Galois and Cartan Cohomology of Real Groups

Jeffrey Adams and Olivier Taïbi

January 12, 2018

Abstract

Suppose G is a complex, reductive algebraic group. A real form of G is an antiholomorphic involutive automorphism σ, so $G(\mathbb{R}) = G(\mathbb{C})^\sigma$ is a real Lie group. Write $H^1(\sigma, G)$ for the Galois cohomology (pointed) set $H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), G)$. A Cartan involution for σ is an involutive holomorphic automorphism θ of G, commuting with σ, so that $\theta\sigma$ is a compact real form of G. Let $H^1(\theta, G)$ be the set $H^1(\mathbb{Z}_2, G)$ where the action of the nontrivial element of $\mathbb{Z}_2$ is by θ. By analogy with the Galois group we refer to $H^1(\theta, G)$ as Cartan cohomology of G with respect to θ. Cartan's classification of real forms of a connected group, in terms of their maximal compact subgroups, amounts to an isomorphism $H^1(\sigma, G_{\mathrm{ad}}) \simeq H^1(\theta, G_{\mathrm{ad}})$ where G_{ad} is the adjoint group. Our main result is a generalization of this: there is a canonical isomorphism $H^1(\sigma, G) \simeq H^1(\theta, G)$.

We apply this result to give simple proofs of some well-known structural results: the Kostant-Sekiguchi correspondence of nilpotent orbits; Matsuki duality of orbits on the flag variety; conjugacy classes of Cartan subgroups; and structure of the Weyl group. We also use it to compute $H^1(\sigma, G)$ for all simple, simply connected groups, and to give a cohomological interpretation of strong real forms. For the applications it is important that we do not assume G is connected.

1 Introduction

Suppose G is a complex, reductive algebraic group. A *real form* of G is an antiholomorphic involutive automorphism σ of G, in which case $G(\mathbb{R}) = G(\mathbb{C})^\sigma$ is a real Lie group. See Section 3 for more details. Let $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ and write $H^i(\Gamma, G)$ for the Galois cohomology of G (if G is nonabelian $i \le 1$). If we want to specify how the nontrivial element of Γ acts we will write $H^i(\sigma, G)$. The

2000 Mathematics Subject Classification: 11E72 (Primary), 20G10, 20G20
Jeffrey Adams is supported in part by National Science Foundation Grant #DMS-1317523
Olivier Taïbi is supported by ERC Starting Grant 306326.

equivalence (i.e., conjugacy) classes of real forms of G, which are inner to σ (see Section 3) are parametrized by $H^1(\sigma, G_{\mathrm{ad}})$ where G_{ad} is the adjoint group.

On the other hand, at least for G connected, Cartan classified the real forms of G in terms of holomorphic involutions as follows. We say a *Cartan involution* for σ is a holomorphic involutive automorphism θ, commuting with σ, so that $\sigma^c = \theta\sigma$ is a compact real form. If G is connected then θ exists, and is unique up to conjugacy by G^σ. Following Mostow we prove a similar result in general. See Section 3.

Let $H^i(\mathbb{Z}_2, G)$ be the group cohomology of G where the nontrivial element of $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$ acts by θ. As above we denote this $H^i(\theta, G)$, and we refer to this as Cartan cohomology of G. Conjugacy classes of involutions which are inner to θ are parametrized by $H^1(\theta, G_{\mathrm{ad}})$.

Thus the equivalence of the two classifications of real forms amounts to an isomorphism (for connected G) of the first Galois and Cartan cohomology spaces $H^1(\sigma, G_{\mathrm{ad}}) \simeq H^1(\theta, G_{\mathrm{ad}})$. It is natural to ask if the same isomorphism holds with G in place of G_{ad}. For our applications it is helpful to know the result for disconnected groups as well.

Theorem 1.1 *Suppose G is a complex, reductive algebraic group (not necessarily connected), and σ is a real form of G. Let θ be a Cartan involution for σ. Then there is a canonical bijection of pointed sets $H^1(\sigma, G) \simeq H^1(\theta, G)$.*

The proof will be given in Section 4.

The interplay between the σ and θ pictures plays a fundamental role in the structure and representation theory of real groups, going back at least to Harish Chandra's formulation of the representation theory of $G(\mathbb{R})$ in terms of $(\mathfrak{g}, K)$-modules. The theorem is an aspect of this, and we give several applications.

Suppose X is a homogeneous space for G, equipped with a real form σ_X which is compatible with σ. Then the space of $G(\mathbb{R})$-orbits on $X(\mathbb{R}) = X^{\sigma_X}$ can be understood in terms of the Galois cohomology of the stabilizer of a point in X. Similar remarks apply to computing G^θ-orbits on X^{θ_X}. Note that these stabilizers may be disconnected, even if G is connected. See Proposition 5.4.

We use this principle to give simple proofs of several well-known results, including the Kostant-Sekiguchi correspondence [23] and Matsuki duality [20]. Let $G(\mathbb{C})$ be a connected complex reductive group, with real form σ and corresponding Cartan involution θ. Let $G(\mathbb{R}) = G(\mathbb{C})^\sigma$, and $K(\mathbb{C}) = G(\mathbb{C})^\theta$. Let $\mathfrak{g}_0 = \mathfrak{g}^\sigma$ and $\mathfrak{p} = \mathfrak{g}^{-\theta}$. The Kostant-Sekiguchi correspondence is a bijection between the nilpotent $G(\mathbb{R})$-orbits on $\mathfrak{g}_0$ and the nilpotent $K(\mathbb{C})$-orbits on $\mathfrak{p}$. Matsuki duality is a bijection between the $G(\mathbb{R})$ and $K(\mathbb{C})$ orbits on the flag variety of G. See Propositions 6.1.5 and 6.2.8.

On the other hand Theorem 5.8 applied to the space of Cartan subgroups gives a simple proof of another result of Matsuki: there is a bijection between $G(\mathbb{R})$-conjugacy classes of Cartan subgroups of $G(\mathbb{R})$ and K-conjugacy classes of θ-stable Cartan subgroups of G [20]. Also a well-known result about two versions of the rational Weyl group (Proposition 6.3.2) follows.

If G is connected Borovoi proved $H^1(\sigma, G) \simeq H^1(\sigma, H_f)/W_i$ where H_f is a fundamental Cartan subgroup, and W_i is a certain subgroup of the Weyl group [9]. Essentially the same proof carries over to give $H^1(\theta, G) \simeq H^1(\theta, H_f)/W_i$. We prove this as a consequence of Theorem 1.1 (Proposition 7.4).

