generic fiber G of $\underline{G}$ as an inner twist $(\Xi, \overline{\overline{z}})$ of G^*.

Remark 7.1.1. *The fact that any connected reductive group G over F arises in this way is a consequence of [Kal, Lemmas A.1 and 3.6.1].*

More directly, that is without making use of [Kal, Lemma A.1], Steinberg's theorem on rational conjugacy classes in quasi-split semisimple simply connected algebraic groups ([Ste65]) implies that if we start with a reductive group G and a maximal torus T of G, then it can be realized as an inner twist $(G^*, \Xi, \overline{\overline{z}})$ with $\overline{\overline{z}}$ taking values in $\Xi^{-1}(T_{\mathrm{ad}}(\overline{F}))$.

We now use the constructive proof of Theorem 4.4.2 at the end of section 4.4. Let $E_1 = E$ and $S_1 = S$ and choose a finite Galois extension E_2 of F which is totally complex and such that for every $v \in S$ non-archimedean,

$$N_{E_2/E}\left(\prod_{w|v} \mathcal{O}(E_{2,w})^\times\right)$$

is contained in the subgroup of N-th powers in $\prod_{w|v} \mathcal{O}(E_w)^\times$. Finally, let E_3 be any finite Galois extension of F containing the Hilbert class field of E_2. Choose global fundamental classes $\overline{\alpha}_1$, $\overline{\alpha}_2$, $\overline{\alpha}_3$ such that $\overline{\alpha}_k = \mathrm{AW}_k^2(\overline{\alpha}_{k+1})$ for $k \in \{1,2\}$ and $\overline{\alpha}_3$ is normalized, i.e. $\overline{\alpha}_3(1,1) = 1$. Fix finite sets of places $S_3 \supset S_2 \supset S$ as in section 2. For each $v \in S_3$ fix a place $\dot{v}_3 \in S_{E_3}$. Choose local fundamental classes $\alpha_{k,v}$ for $v \in S$ and $k \in \{1,2,3\}$. Choose sets of representatives $(R_{k,v})_{1 \le k \le 3, v \in S}$ as in section 4.2, or rather, choose their image $\overline{R}_{k,v}$ in $\mathrm{Gal}(E_3/F)$. These families $(S_k)_{k \le 3}$, $(\overline{\alpha}_k)_{k \le 3}$, $(\alpha_{k,v})_{k \le 3, v \in S}$, $(\overline{R}_{k,v})_{k \le 3, v \in S}$ can be extended to $k \ge 0$ and $v \in V$, as explained in sections 4.1, 4.2 and 4.4. Moreover $\{\dot{v}_3\}_{v \in S}$ can be lifted and extended to yield $\dot{V}$ as in section 2.

Now choose $\overline{\beta}_3^{(0)} : \mathrm{Gal}(E_3/F) \to \mathrm{Maps}(S_{E_3}, C(E_3))$ such that $\mathrm{d}(\overline{\beta}_3^{(0)}) = \overline{\alpha}_3/\overline{\alpha_3'}$. Choose $\beta_2^{(1)} : \mathrm{Gal}(E_2/F) \to \mathrm{Maps}(S_{E_2}, I(E_2, S_2))$ lifting $\mathrm{AWES}_2^1(\overline{\beta}_3^{(0)})$ such that $\beta_2^{(1)}(1) = 1$ and $\beta_1^{(2)} := \mathrm{AWES}_1^1\left(\beta_2^{(1)}\right)$ takes values in $\mathrm{Maps}(S_{E_1}, I(E,S))$. Let $\alpha_1 = \alpha_1' \times \mathrm{d}(\beta_1^{(2)})$. At the end of section 4.4 we constructed a family $(\beta_k)_{k \ge 0}$ such that there exists $\epsilon_2' \in \mathrm{Maps}(S_{E_2}, \widehat{\mathcal{O}(E_2)}^\times)$ satisfying $\beta_2|_{S_{E_2}} = \beta_2^{(1)} \times \mathrm{d}(\epsilon_2')$, more precisely ϵ_2' is the restriction to S_{E_2} of

$$\lim_{n\to+\infty} \prod_{2\leq i\leq n} \mathrm{AWES}_2^0 \circ \cdots \circ \mathrm{AWES}_{i-1}^0(\epsilon_i).$$