Let Z be the center of G and let Z_{tor} be its torsion subgroup. Associated to a real form σ is its *central invariant*, denoted $\mathrm{inv}(\sigma) \in Z^\sigma_{\mathrm{tor}}/(1+\sigma)Z_{\mathrm{tor}}$ (Definition 8.7). The formulation of a precise version of the Langlands classification of irreducible representations requires the notion of *strong real form*, refining that of a real form, and its *central invariant*, which is an element of Z^σ_{tor} lifting the central invariant of the underlying real form (Definition 8.11).

Theorem 1.2 (Proposition 8.14) *Suppose σ is a real form of G. Choose a representative $z \in Z^\sigma_{\mathrm{tor}}$ of $\mathrm{inv}(\sigma) \in Z^\sigma_{\mathrm{tor}}/(1+\sigma)Z_{\mathrm{tor}}$. Then there is a bijection*

$$H^1(\sigma, G) \overset{1-1}{\longleftrightarrow} G\text{-conjugacy classes of strong real forms } \sigma \text{ with } \mathrm{inv}(\sigma) = z$$

This bijection is useful in both directions. On the one hand it is not difficult to compute the right hand side, thereby computing $H^1(\sigma, G)$. Over a p-adic field $H^1(\sigma, G) = 1$ if G is simply connected. Over $\mathbb{R}$ this is not the case, and we use Theorem 1.1 to compute $H^1(\sigma, G)$ for all such groups. See Section 3 and the tables in Section 10. We used the Atlas of Lie Groups and Representations software for some of these calculations. See [11] for another approach.

In the context of the Langlands classification it would be more natural if strong real forms were described in terms of classical Galois cohomology. Theorem 1.2 provides such an interpretation. See Corollary 8.15.

The authors would like to thank Michael Rapoport for asking about the interpretation of strong real forms in terms of Galois cohomology, and apologize it took so long to get back to him. We are also grateful to Tasho Kaletha for several helpful discussions during the writing of this paper and of [15], and Skip Garibaldi for a discussion of the Galois cohomology of the spin groups. Finally we thank the two referees for carefully reading the manuscript and making a number of helpful suggestions.

2 Preliminaries on Group Cohomology

See [24] for an overview of group cohomology.

For now suppose τ is an involutive automorphism of an abstract group G. Define $H^i(\tau, G)$ as the set $H^i(\mathbb{Z}_2, G)$ where the nontrivial element of $\mathbb{Z}_2$ acts by τ [1]. If G is abelian these are groups and are defined for all $i \geq 0$. Otherwise these are pointed sets, and defined only for $i = 0, 1$. Let

$$Z^1(\tau, G) = G^{-\tau} = \{g \in G \mid g\tau(g) = 1\}.$$

[1] There is a small notational issue here. If $\tau = 1$ (the identity automorphism of G), $H^1(1, G)$ denotes the group $H^1(\mathbb{Z}_2, G)$ with $\mathbb{Z}_2$ acting trivially.

Then we have the standard identifications

$$H^0(\tau, G) = G^\tau,\ H^1(\tau, G) = Z^1(\tau, G)/\sim$$

where $\sim$ is the equivalence relation $g \sim xg\tau(x^{-1}) \quad (x \in G)$. For $g \in G^{-\tau}$ let $cl(g)$ be the corresponding class in $H^1(\tau, G)$.

If G is abelian we also have the Tate cohomology groups $\widehat{H}^i(\tau, G)$ $(i \in \mathbb{Z})$. These satisfy

$$\widehat{H}^0(\tau, G) = G^\tau/(1+\tau)G, \quad \widehat{H}^1(\tau, G) = H^1(\tau, G),$$

and (since the group generated by τ is cyclic), $\widehat{H}^i(\tau, G) \simeq \widehat{H}^{i+2}(\tau, G)$ for all i, and these isomorphisms are canonical.

Suppose $1 \to A \to B \to C \to 1$ is an exact sequence of groups with an involutive automorphism τ. Then there is an exact sequence

(2.1)

$$1 \to H^0(\tau, A) \to H^0(\tau, B) \to H^0(\tau, C) \to H^1(\tau, A) \to H^1(\tau, B) \to H^1(\tau, C)$$

Furthermore if $A \subset Z(B)$ ($Z(*)$ denotes the center of a group) then there is one further step $\to H^2(\tau, A) = A^\tau/(1+\tau)A$.

We will need the following generalization of $H^1(\tau, G)$.

Definition 2.2 *Suppose τ is an involutive automorphism of G, and A is a subset of $Z(G)$. Define*

(2.3)(a) $$Z^1(\tau, G; A) = \{g \in G \mid g\tau(g) \in A\}$$

and

(2.3)(b) $$H^1(\tau, G; A) = Z^1(\tau, G; A)/[g \sim tg\tau(t^{-1})\ (t \in G)].$$

These are pointed sets if $1 \in A$. The map $g \mapsto g\tau(g)$ factors to a map from $H^1(\tau, G; A)$ to A.

For $z \in Z$ we write $H^1(\tau, G; z)$ instead of $H^1(\tau, g; \{z\})$. Taking $A = \{1\}$ gives ordinary cohomology $H^1(\tau, G)$. Write $cl(g)$ for the image of $g \in Z^1(\tau, G; A)$ in $H^1(\tau, G; A)$.

We make use of twisting in nonabelian cohomology [24, Section III.4.5]. Let $Z = Z(G)$. For $g \in G$ let $\mathrm{int}(g)$ be the inner automorphism $\mathrm{int}(g)(h) = ghg^{-1}$. Fix an involutive automorphism τ of G, and $z \in Z$. Note that $\mathrm{int}(g) \circ \tau$ is an involution if and only if $g \in Z^1(\tau, G; Z)$.

Lemma 2.4 *Suppose $\tau' = \mathrm{int}(g) \circ \tau$ for some $g \in Z^1(\tau, G; Z)$. Let $w = g\tau(g) \in Z$. Then the map $h \mapsto hg^{-1}$ induces an isomorphism*

$$H^1(\tau, G; z) \to H^1(\tau', G; zw^{-1}).$$

If $H^1(\tau, Z) = 1$, this isomorphism is independent of the choice of $g \in Z^1(\tau, G; w)$ satisfying $\tau' = \mathrm{int}(g) \circ \tau$.

In particular $H^1(\tau,G) \simeq H^1(\tau',G)$ if $\tau' = \mathrm{int}(g) \circ \tau$, where $g \in Z^1(\tau,G)$, and this isomorphism is canonical if $H^1(\tau,Z)=1$.

Finally suppose τ' is conjugate to τ by an inner automorphism of G. Then $H^1(\tau,G) \simeq H^1(\tau',G)$, and this isomorphism is canonical if

$$\ker\left(H^1(\tau,Z) \to H^1(\tau,G)\right) = 1.$$

We omit the elementary proof.

Write $[\tau]$ for the G-conjugacy class of τ.

Definition 2.5 *Assume* $\ker\left(H^1(\tau,Z) \to H^1(\tau,G)\right) = 1$. *Given a G-conjugacy class $[\tau]$ of involutive automorphisms of G, define $H^1([\tau],G) = H^1(\tau,G)$.*

This is well-defined by Lemma 2.4.

3 Real Forms and Cartan involutions

In the rest of the paper, unless otherwise noted, G will denote a complex, reductive algebraic group. Except in a few places we do not assume G is connected. Write G^0 for the identity component.

We identify G with its complex points $G(\mathbb{C})$ and use these interchangeably. We may view G either as an algebraic group or as a complex Lie group. The identity component of G as an algebraic group is the same as the topological identity component when viewed as a Lie group, and the component group G/G^0 is finite.

A *real form* of G is a real algebraic group H endowed with an isomorphism $\phi : H_{\mathbb{C}} \simeq G$, where $H_{\mathbb{C}}$ denotes the base change of H from $\mathbb{R}$ to $\mathbb{C}$. By an *algebraic, conjugate linear, involutive automorphism* of $H_{\mathbb{C}}$ we mean an algebraic, involutive automorphism of $H_{\mathbb{C}}$ (considered as a scheme over $\mathbb{R}$) such that the induced morphism between rings of algebraic functions on H is conjugate linear, and compatible with the morphisms defining the group structure on H. Naturally associated to a real form H is an algebraic, conjugate linear, involutive automorphism σ_H of $H_{\mathbb{C}}$. Transporting σ_H to G via ϕ this is equivalent to having an algebraic, conjugate linear, involutive automorphism σ of G. Conversely, by Galois descent any such automorphism of G comes from a real form (H,ϕ), which is unique up to unique isomorphism. See [12, §6.2, Example B and §6.5] for details in a much more general situation.

It is convenient to work with a more elementary notion of real form, using only the structure of G as a complex Lie group. Any algebraic, conjugate linear, involutive automorphism of G induces an antiholomorphic involutive automorphism of G. In fact every antiholomorphic automorphism arises this way:

Lemma 3.1 *Let G be a complex reductive algebraic group. Then any antiholomorphic involutive automorphism of G is induced by a unique algebraic conjugate linear involutive automorphism of $G(\mathbb{C})$.*

The reductive hypothesis is necessary. For example suppose $G = \mathbb{C}\times\mathbb{C}^\times$ and $\phi(z,w) = (\overline{z}, e^{\overline{z}}\overline{w}^{-1})$. Then ϕ is an antiholomorphic involutive automorphism of G, but it is not algebraic.

Proof. First will use the fact that every holomorphic representation of G is algebraic, so we prove this first.

(1) Consider a finite-dimensional complex vector space V and a holomorphic morphism $\rho : G \to \mathrm{GL}(V)$. Let T be a maximal torus of G^0. Then $T \simeq \mathrm{GL}_1^r$ for some $r \geq 0$, and by a well-known elementary argument we have a canonical decomposition $V = \bigoplus_\chi V_\chi$ where the sum is over the algebraic morphisms $\chi : T \to \mathrm{GL}_1$ and $V_\chi = \{v \in V \mid \forall t \in T,\, \rho(t)(v) = \chi(t)v\}$. Fix a Borel subgroup B of G^0 containing T. Let N be the unipotent radical of B. Let α be a root of T acting by conjugation on N, and U_α the corresponding one-dimensional additive algebraic subgroup of N. For any $X \in \mathrm{Lie}(U_\alpha)$, $\mathrm{d}\rho(X) \in \mathrm{End}(V)$ maps V_χ to $V_{\chi+\alpha}$, and so $\mathrm{d}\rho(X)$ is nilpotent. It follows that $\rho|_{U_\alpha}$ is algebraic. Choose an ordering $\alpha_1, \dots, \alpha_k$ of the set of roots of T in N. Then the product map $U_{\alpha_1} \times \cdots \times U_{\alpha_k} \to N$ is an isomorphism [6, Proposition 14.4], and so $\rho|_N$ is algebraic. Let $\overline{N}$ the unipotent radical of the Borel subgroup $\overline{B}$ of G^0 opposite to B with respect to T. There exists $g \in G^0$ such that $\overline{B} = gBg^{-1}$, and so $\rho|_{\overline{N}}$ is also algebraic. The product map $\overline{N} \times T \times N \to G^0$ is an open embedding (in the algebraic sense) [6, Corollary 14.14], and using translation we see that for any $g \in G$, there is a Zariski-open U of G containing g such that $\rho|_U$ is algebraic. The target $\mathrm{GL}(V)$ is separated, so by glueing we obtain that ρ is algebraic.

(2) Fix a representation $\rho : G \to \mathrm{GL}(V)$, where V is a complex vector space of finite dimension, such that ρ is a closed immersion [6, Proposition 1.10]. Suppose that $\varphi : G \to G$ is an antiholomorphic involutive automorphism. Choose an arbitrary real structure on V, and let σ_V denote complex conjugation $\mathrm{GL}(V) \to \mathrm{GL}(V)$ with respect to this real structure. Then $\sigma_V \circ \rho \circ \varphi$ is a holomorphic representation of G, so it is algebraic and φ is algebraic conjugate linear.

$\square$

Lemma 3.1 justifies the following elementary definition of real forms.

Definition 3.2 *A real form of G is an antiholomorphic involutive automorphism σ of G. Two real forms are equivalent if they are conjugate by an inner automorphism. Write $[\sigma]$ for the equivalence class of σ.*

We say two real forms σ_1, σ_2 are inner to each other, or in the same inner class, if $\sigma_1\sigma_2^{-1}$ is an inner automorphism of G. This is well defined on the level of equivalence classes.

See Remark 8.2 for a subtle point regarding this notion of equivalence.

If σ is a real form of G, let $G(\mathbb{R}) = G^\sigma$ be the fixed points of σ. This is a real Lie group, with finitely many connected components.

We turn now to compact real forms and Cartan involutions. If G is connected these results are well-known. The general case is due to Mostow [21].

Definition 3.3 *A real form σ of G is said to be a* compact real form *if G^σ is compact and meets every component of G.*

Mostow defines a compact real form (cf. [21, Section 2]) of G to be a compact subgroup G_K such that $\mathrm{Lie}(G) = \mathrm{Lie}(G_K) \oplus i\mathrm{Lie}(G_K)$ and G_K meets every component of G. Let us check that our definition is equivalent to this.