Therefore $\beta_1|_{S_E} = \mathrm{AWES}_1^1(\beta_2) = \beta_1^{(2)} \times \mathrm{d}(x)$ where $x = \mathrm{AWES}_1^0(\epsilon_2')$ is a map

$$S_E \to N_{E_2/E}\left(\widehat{\mathcal{O}(E_2)}^{\times}\right).$$

In particular for every non-archimedean $v \in S$ there exists a map $y_v : S_E \to \prod_{w|v} \mathcal{O}(E_w)^\times$ such that $y_v^N = \mathrm{pr}_v(x)$. For $v \in S$ archimedean, simply let $y_v = 1$. Recall that $N = \exp(Z)$. Going back to the construction of N'-th roots in Propositions 5.1.1, 5.1.2 and 5.1.3, we see that for any choice of N-th root $\sqrt[N]{\beta_1^{(2)}} : \mathrm{Gal}(E/F) \to \mathrm{Maps}(S_E, \mathcal{I}(E, S \cup N))$, we can choose the N-th root $\sqrt[N]{\beta_1}$ so that for all $v \in S$,

$$\mathrm{pr}_v\left(\sqrt[N]{\beta_1}\right)|_{S_E} = \mathrm{pr}_v\left(\sqrt[N]{\beta_1^{(2)}}\right) \times \mathrm{d}\left(y_v\right).$$

If α_1 is chosen to form $\overline{z}$ in (7.1.1), the generic fiber G of $\underline{G}$ is endowed with a global rigidifying datum $(G^*, \Xi, z, \mathfrak{w})$ where $z = \iota(\Lambda)$. For $v \in V$, the localization of this rigidifying datum at v is $(G^*_{F_v}, \Xi_v, z_v, \mathfrak{w}_v)$ where $\Xi_v = \Xi_{\overline{F_v}}$ and $z_v = \mathrm{pr}_{\dot{v}}(z \circ \mathrm{loc}_v)$.

Let $z_v' = \iota_v(l_v(\Lambda))$ and fix a rigid inner twist (G_v', Ξ_v') of $G^*_{F_v}$ by z_v', which is well-defined up to conjugation by $G_v'(F_v)$ (see [Kal16, Fact 5.1]). We now compare the rigid inner twists (G_{F_v}, Ξ_v) and (G_v', Ξ_v') of $G^*_{F_v}$. Recall (Proposition 5.4.5) that

$$\mathrm{pr}_{\dot{v}}(z \circ \mathrm{loc}_v) = \iota_v(l_v(\Lambda)) \times \mathrm{d}(\kappa_v(\Lambda))$$

where $\kappa_v(\Lambda) = \mathrm{pr}_{\dot{v}}\left(\sqrt[N]{\beta_1}\right) \underset{E/F}{\sqcup} N\Lambda \in T(\overline{F_v})$. Therefore we have an isomorphism of rigid inner twists of $G^*_{F_v}$

$$(f_v, \kappa_v(\Lambda)) : (G_{F_v}, \Xi_v, z_v) \xrightarrow{\sim} (G_v', \Xi_v', z_v')$$

where f_v is obtained from $\Xi_v' \circ \mathrm{Ad}(\kappa_v(\Lambda)) \circ \Xi_v^{-1}$ by Galois descent. Thus $f_v : G_{F_v} \simeq G^*_{F_v}$ identifies the rigidifying datum $(G^*_{F_v}, \Xi_v, z_v, \mathfrak{w}_v)$ for G_{F_v} with the rigidifying datum $(G^*_{F_v}, \Xi_v', z_v', \mathfrak{w}_v)$ for G_v'.

- For $v \in V \smallsetminus S$, $l_v(\Lambda_v) = 0$ and we can simply take $G_v' = G^*_{F_v}$ and $\Xi_v' = \mathrm{Id}$. In particular G_{F_v} is quasi-split and we can simply take as rigidifying datum the pull-back $f_v^*(\mathfrak{w}_v)$ of the Whittaker datum $\mathfrak{w}_v$. The image $\overline{\kappa}_v(\Lambda)$ of $\kappa_v(\Lambda)$ in $\overline{T}(\overline{F_v})$ equals

$$\mathrm{pr}_{\dot{v}}(\beta_1) \underset{E/F}{\sqcup} \Lambda \in \underline{\overline{T}}(\mathcal{O}(E_{\dot{v}}))$$