Lemma 3.4 *For any complex reductive group G, the map $\sigma \mapsto G^\sigma$ is a bijection between the set of compact real forms of G, in the sense of Definition 3.3, to the set of compact real forms of G in the sense of* [21].

Proof. If σ is any real form of G, then $\dim_{\mathbb{R}} G^\sigma = \dim_{\mathbb{R}} \mathrm{Lie}(G^\sigma) = \dim_{\mathbb{R}} \mathrm{Lie}(G)^\sigma$, and since $\mathrm{Lie}(G) = (\mathrm{Lie}(G))^\sigma \oplus i(\mathrm{Lie}(G))^\sigma$ we obtain $\dim_{\mathbb{R}} G^\sigma = \dim_{\mathbb{C}} \mathrm{Lie}(G) = \dim_{\mathbb{C}} G$. Choose a faithful algebraic representation $\rho : G \hookrightarrow \mathrm{GL}(V)$. If K is any compact subgroup of G, then V admits a hermitian form for which $\rho(K)$ is unitary. In particular we see that $\mathrm{Lie}(K) \cap i\mathrm{Lie}(K) = 0$. These two facts imply that for any compact real form σ of G, G^σ is a compact real form of G in the sense of [21].

Let us now check that $\sigma \mapsto G^\sigma$ is injective. The action of σ on G^0 is determined by its action on $\mathrm{Lie}(G) = \mathrm{Lie}(G^\sigma) \oplus i\mathrm{Lie}(G^\sigma)$. Once $\sigma|_{G^0}$ is determined, σ is determined by the requirement that it fixes G^σ pointwise, since G^σ meets every connected component of G.

Finally we show that $\sigma \mapsto G^\sigma$ is surjective. Suppose K is a compact real form of G in the sense of [21]. Choose ρ and a hermitian form on V as above. Choosing an orthonormal basis for V, we can view ρ as a closed embedding $G \to \mathrm{GL}_n(\mathbb{C})$ such that $\rho(K) \subset U(n)$. Let $\tau(g) = {}^t\bar{g}^{-1}$ ($g \in GL_n(\mathbb{C})$). Then $\rho(G^0)$ is stable under τ, since $\mathrm{Lie}(\rho(G)) = \mathrm{Lie}(\rho(K)) \oplus i\mathrm{Lie}(\rho(K))$, and $d\tau$ fixes $\mathrm{Lie}(\rho(K)) \subset \mathfrak{u}(n)$ pointwise. Furthermore $\rho(G)$ is stable under τ since τ fixes $\rho(K)$ pointwise, and $G = G^0K$. Pull back τ to G to define $\sigma = \rho^{-1} \circ \tau \circ \rho$. This is a compact real form of G, and $K \subset G^\sigma$. By the Cartan decomposition [21, Lemma 2.1] $G^\sigma \cap G^0 = K \cap G^0$, and this implies $G^\sigma = K$. $\square$

Using Lemma 3.4 we will refer to σ or $K = G^\sigma$ as a compact real form of G.

We turn next to the Cartan decomposition of the complex group G. We first define the Cartan decomposition of a general real Lie group.

Definition 3.5 *We say a real Lie group G has a Cartan decomposition $(K, \mathfrak{p})$ if K is a compact subgroup of G, $\mathfrak{p}$ is a subspace of $\mathrm{Lie}(G)$ stable under $\mathrm{Ad}(K)$, and the map $(k, X) \mapsto k\exp(X)$ is a diffeomorphism from $K \times \mathfrak{p}$ onto G.*

It is easy to see that K is necessarily a *maximal* compact subgroup of G.

The Cartan decomposition holds for a complex reductive group, viewed as a real group by restriction of scalars:

Lemma 3.6 (Mostow [21, Lemma 2.1]**)** *Suppose σ is a compact real form of G. Let $K = G^\sigma$ and $\mathfrak{p} = \mathrm{Lie}(G)^{-\sigma} = i\mathrm{Lie}(K)$. Then $(K, \mathfrak{p})$ is a Cartan decomposition of G considered as a real Lie group. Furthermore K is a maximal compact subgroup of G.*

Although we will not use it, it is not difficult to check that the complexification functor [24, Section III.4.5], from the category of compact Lie groups to that of complex reductive groups endowed with a compact real form, induces a bijection on the level of isomorphism classes. A key step in the proof is the existence of compact real forms:

Theorem 3.7 (Weyl, Chevalley, Mostow [21, Lemma 6.1]**)** *Every complex reductive group has a compact real form.*

We turn next to uniqueness of the compact form. See [21, Theorem 3.1], and [14, Ch. XV] for a proof which handles one case overlooked in [21].

Theorem 3.8 (Cartan, Hochschild, Mostow [14, Ch. XV]**)** *Let σ be a compact real form of a complex reductive group G, and set $K = G^\sigma$. Let L be a compact subgroup of G. Then there exists $g \in G^0$ such that $gLg^{-1} \subset K$. All compact real forms of G are conjugate under G^0.*

Fix a compact real form K of G. The center $Z(G^0)$ of G^0 is a normal subgroup of G. It follows from the Cartan decomposition that $Z(G^0) = Z(K^0)A$ where $A = \exp(i\mathrm{Lie}(Z(K^0))) \subset \exp(\mathfrak{p})$ is a vector group (see [21, Lemma 2.4]). Since $G = KG^0$ we have (writing superscript for invariants): $Z(G^0)^K = Z(G^0)^G$, independent of the choice of K. Also $K/K^0 \simeq G/G^0$ acts on $Z(G^0)$, normalizing A, and

$$Z(G) \cap G^0 = Z(G^0)^{G/G^0} = Z(K^0)^{K/K^0} A^{G/G^0} = (Z(K) \cap K^0) A^{G/G^0}. \tag{3.9}$$

Lemma 3.10 *Suppose K is a compact real form of G. Then the Cartan decomposition of $\mathrm{Norm}_G(K)$ is $\mathrm{Norm}_G(K) = KA^{G/G^0}$.*

Proof. Since $G = K\exp(\mathfrak{p})$, it suffices to show that $\mathrm{Norm}_G(K) \cap \exp(\mathfrak{p}) = A^{G/G^0}$. Let $X \in \mathfrak{p}$ be such that $\exp(X)$ normalizes K. For $k \in K$, there exists $k' \in K$ such that $\exp(X)k\exp(-X) = k'$. This can be rewritten as

$$k\exp(-X) = k'\exp(-\mathrm{Ad}(k')^{-1}(X))$$

so by uniqueness of the Cartan decomposition, $k' = k$ and $\mathrm{Ad}(k)(X) = X$, so X is invariant under K. The fact that X is invariant under K^0 means that $X \in \mathrm{Lie}(A)$, and since K meets every connected component of G, $X \in \mathrm{Lie}(A)^{G/G^0}$.
$\square$