  and so $\mathrm{Ad}(\kappa_v(\Lambda))$ is an automorphism of the reductive group scheme $\underline{G^*}_{\mathcal{O}(E_{\dot{v}})}$. Since Ξ_v is obtained as the generic fiber of an isomorphism $\underline{G^*}_{\mathcal{O}(E_{\dot{v}})} \simeq \underline{G}_{\mathcal{O}(E_{\dot{v}})}$, we see that f_v descends from an isomorphism $\underline{G}_{\mathcal{O}(E_{\dot{v}})} \simeq \underline{G^*}_{\mathcal{O}(E_{\dot{v}})}$ and so f_v can be extended to an isomorphism of reductive models $\underline{G}_{\mathcal{O}(F_v)} \simeq \underline{G^*}_{\mathcal{O}(F_v)}$. This shows that $f_v^*(\mathfrak{w}_v)$ is compatible with the $G(F_v)$-conjugacy class of hyperspecial maximal compact subgroups represented by $\underline{G}(\mathcal{O}(F_v))$. Note that this holds even for $v \notin S$ dividing N.

- For $v \in S$, one can compute the element $\kappa_v(\Lambda)$ up to an element of $T(F_v)$, since

$$\mathrm{d}\,(y_v) \underset{E/F}{\sqcup} N\Lambda = N_{E/F}\left(y_v \underset{E/F}{\sqcup} N\Lambda\right) \in T(F_v)$$

and so $\mathrm{d}(\kappa_v(\Lambda)) = \mathrm{d}(\kappa'_v(\Lambda))$ where

$$\kappa'_v(\Lambda) = \mathrm{pr}_{\dot{v}}\left(\sqrt[N]{\beta_1^{(2)}}\right) \underset{E/F}{\sqcup} N\Lambda$$

is computable. Thus $(f_v, \kappa'_v(\Lambda))$ is also an isomorphism of rigid inner twists of $G^*_{F_v}$. Note that to compute f_v it is enough to compute the image of $\kappa'_v(\Lambda)$ in $\overline{T}(\overline{F_v})$, i.e.

$$\mathrm{pr}_{\dot{v}}\left(\beta_1^{(2)}\right) \underset{E/F}{\sqcup} \Lambda \in \overline{T}(E_{\dot{v}})$$

and so in practice it is not necessary to compute an N-th rooth of $\beta_1^{(2)}$.

7.2 A simple example

Let us illustrate this on a simple example, where almost no computation of cocycles is needed.

7.2.1 Definition of the group G

Let $F = \mathbb{Q}(s)$ with $s^2 = 3$. Let D be a quaternion algebra over F such that D is definite at both real places of F, and split at all non-archimedean places of F. Let $N_D \in \mathrm{Sym}^2(D^*)$ be the reduced norm, and G the reductive group scheme over F defined by

$$G(R) = \{x \in R \otimes_F D \,|\, N_D(x) = 1 \text{ in } R\}$$

for any F-algebra R.

7.2.2 A reductive model of G

The class group of F is trivial, and the narrow class group of F is $\mathbb{Z}/2\mathbb{Z}$, corresponding to the totally complex and everywhere unramified extension $E = F(\zeta)$ of F, where $\zeta^2 - s\zeta + 1 = 0$ (ζ is a primitive 12-th root of unity). The class group of E is also trivial.

Write σ for the non-trivial $\mathcal{O}(F)$-automorphism of $\mathcal{O}(E)$. Let S be the set of real places of F, so that $S = \{v_+, v_-\}$ where the image of s in $F_{v_+} = \mathbb{R}$ is positive. We still denote by v_+, v_- the unique complex places of E above v_+, v_-. The group $\mathcal{O}(E)^\times$ is generated by ζ and $\zeta - 1$, which has infinite order. The group $\mathcal{O}(F)^\times$ is generated by -1 and $2 - s = N_{E/F}(\zeta - 1)$, which has infinite order.

Let $\underline{G^*} = \mathrm{SL}_2$ over $\mathcal{O}(F)$ and let $\underline{T} \subset \underline{G^*}$ be the torus defined by

$$\underline{T}(R) = \left\{ \begin{pmatrix} x & -y \\ y & x + sy \end{pmatrix} \middle| \, x, y \in R, \, x^2 + sxy + y^2 = 1 \right\}$$

for any $\mathcal{O}(F)$-algebra R. Then $\underline{T}$ splits over $\mathcal{O}(E)$. Let $\underline{Z} \simeq \mu_2$ be the center of $\underline{G^*}$ and $\overline{\underline{T}} = \underline{T}/\underline{Z}$. The element $(x = s, y = -2) \in \underline{T}(\mathcal{O}_F)$ maps to the unique element of order 2 in $\overline{T}(F)$, and so we have a 1-cocyle

$$\overline{z} : \sigma \mapsto \overline{(x = s, y = -2)} \in \mathrm{PGL}_2(\mathcal{O}(F)).$$