Lemma 3.11 *Let σ be a compact real form of a real reductive group G. Let H be a σ-stable algebraic subgroup of G. Then H is reductive and $\sigma|_H$ is a compact real form of H.*

Proof. The algebraic group H is clearly linear. The unipotent radical U of H is stable under σ and connected, and so U^σ is Zariski-dense in U. Any unipotent element of G^σ is trivial, thus $U = \{1\}$ and H is reductive. Clearly H^σ is compact, and we are left to show that H^σ meets every connected component of H. For $h \in H$ write $h = k\exp(X)$ where $k \in G^\sigma$ and $X \in \mathfrak{p}$. Then $\exp(2X) = \sigma(h)^{-1}h \in H$, and thus $\exp(2nX) \in H$ for all $n \in \mathbb{Z}$. Since H is Zariski-closed in G this implies $\exp(tX) \in H$ for all $t \in \mathbb{C}$, which implies $X \in \mathfrak{h}^{-\sigma}$, $k \in H^\sigma$, and H^σ meets every component of H. $\square$

This argument is classical.

Definition 3.12 *Suppose σ is a real form of a complex reductive group G. A* Cartan involution *for σ is a holomorphic involutive automorphism θ of G, commuting with σ, such that $\theta\sigma$ is a compact real form of G.*

By Lemma 3.1 applied to σ and $\theta\sigma$, any Cartan involution is algebraic. In fact a simple variant of the proof of Lemma 3.1 shows directly that any holomorphic automorphism of a complex reductive group is automatically algebraic.

Theorem 3.13 *Let G be a complex reductive group, possibly disconnected.*

(1) Suppose σ is a real form of G.

- *(a) There exists a Cartan involution θ for σ, unique up to conjugation by an inner automorphism from $(G^\sigma)^0$.*
- *(b) Suppose (H, θ_H) is a pair consisting of a σ-stable reductive subgroup of G and a Cartan involution θ_H for $\sigma|_H$. Then there exists a Cartan involution θ for G such that $\theta(H) = H$ and $\theta|_H = \theta_H$.*

(2) Suppose θ is a holomorphic, involutive automorphism of G.

- *(a) There is a real form σ of G such that θ is a Cartan involution for σ, unique up to conjugation by an inner automorphism from $(G^\theta)^0$.*
- *(b) Suppose (H, σ_H) is a pair consisting of a θ-stable reductive subgroup of G and a real form σ_H such that $\theta|_H$ is a Cartan involution for σ_H. Then there exists a real form σ of G such that $\sigma(H) = H$ and $\sigma|_H = \sigma_H$.*

For applications to the classification of real forms and to homogeneous spaces, the fact that the statement of Theorem 3.13 is symmetric in σ and θ is crucial.

We will deduce Theorem 3.13 from the next Lemma, whose proof is adapted from [21, Theorem 4.1].

Lemma 3.14 *Suppose τ is an involutive automorphism of G, either holomorphic or anti-holomorphic.*

(1) There exists a compact real form σ^c of G which commutes with τ.

(2) *Suppose H is a τ-stable reductive subgroup of G, σ_H^c is a compact real form of H, and $\tau|_H$ commutes with σ_H^c. Then we can find σ^c satisfying (1) so that σ^c restricted to H equals σ_H^c.*

Proof. Thanks to Theorem 3.7 choose any compact real form σ_1^c of G and set $K_1 = G^{\sigma_1^c}$, $\mathfrak{p}_1 = \mathrm{Lie}(G)^{-\sigma_1^c}$, and $P_1 = \exp(\mathfrak{p}_1)$. Then $\tau(K_1)$ is another compact real form of G, so by Theorem 3.8 there exists $g \in G^0$ so that

$$\text{(3.15)(a)} \qquad \tau(K_1) = gK_1g^{-1}.$$

Applying τ to both sides we see $\tau(g)g \in \mathrm{Norm}_G(K_1)$. By Lemma 3.10 we can write

$$\text{(3.15)(b)} \qquad \tau(g)g = ak \quad (a \in A^{G/G^0}, k \in K_1).$$

By (a) $g^{-1}\tau(K_1)g = K_1$, i.e. $\mathrm{int}(g^{-1}) \circ \tau$ stabilizes K_1. Since this isomorphism is holomorphic or antiholomorphic and $\mathfrak{p}_1 = i\mathrm{Lie}(K_1)$, this implies $g^{-1}\tau(P_1)g = P_1$. By the Cartan decomposition $G = K_1P_1$ we may assume $g \in P_1$, in which case $g^{-1}\tau(g)g \in P_1$. Plugging in (b) we conclude $g^{-1}ak \in P_1$, which by uniqueness of the Cartan decomposition implies $k = 1$, so

$$\text{(3.15)(c)} \qquad \tau(g)g \in A^{G/G^0}.$$

Set $a = \tau(g)g \in Z(G)$. Then $\tau(a) = g\tau(g) = gag^{-1} = a$. After replacing g with $ga^{-\frac{1}{2}}$ we may assume $\tau(g) = g^{-1}$ (we are writing $\frac{1}{2}$ for the unique square root in P_1). We observe that $g^{-1}\tau(g^{\frac{1}{2}})g$ is an element of P_1 and its square equals $g^{-1}\tau(g)g = g^{-1}$, therefore $\tau(g^{\frac{1}{2}}) = g^{-\frac{1}{2}}$.

Now let $\sigma^c = \mathrm{int}(g^{\frac{1}{2}}) \circ \sigma_1^c \circ \mathrm{int}(g^{-\frac{1}{2}})$, $K = G^{\sigma^c} = g^{\frac{1}{2}}K_1g^{-\frac{1}{2}}$, and $\mathfrak{p} = \mathrm{Lie}(G)^{-\sigma^c}$. Then

$$\tau(K) = \tau(g^{\frac{1}{2}})\tau(K_1)\tau(g^{-\frac{1}{2}}) = g^{-\frac{1}{2}}gK_1g^{-1}g^{\frac{1}{2}} = K.$$

This also implies $\tau(\mathfrak{p}) = \mathfrak{p}$, and τ commutes with σ^c, as one can check using the Cartan decomposition.