Since PGL_2 is also the automorphism group of the matrix algebra M_2, we obtain an Azumaya algebra $\mathcal{O}(D)$ over $\mathcal{O}(F)$ by twisting $\mathrm{M}_2(\mathcal{O}(F))$ using $\overline{z}$. Explicitly, it has basis $(1, Z, I, ZI)$ over $\mathcal{O}(F)$, where

$$Z = \begin{pmatrix} 0 & -1 \\ 1 & s \end{pmatrix}, \; I = \begin{pmatrix} 0 & 2\zeta - s \\ 2\zeta - s & 0 \end{pmatrix}.$$

We have $Z^{12} = 1$ and $I^2 = -1$. Let $D = F \otimes_{\mathcal{O}(F)} \mathcal{O}(D)$. Let $\underline{G}$ be the inner twist of $\underline{G^*}$ by $\overline{z}$, so that

$$\underline{G}(R) = \left\{ x \in R \otimes_{\mathcal{O}(F)} \mathcal{O}(D) \middle| N_D(x) = 1 \right\}$$

for any $\mathcal{O}(F)$-algebra R.

7.2.3 The group G as a rigid inner twist

If we identify $Y = X_*(T)$ with $\mathbb{Z}$, then $\overline{Y} = X_*(\overline{T})$ is identified with $\frac{1}{2}\mathbb{Z}$. Let $\Lambda \in \overline{Y}[\dot{S}_E]_0^{N_{E/F}}$ be defined by $\Lambda(v_+) = 1/2$ and $\Lambda(v_-) = -1/2$. An easy computation shows that one can choose the Tate cocycle α_1 for E/F such that

$$\alpha_1(\sigma, \sigma)(v_+)/\alpha_1(\sigma, \sigma)(v_-) = -1$$

and so $\overline{z} = \alpha_1 \underset{E/F}{\sqcup} \Lambda$. Using $z = \iota(\Lambda)$, we obtain a realization of G as a rigid inner twist (Ξ, z) of G^*.

7.2.4 Choice(s) of Whittaker data

Let ψ be the unitary character of $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$ such that $\psi_\infty(x) = \exp(2i\pi x)$, so that for every prime p we have $\ker(\psi_p) = \mathbb{Z}_p$. Fortunately the different ideal of $F/\mathbb{Q}$ is principal, generated by $2s$, and so for any choice of sign the global Whittaker datum $\mathfrak{w}$ for G^*

$$\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in U(\mathbb{A}_F) \mapsto \psi(\pm \mathrm{Tr}_{F/\mathbb{Q}}(x/(2s))) \tag{7.2.1}$$

is compatible with the model $\underline{G^*}_{\mathcal{O}(F_v)}$ at *every* finite place v of F. Therefore the global rigidifying datum $\mathcal{D} = (G^*, \Xi, z, \mathfrak{w})$ for G is such that for any finite place v of F, the localization $\mathcal{D}_v$ is unramified and compatible with the $G(F_v)$-conjugacy class of hyperspecial maximal compact subgroups $\underline{G}(\mathcal{O}(F_v))$.

7.2.5 Real places

At any real place v of F, we could compute explicit coboundaries expressing local-global compatibility, but this is not necessary since the parametrization of Arthur-Langlands packets for the compact Lie groups $G(F_v) \simeq \mathrm{SU}(2)$ is simply determined by the Whittaker datum $\mathfrak{w}_v$ and the *cohomology class* of z_v in $H^1(P_v \to \mathcal{E}, Z \to T)$ (see [Kal16, §5.6] and [Taïa, §3.2]), which only depends on $l_v(\Lambda)$. This simplification is particular to anisotropic real groups, for which Langlands packets have at most one element.