Now suppose we are given (H, σ_H^c) as in (2), and set $K_H = H^{\sigma_H^c}$. In the first step of the preceding argument choose σ_1^c so that $K_H \subset K_1$, using Theorem 3.8 (then $K_H = K_1 \cap H$ since K_H is a maximal compact subgroup of H). Suppose $h \in K_H$. Choosing $g \in P_1$ as above, recall $(\mathrm{int}(g^{-1}) \circ \tau)(K_1) = K_1$, so let $k = g^{-1}\tau(h)g \in K_1$. Since $\tau|_H$ commutes with σ_H^c, $\tau(K_H) = K_H \subset K_1$, so $\tau(h) \in K_1$. Write

$$\text{(3.15)(d)} \qquad kg^{-1} = \tau(h) \cdot \tau(h^{-1})g^{-1}\tau(h).$$

By uniqueness of the Cartan decomposition we conclude $g\tau(h) = \tau(h)g$ for all $h \in K_H$. Since τ is an automorphism of K_H we see $gh = hg$ for all $h \in K_H$. Since $\mathrm{int}(K_H) \subset \mathrm{int}(K_1)$ acts on P_1, this implies that $g^{\frac{1}{2}}h = hg^{\frac{1}{2}}$ for all $h \in K_H$ as well. Define σ^c, K and P as before. Then $K_H = K \cap H$ and $\sigma^c(h) = h$ for all $h \in K_H$. Now $(\sigma^c)^{-1} \circ \sigma_H^c : H \to G$ is a holomorphic automorphism which is the identity on K_H, thus it is the identity on H (recall that $\mathrm{Lie}(H) = \mathrm{Lie}(K_H) \oplus i\mathrm{Lie}(K_H)$ and that K_H meets every connected component of H).

$\square$

Proof of Theorem 3.13. For existence in (1)(a) apply Lemma 3.14 to $\tau = \sigma$ to construct a compact real form σ^c, commuting with σ, and set $\theta = \sigma\sigma^c$. For (1)(b) apply Lemma 3.14(2) with $\tau = \sigma$, $\sigma^c_H = \sigma|_H\theta_H$ to construct σ^c, commuting with σ, and let $\theta = \sigma\sigma^c$.

We now prove the uniqueness statement in (1)(a). Suppose θ, θ_1 commute with σ, and $\sigma^c = \sigma\theta$ and $\sigma^c_1 = \sigma\theta_1$ are compact real forms. By Theorem 3.8 there exists $g \in G^0$ so that

$$\sigma^c_1 = \mathrm{int}(g) \circ \sigma^c \circ \mathrm{int}(g^{-1}) = \mathrm{int}(g\sigma^c(g^{-1})) \circ \sigma^c.$$

Let $G = K\exp(\mathfrak{p})$ be the Cartan decomposition with respect to σ^c. Then we can take $g = \exp(X)$ for $X \in \mathfrak{p}$, so $g\sigma^c(g^{-1}) = \exp(2X)$. Since σ^c and σ^c_1 commute with σ, so does $\mathrm{int}(g\sigma^c(g^{-1})) = \mathrm{int}(\exp(2X))$, so by (3.9)

$$\exp(2\sigma(X))\exp(-2X) \in Z(G) \cap G^0 = (Z(K) \cap K^0)A^{G/G^0}.$$

Applying the Cartan decomposition for σ^c again we conclude

$$\exp(2\sigma(X))\exp(-2X) \in A^{G/G^0},$$

so $\sigma(X) - X \in \mathrm{Lie}(A^{G/G^0})$. We are free to multiply g by an element of $Z(G) \cap G^0$, which contains A^{G/G^0}. In particular we can replace X with $X + (\sigma(X) - X)/2 \in \mathfrak{p}^\sigma$. Then $g \in \exp(\mathfrak{p}^\sigma) \in (G^\sigma)^0$.

The proof of (2) is similar. We apply Lemma 3.14 with $\tau = \theta$. For existence in (2)(a) apply part (1) of the Lemma to construct σ^c, commuting with θ, and let $\sigma = \theta\sigma^c$. For (2)(b) apply part (2) of the Lemma with $\sigma^c_H = \sigma_H\theta|_H$ to construct σ^c, commuting with θ, and let $\sigma = \theta\sigma^c$. We omit the proof of the conjugacy statement, which is similar to case (1)(a). $\square$

Remark 3.16 *Let G be a complex reductive group, σ a real form and θ a Cartan involution for (G, σ). Then the images of G^σ, G^θ and $G^\sigma \cap G^\theta$ in G/G^0 coincide. In fact it is easy to check that the restriction of the Cartan decomposition for $(G, \sigma\theta)$ to G^σ (resp. G^θ) is a Cartan decomposition, showing that $G^\sigma \cap G^\theta$ intersects all connected components of G^σ (resp. G^θ).*

Let $\mathrm{Int}(G)$ be the group of inner automorphisms of G, $\mathrm{Aut}(G)$ the (holomorphic) automorphisms, and set $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Int}(G)$. Let $\mathrm{Int}^0(G)$ be the subgroup of $\mathrm{Int}(G)$ consisting of automorphisms induced by elements of G^0, so that $\mathrm{Int}^0(G) \simeq G^0/(Z(G) \cap G^0)$.

Corollary 3.17 *The correspondence between real forms and Cartan involutions induces a bijection between*

(3.18)(a) $\quad\{$*antiholomorphic involutive automorphisms of* $G\}/\mathrm{Int}^0(G)$

and

(3.18)(b) $\{$*holomorphic involutive automorphisms of* $G\}/\mathrm{Int}^0(G)$.

Both quotients are by the conjugation action of inner automorphisms coming from G^0. *The same statement holds with* $\mathrm{Int}^0(G)$ *replaced by any group* $\mathcal{A}$ *satisfying* $\mathrm{Int}^0(G) \subset \mathcal{A} \subset \mathrm{Aut}(G)$.

If $\mathrm{Int}^0(G)$ is replaced by $\mathrm{Int}(G) = G_{\mathrm{ad}}$ then (a) is the set of equivalence classes of real forms of G (Definition 3.2). We use this bijection to identify an equivalence class of real forms with an equivalence class of Cartan involutions as in (b).

4 Borel-Serre's Theorem

In this section only G denotes a *real* Lie group. Since it requires no extra effort we work in the following generality.

Recall we have a Cartan decomposition in the case that G is the group $H(\mathbb{C})$ of complex points of a reductive group H viewed as a real group (Definition 3.5): for any compact real form σ^c of H, we have $H = H^{\sigma^c}\exp(\mathrm{Lie}(H)^{-\sigma^c})$. Although we will not use this fact, it is easy to deduce that if σ is a real form of a complex reductive group H, then for any Cartan involution θ of (H,σ), the Lie group $H(\mathbb{R}) = H^\sigma$ has a Cartan decomposition $H(\mathbb{R}) = H(\mathbb{R})^\theta \exp(\mathrm{Lie}(H(\mathbb{R}))^{-\theta})$.