In order to formulate the local Langlands correspondence at each real place v of F it is necessary to identify an algebraic closure of the base field F_v, occurring in the definition of the Weil group W_{F_v}, with the coefficient field $\mathbb{C}$. We have natural algebraic closures E_{v+} and E_{v_-} of F_{v_+} and F_{v_-}. Choose $\tau_+ : \zeta \mapsto \exp(2i\pi/12)$ (resp. $\tau_- : \zeta \mapsto \exp(5 \times 2i\pi/12)$) identifying E_{v_+} (resp. E_{v_-}) with $\mathbb{C}$. There is a natural identification θ_+ (resp. θ_-) of $G^*_{F_{v_+}}$ (resp. $G^*_{F_{v_-}}$) with the usual split group SL_2 over $\mathbb{R}$, compatibly with the canonical isomorphisms $F_{v_+} = \mathbb{R}$ and $F_{v_-} = \mathbb{R}$. Let $\sqrt{3}$ be the positive square root of 3 in $\mathbb{R}$, so that $\tau_+(s) = \sqrt{3}$ and $\tau_-(s) = -\sqrt{3}$. In particular for any choice of sign in (7.2.1), the Whittaker data $(\theta_+)_*(\mathfrak{w}_{v_+})$ and $(\theta_-)_*(\mathfrak{w}_{v_-})$ differ. Associated to $\mathfrak{w}_+$ is a Borel subgroup B_+ of $G^*_{F_{v_+}} \times_{F_{v_+}} \mathbb{C}$ containing $T_{F_{v_+}} \times_{F_{v_+}} \mathbb{C}$ (see [Taïa]), corresponding to the generic discrete series representations of $G^*(F_{v_+})$. Using τ_+ we see B_+ as a Borel subgroup of $G^*_{E_{v_+}}$, and since T is defined over F and split over E we see that B_+ comes from a well-defined Borel subgroup of G^*_E containing T_E, which we still denote by B_+. Similarly, we have a Borel subgroup B_- of G^*_E containing T_E. Up to changing the sign in (7.2.1), we can assume that B_+ is such that the unique root of T_E in B_+ is $\alpha_+ : (x, y) \mapsto (x + \zeta y)^2$. Let us determine B_- using θ_+ and θ_-. For this we need to conjugate $\theta_+(T_{F_{v_+}})$ and

$\theta_-(T_{F_{v_-}})$ by an element of $\mathrm{SL}_2(\mathbb{R})$. The matrix

$$g = \begin{pmatrix} 1 & -\sqrt{3} \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$$

conjugates $\theta_-(T_{F_{v_-}})$ into $\theta_+(T_{F_{v_+}})$, mapping $\theta_-(x,y)$ to $\theta_+(x-\sqrt{3}y, y)$. Since $(\theta_+)_*(\mathfrak{w}_+)$ and $(\theta_-)_*(\mathfrak{w}_-)$ differ, the root α_- of T_E in B_- is *not* equal to

$$(\tau_-)^{-1} \circ \tau_+ \circ \alpha_+ \circ \tau_+^{-1} \circ (\theta_+)_{\mathbb{C}}^{-1} \circ \mathrm{Ad}(g) \circ (\theta_-)_{\mathbb{C}} \circ \tau_-,$$

which equals α_+. Therefore $\alpha_- \neq \alpha_+$ and $B_- \neq B_+$. Note that other choices for τ_+, τ_- would lead to other Borel subgroups, and some choices would give equal Borel subgroups.

Let us now consider Arthur-Langlands packets of unitary representations of $G(F_{v_+})$ and $G(F_{v_-})$. We refer to [Taïa, §3.2.2] for the parametrization of "cohomological" Arthur-Langlands packets for inner forms of symplectic or special orthogonal groups, following Shelstad, Adams-Johnson and Kaletha. The present case is much simpler. Note also that since $G(F_{v_+})$ and $G(F_{v_-})$ are compact, any non-empty Arthur-Langlands packet is "cohomological", i.e. is a packet of Adams-Johnson representations. For $v \in \{v_+, v_-\}$ there is only one Arthur-Langlands parameter

$$W_{F_v} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L G$$

which is non-trivial on $\mathrm{SL}_2(\mathbb{C})$ and yields a non-empty packet, namely the principal representation

$$\mathrm{SL}_2(\mathbb{C}) \to \widehat{G} \simeq \mathrm{PGL}_2(\mathbb{C}),$$

with corresponding packet containing the trivial representation with multiplicity one. Any other Arthur-Langlands parameter yielding a non-empty packet of representations is tempered and discrete, and so up to conjugation by $\widehat{G}$ it is of the form

$$\begin{aligned} \varphi_{k_+} : W_{F_{v_+}} &\longrightarrow \mathrm{PGL}_2(\mathbb{C}) \\ z \in E_{v_+}^\times &\longmapsto \begin{pmatrix} \tau_+(z/\bar{z})^{k_++1} & 0 \\ 0 & 1 \end{pmatrix} \\ j &\longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

for some $k_+ \in \mathbb{Z}_{\geq 0}$, and similarly discrete tempered parameters for $G_{F_{v_-}}$ are parametrized by integers $k_- \geq 0$, using τ_-. Above j is any element of $W_{F_{v_+}} \smallsetminus E_{v_+}^\times$ such that $j^2 = -1$. Note that we have put φ_{k_+} in dominant form for the upper-triangular Borel subgroup $\mathcal{B}$ of $\widehat{G}$. Using B_+ we have an identification between the group $\mathcal{T}$ of diagonal matrices in $\mathrm{PGL}_2(\mathbb{C})$ and $\widehat{T} = X^*(T) \otimes_{\mathbb{Z}} \mathbb{C}^\times$. So we can identify $l_{v_+}(\Lambda) = \Lambda(v_+) \in X_*(\overline{T})^{N_{E/F}}$ with an element of $X^*(\overline{\mathcal{T}})$, where $\overline{\mathcal{T}}$ is the preimage of $\mathcal{T}$ in $\widehat{\overline{G}} = \mathrm{SL}_2(\mathbb{C})$. The preimage $\mathcal{S}^+_{\varphi_{k_+}}$ of $\mathcal{S}_{\varphi_{k_+}} = \mathrm{Cent}(\varphi_{k_+}, \widehat{G})$ in $\widehat{\overline{G}}$ has 4 elements and is generated by

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \in \overline{\mathcal{T}}.$$

The class of $l_{v_+}(\Lambda)$ modulo $(1-\sigma)X_*(T)$ defines a character of $\mathcal{S}^+_{\varphi_{k_+}}$. There is a unique element π_{v_+,k_+} in the Arthur-Langlands packet attached to (the $\widehat{G}$-conjugacy class of) φ_{k_+}, that is the unique irreducible representation of $G(F_{v_+})$ in dimension $k_+ + 1$. The character $\langle \cdot, \pi_{v_+,k_+} \rangle$ of $\mathcal{S}^+_{\varphi_{k_+}}$ is the one defined by $l_{v_+}(\Lambda)$.

Similarly, each discrete series L-packet for $G_{F_{v_-}}$ has a unique element π_{v_-,k_-}, and a character $\langle \cdot, \pi_{v_-,k_-} \rangle$ of $\mathcal{S}^+_{\varphi_{k_-}}$ coming from the character $l_{v_-}(\Lambda) = \Lambda(v_-)$ of $\overline{\mathcal{T}}$. Note that since B_- and B_+ differ and $\Lambda(v_-) = -\Lambda(v_+)$, the characters of $\overline{\mathcal{T}}$ corresponding to $\Lambda(v_+)$ and $\Lambda(v_-)$ are equal.

7.2.6 Automorphic representations

To lighten notation we let $K = \underline{G}(\widehat{\mathcal{O}(F)})$. We can now formulate precisely the endoscopic decomposition of the space of $G(\mathbb{R} \otimes_{\mathbb{Q}} F)$-finite functions on $G(F)\backslash G(\mathbb{A}_F)/K$, with commuting actions of $G(\mathbb{R} \otimes_{\mathbb{Q}} F)$ and of the Hecke algebra in level K. Let V_+ (resp. V_-) be the irreducible representation of $G(F_{v_+})$ (resp. $G(F_{v_-})$) of dimension $k_+ + 1$ (resp. $k_- + 1$). Note that $V_\pm$ is obtained by restricting an irreducible algebraic representation of $G_{E_{v_\pm}}$. Recall [Gro99] that we can cut out the $V_+ \otimes V_-$-isotypical subspace inside the space of all automorphic forms for G, and define the space $M_{k_+,k_-}(K)$ of automorphic forms of weight (k_+, k_-) and level K as the space of $G(F)$-equivariant functions

$$G(\mathbb{A}_{F,f})/K \to V_+ \otimes V_-,$$

which is a finite-dimensional vector space over $\mathbb{C}$ endowed with a semi-simple action of the commutative Hecke algebra in level K. Moreover it is easy to check that $M_{k_+,k_-}(K)$ has a natural E-structure.

The automorphic multiplicity formula for SL_2 and its inner forms was proved in [LL79], although at the time there was no general definition of transfer factors, let alone Kaletha's normalization of transfer factors for inner forms. Formally we can use the main result of [Taïa], but of course a careful reading of [LL79] and a comparison of transfer factors with the later definition in [LS87] and [Kal16], [Kal] should give a more direct proof. In the present case, automorphic representations for G in level K fall into three categories:

- the trivial representation,

- representations corresponding to self-dual automorphic cuspidal representations of PGL_3/F which are algebraic regular at both infinite places and unramified at all finite places,
- representations "automorphically induced" from certain algebraic Hecke characters for E.