More general real Lie groups G admit a Cartan decomposition, including many non-linear ones (for example the *finite* covers of $\mathrm{SL}_2(\mathbb{R})$) or non-reductive ones (for example $G = H(\mathbb{R})$ where H is a real linear algebraic group). On the other hand the universal cover $\widetilde{G}$ of $\mathrm{SL}_2(\mathbb{R})$ has a decomposition $\widetilde{G} = L\exp(\mathfrak{p})$ where $L \simeq \mathbb{R}$ is the universal cover of the circle, hence noncompact. For a generalization of the Cartan decomposition to any real Lie group having finitely many connected components see [14, Ch. XV] or [21, Theorem 3.2].

Proposition 4.1 *Suppose* G *is a real Lie group admitting a Cartan decomposition* $(K,\mathfrak{p})$. *Let* τ *be an involutive automorphism of* G *which preserves* K *and* $\mathfrak{p}$. *Let* $Z_K = Z(G) \cap K$. *The inclusion map* $K \to G$ *induces an isomorphism*

(4.2) $$H^1(\tau, K; Z_K) \simeq H^1(\tau, G; Z_K)$$

which respects the maps to Z_K.

The proof is adapted from [8, Théorème 6.8] (see also [24, Section III.4.5]). This specializes to Borel-Serre's Theorem (see (4.6)).

Proof. It is enough to prove this when Z_K is replaced by $\{z\} \subset Z_K$ where z is any single element of Z_K. The left hand side of (4.2) is

(4.3)(a) $$\{k \in K \mid k\tau(k) = z\}/[k \sim tk\tau(t^{-1}) \quad (t \in K)]$$

and the right hand side is

(4.3)(b) $$\{g \in G \mid g\tau(g) = z\}/[g \sim tg\tau(t^{-1}) \quad (t \in G)].$$

Consider the map ϕ from (a) to (b) induced by inclusion.

We first show that ϕ is surjective. Suppose $g \in G$ satisfies $g\tau(g) = z$. Let $P = \exp(\mathfrak{p})$, and write $g = kp$ with $k \in K, p \in P$. Then $kp\tau(kp) = z$, which can be written

$$k\tau(k) \cdot \tau(k^{-1})p\tau(k) = z \cdot \tau(p^{-1}).$$

By uniqueness of the Cartan decomposition we conclude $k\tau(k) = z$ and $\tau(k^{-1})p\tau(k) = \tau(p^{-1})$. The latter condition is equivalent to $kpk^{-1} = \tau(p^{-1})$. The set of $p \in P$ satisfying this condition is the exponential of the subspace $\{Y \in \mathfrak{p} \mid \mathrm{Ad}(k)Y = -\tau(Y)\}$. Therefore $p = q^2$ for some $q \in P$ satisfying $kq = \tau(q^{-1})k$. Then $g = kq^2 = (kq)q = \tau(q^{-1})kq$. Therefore ϕ takes $cl(k)$ in (a) to $cl(g)$ in (b).

We now show that ϕ is injective. Suppose $k, k' \in K$, $k\tau(k) = k'\tau(k') = z$, and $k' = tk\tau(t^{-1})$ for some $t \in G$. Write $t^{-1} = xp$ with $x \in K, p \in P$. Then $k' = p^{-1}x^{-1}k\tau(x)\tau(p)$, i.e.

$$k' \cdot (k')^{-1}pk' = x^{-1}k\tau(x) \cdot \tau(p)$$

By uniqueness of the Cartan decomposition we conclude $k' = x^{-1}k\tau(x)$ with $x \in K$, i.e. k and k' are equivalent in (a). □

Corollary 4.4 *Suppose G is a real Lie group admitting a Cartan decomposition $(K, \mathfrak{p})$, and as before let $Z_K = Z(G) \cap K$. Let τ, μ be involutive automorphisms of G which preserve K and $\mathfrak{p}$, and assume that $\tau|_K = \mu|_K$. Then there are bijections of pointed sets*

$$H^1(\tau, G; Z_K) \simeq H^1(\tau|_K, K, Z_K) \simeq H^1(\mu, G; Z_K)$$

compatible with the maps to Z_K. In particular there is a canonical bijection of pointed sets

$$H^1(\tau, G) \simeq H^1(\mu, G). \tag{4.5}$$

Now let G be a complex reductive group, viewed as a real group. Recall (Section 3) G has a compact real form σ^c, and a Cartan decomposition $G = K\exp(\mathfrak{p})$. Hence Proposition 4.1 applies. Taking $\tau = \sigma^c$ and restricting to the fibres of $\{1\} \subset Z_K$ gives Borel-Serre's Theorem [8, Théorème 6.8], [24, Section III.4.5]

$$H^1(\sigma^c, K) \simeq H^1(\sigma^c, G). \tag{4.6}$$

This admits the following natural generalization to arbitrary real forms.

Corollary 4.7 *Suppose G is a complex, reductive algebraic group G, σ is a real form of G, and θ is a Cartan involution for σ. Let $\sigma^c = \sigma\theta$.*

There are canonical bijections of pointed sets

$$H^1(\theta, G; Z^{\sigma^c}) \simeq H^1(\theta, G^{\sigma^c}; Z^{\sigma^c}) = H^1(\sigma, G^{\sigma^c}; Z^{\sigma^c}) \simeq H^1(\sigma, G; Z^{\sigma^c}).$$

In particular there is a canonical bijection of pointed sets:

$$H^1(\theta, G) \simeq H^1(\sigma, G).$$

This follows from Corollary 4.4 for the Cartan decomposition of G induced by σ^c, using the fact that σ and θ agree on $K = G^{\sigma^c}$.

Example 4.8 Suppose G is connected and θ is the identity, so $G(\mathbb{R})$ is a connected compact Lie group. In this case $H^1(\theta, G)$ is the set of conjugacy classes of involutions in G, which is is in bijection with H_2/W, where H is a Cartan subgroup, H_2 is the group of involutions in H and W is the Weyl group.

On the other hand $H^1(\theta, G(\mathbb{R}))$ is the set of conjugacy classes of involutions in $G(\mathbb{R})$, which is $H(\mathbb{R})_2/W$. Since $H(\mathbb{R})$ is compact this is equal to H_2/W. So we recover [24, Theorem 6.1]: $H^1(\sigma, G) \simeq H^1(\theta, G(\mathbb{R})) = H(\mathbb{R})_2/W$.