The multiplicity formula is non-trivial only in the third case. Making it explicit allows one to enumerate representations in the (most interesting) second case.

7.2.7 Global endoscopic parameters

Let $\chi : C(E) \to \mathbb{C}^\times$ be a continuous unitary character which is trivial on $C(F) = C(E)^{\mathrm{Gal}(E/F)}$. In particular, $\chi^\sigma = \chi^{-1}$. Using χ we can form the parameter

$$\begin{aligned} \varphi_\chi : W_{E/F} &\longrightarrow \mathrm{PGL}_2(\mathbb{C}) \\ z \in C(E) &\longmapsto \begin{pmatrix} \chi(z) & 0 \\ 0 & 1 \end{pmatrix} \\ \tilde{\sigma} &\longmapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

where $\tilde{\sigma} \in W_{E/F}$ is any lift of $\sigma \in \mathrm{Gal}(E/F)$. The parameters φ_χ and $\varphi_{\chi^{-1}}$ are conjugated by $\mathrm{PGL}_2(\mathbb{C})$. We only consider characters χ such that the restriction of φ_χ to the Weil groups at both real places of F are discrete, i.e. we impose that $\chi_{v_+} = \chi|_{E^\times_{v_+}}$ and $\chi_{v_-} = \chi|_{E^\times_{v_-}}$ are non-trivial. Therefore there are $a_+, a_- \in \mathbb{Z} \smallsetminus \{0\}$ such that

$$\chi_{v_+}(z) = \tau_+(z/\bar{z})^{a_+}, \quad \chi_{v_-}(z) = \tau_-(z/\bar{z})^{a_-}.$$

Moreover we impose that χ is everywhere unramified, i.e. at every finite place w of E, χ_w is trivial on $\mathcal{O}(E_w)^\times$. Since E has class number 1 the map

$$E^\times_{v_+} \times E^\times_{v_-} \times \prod_{w \text{ finite}} \mathcal{O}(E_w)^\times \to C(E)$$

is surjective, and its kernel is $\mathcal{O}(E)^\times$. Thus for $a_+, a_- \in \mathbb{Z} \smallsetminus \{0\}$ there is at most one everywhere unramified χ as above, and there exists one if and only if $\chi_{v_+} \times \chi_{v_-}$ is trivial on $\mathcal{O}(E)^\times$, which is generated by ζ and $\zeta - 1$. A simple computation shows that this is equivalent to

$$a_+ + 5a_- = 0 \mod 12.$$

For such a character χ, at a finite place w of E we have:

- If w is fixed by σ (inert case), then there is a uniformizer $\varpi_w \in \mathcal{O}(F)$, and so χ_w is trivial.

- If w is not fixed by σ (split case), then if $\varpi_w \in \mathcal{O}(E)$ is a uniformizer, we have

$$\chi_w(\varpi_w) = \chi_{v_+}(\varpi_w)^{-1}\chi_{v_-}(\varpi_w)^{-1}.$$

This concludes the description of all endoscopic global parameters for G which are discrete at both real places and unramified at all finite places. They are parametrized by pairs $(a_+, a_-) \in (\mathbb{Z} \smallsetminus \{0\})^2$ such that $a_+ + 5a_- = 0 \mod 12$, modulo $(a_+, a_-) \sim (-a_+, -a_-)$.

Let χ be a character as above. Then the centralizer $\mathcal{S}_{\varphi_\chi}$ of φ_χ is

$$\left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \subset \mathcal{T}$$

and so it coincides with the local centralizers at v_+, v_-. Up to replacing χ by χ^{-1}, we are in exactly one of the following cases:

- $a_+ > 0$ and $a_- > 0$, i.e. $\chi_{v_+}(z) = \tau_+(z/\bar{z})^{k_+ + 1}$ for $k_+ \geq 0$ and $\chi_{v_-}(z) = \tau_-(z/\bar{z})^{k_- + 1}$ for $k_- \geq 0$. Then $\langle \cdot, \pi_{v_+, k_+} \rangle \times \langle \cdot, \pi_{v_-, k_-} \rangle$ is the non-trivial character of $\mathcal{S}_{\varphi_\chi}$.
- $a_+ > 0$ and $a_- < 0$, i.e. $\chi_{v_+}(z) = \tau_+(z/\bar{z})^{k_+ + 1}$ for $k_+ \geq 0$ and $\chi_{v_-}(z) = \tau_-(z/\bar{z})^{-k_- - 1}$ for $k_- \geq 0$. Then $\langle \cdot, \pi_{v_+} \rangle \times \langle \cdot, \pi_{v_-} \rangle$ is the trivial character of $\mathcal{S}_{\varphi_\chi}$.

By the multiplicity formula, in weight (k_+, k_-) and level $\underline{G}(\widehat{\mathcal{O}(F)})$, there is at most one endoscopic automorphic representation, and there is one if and only if

$$(k_+ + 1) - 5(k_- + 1) = 0 \mod 12. \tag{7.2.2}$$

In low weight, we have computed Hecke operators for small primes and verified this condition.

7.2.8 Comments

The class number

$$\mathrm{card}\left(G(F) \backslash G(\mathbb{A}_{F,f}) / \underline{G}(\widehat{\mathcal{O}(F)}) \right) = 1$$

as one can check when computing a Hecke operator at any finite place, by strong approximation. Note that $\underline{G}$ is *not* the only reductive model of G, even up to the action of $G_{\mathrm{ad}}(F)$. By splitting the Azumaya algebra $\mathcal{O}(D)$ modulo $(2) = (s-1)^2$, we can compute an $(s-1)$-Kneser neighbour of $\mathcal{O}(D)$, that is another maximal order $\mathcal{O}'(D)$ of D, having basis over $\mathcal{O}(F)$

$$1, Z + sI, (1-s)(s + ZI), (1-s)^{-1}(1 + I + sZI).$$

It gives rise to a second model $\underline{G}'$ of G, which is not isomorphic to $\underline{G}$ since one can compute using reduction theory that $\underline{G}(\mathcal{O}_F)$ is a dihedral group of order 24 (generated by Z and I, with $IZI^{-1} = Z^{-1}$), whereas $\underline{G}'(\mathcal{O}(F))$ is isomorphic to $\mathrm{SL}_2(\mathbb{F}_3)$ (an isomorphism is given by reduction modulo s). One can also check that the class number

$$\mathrm{card}\left(G_{\mathrm{ad}}(F)\backslash G_{\mathrm{ad}}(\mathbb{A}_{F,f})/\underline{G}_{\mathrm{ad}}(\widehat{\mathcal{O}(F)})\right) = 2,$$

and so $\underline{G}$ and $\underline{G}'$ are up to isomorphism the only two reductive models of G over $\mathcal{O}(F)$. So we have two distinct notions of "level one" for automorphic representations for G, and although the relevant Arthur-Langlands parameters are the same in both cases, the automorphic multiplicities differ. More precisely, any algebraic Hecke character χ for E as above contributes an automorphic representation for G either in level $\underline{G}(\widehat{\mathcal{O}(F)})$ or in level $\underline{G}'(\widehat{\mathcal{O}(F)})$.

7.2.9 Higher rank

Alternatively, one could explicitly compute the geometric transfer factors defined in [LL79] for G and the endoscopic group H isomorphic to the unique anisotropic torus over F of dimension 1 which is split by E. Although one would lose the interpretation in terms of characters of centralizers of Langlands parameters, this would probably lead to a proof that the multiplicity formula for G in level $\underline{G}(\widehat{\mathcal{O}(F)})$ reduces to (7.2.2).

Note however that the approach in the present paper generalizes easily to higher rank. For example, using the embedding $(\mathrm{SL}_2)^n \hookrightarrow \mathrm{Sp}_{2n}$, it is easy to generalize the above example to the case where G is the inner form of $G^* = \mathrm{Sp}_{2n}$ over F which is definite (i.e. $G(F \otimes_{\mathbb{Q}} \mathbb{R})$ is compact) and split at all finite places. This does not require additional computation, and so one can make explicit Arthur's multiplicity formula (also known in this case, see [Taïa]) in "level one". Moreover, using also *pure* inner forms of quasi-split special orthogonal groups, namely definite special orthogonal groups obtained using copies of $(x, y) \mapsto x^2 + sxy + y^2$ and (in odd dimension) $x \mapsto x^2$, it is possible to carry out the same inductive strategy as in [Taïb], but using definite groups as in [CR15], which makes explicit computations much simpler. Therefore the above example makes it possible to explictly compute automorphic cuspidal self-dual representations for general linear groups over F which are unramified at all finite places and algebraic regular at both real places.

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