Example 4.9 Consider the adjoint group $G = PSL(2, \mathbb{C})$. Let σ_1 be the obvious real form, so that $(G, \sigma_1) = PGL_2/\mathbb{R}$. Via the adjoint action on the Lie algebra which preserves the Killing form, the real algebraic group (G, σ_1) is also isomorphic to the special orthogonal group of a form with signature $(2, 1)$. Therefore $H^1(\sigma_1, G)$ parametrizes, by Galois descent, equivalence classes of non-degenerate quadratic forms on 3-dimensional real vector spaces with positive discriminant. Thus G admits two equivalence classes of real forms, represented by σ_1 as above and a compact form σ_2 such that $G^{\sigma_2} = SO(3)$. Since G is adjoint $|H^1(\sigma, G)| = 2$ for either real form.

Now consider $\tilde{G} = SL(2, \mathbb{C})$. It is the simply connected cover of G and so both σ_i's lift to real forms of $\tilde{G}$, and $\tilde{G}$ also admits two real forms up to equivalence: $(\tilde{G}, \sigma_1)$ is isomorphic to $SL_2/\mathbb{R}$ and $(\tilde{G}, \sigma_2)$ is isomorphic to a special unitary group with signature $(2, 0)$, so that $\tilde{G}^{\sigma_2} \simeq SU(2) \simeq Spin(3)$. From Example 4.8 $|H^1(\sigma_2, \tilde{G})| = 2$. This can also be seen as above by interpreting $H^1(\sigma_2, \tilde{G})$ as parametrizing isomorphism classes of non-degenerate hermitian forms in dimension 2 with positive discriminant.

On the other hand it is well known that $H^1(\sigma_1, \tilde{G}) = 1$, for example see [16, 29.4]. Thus in contrast to the adjoint case, although $(\tilde{G}, \sigma_1)$ and $(\tilde{G}, \sigma_2)$ are inner forms of each other, their cohomology is different. See Lemma 8.10.

5 Rational Orbits

We use the results of the previous section to study rational orbits of G-actions for real reductive groups.

Write

(5.1)(a) $$(G, \tau_G, X, \tau_X)$$

to indicate the following situation, which occurs repeatedly. First of all G is an abstract group equipped with an involutive automorphism τ_G, and X is a set equipped with an involutive automorphism τ_X. Furthermore there is a left action of $g : x \mapsto g \cdot x$ of G on X. We assume (τ_G, τ_X) are *compatible*:

(5.1)(b) $$\tau_X(g \cdot X) = \tau_G(g) \cdot \tau_X(x) \quad (g \in G, x \in X).$$

When X is a homogeneous space the following description of the set of orbits for the action of G^{τ_G} on X^{τ_X} is well-known.

Lemma 5.2 *In the setting of* (5.1) *suppose X is a homogenous space for G. Assume that $X^{\tau_X} \neq \emptyset$, choose $x \in X^{\tau_X}$ and denote by G^x the stabilizer of x. Then we have a bijection*

$$\begin{aligned} X^{\tau_X}/G^{\tau_G} &\to \ker\left(H^1(\tau_G, G^x) \to H^1(\tau_G, G)\right) \\ g \cdot x &\mapsto cl(g^{-1}\tau_G(g)) \end{aligned}$$

We will apply this with G a complex group, X a complex variety, and τ_G and τ_X each acting holomorphically (we will then use the notation σ instead of τ) or anti-holomorphically (we will the use the notation θ instead of τ).

If σ_G is a compact real form of G then X^{σ_X} is either empty of a homogeneous space for G^{σ_G}:

Lemma 5.3 *In the setting of* (5.1)*, suppose G is a complex reductive algebraic group, X is a homogeneous space for G, and σ_G is a compact real form of G. Let $K = G^{\sigma_G}$.*

(1) K acts transitively on X^{σ_X}.

(2) Suppose H is a σ_G-stable subgroup of G, and $H = G^x$ for some $x \in X$. Assume $X^{\sigma_X} \neq \emptyset$. Then $H = G^y$ for some $y \in X^{\sigma_X}$.

Proof.

For (1), if X^{σ_X} is empty there is nothing to prove, so choose $x \in X^{\sigma_X}$. By the previous lemma we have to show that

(a) $$\ker\left(H^1(\sigma_G, G^x) \to H^1(\sigma_G, G)\right)$$

is trivial. By Lemma 3.11 σ_G restricts to a compact real form of G^x, so Proposition 4.1 implies (a) is isomorphic to

(b) $$\ker\left(H^1(\sigma_G, (G^x)^{\sigma_G}) \to H^1(\sigma_G, G^{\sigma_G})\right)$$

which is clearly trivial, proving (1).

For (2) choose $x \in X^{\sigma_X}$. The set of subgroups H in (2) is identified with the set of σ_G-fixed elements of the homogeneous space $G/\mathrm{Norm}_G(G^x)$. By (1) G^{σ_G} acts transitively on this set. Thus for any such H there exists $g \in G^{\sigma_G}$ such that $H = gG^xg^{-1}$. Then $g \cdot x \in X^{\sigma_X}$ and $H = G^{g \cdot x}$. $\square$

We next consider homogeneous spaces for noncompact groups.

Proposition 5.4 *Suppose G is a complex, reductive algebraic group, possibly disconnected, acting transitively on a complex algebraic variety X. Suppose we are given:*

(1) a pair (σ_G, θ_G) consisting of a real form, and a corresponding Cartan involution, of G;

(2) a pair (σ_X, θ_X) of commuting involutions of X, with σ_X antiholomorphic and θ_X holomorphic.

Assume (σ_G, σ_X) are compatible, and so are (θ_G, θ_X) (see (5.1)(b)).

Assume $X^{\sigma_X} \cap X^{\theta_X} \neq \emptyset$. Then the two natural maps

$$X^{\sigma_X}/G^{\sigma_G} \leftarrow (X^{\sigma_X} \cap X^{\theta_X})/(G^{\sigma_G} \cap G^{\theta_G}) \rightarrow X^{\theta_X}/G^{\theta_G}$$

are bijective.

Proof. Choose $x \in X^{\sigma_X} \cap X^{\theta_X}$. Lemma 5.2 applied to $(G, \sigma_G, X, \sigma_X)$ provides an identification

$$X^{\sigma_X}/G^{\sigma_G} \simeq \ker\left(H^1(\sigma_G, G^x) \rightarrow H^1(\sigma_G, G)\right)$$

Similarly, Lemma 5.2 applied to $(G, \theta_G, X, \theta_X)$ gives

$$X^{\theta_X}/G^{\theta_G} \simeq \ker\left(H^1(\theta_G, G^x) \rightarrow H^1(\theta_G, G)\right)$$

Let $\sigma_G^c = \sigma_G\theta_G$. By Lemma 5.3, $G^{\sigma_G^c}$ acts transitively on $X^{\sigma_X\theta_X}$, so that we can also apply Lemma 5.2 to $(G^{\sigma_G^c}, \sigma_G, X^{\sigma_X\theta_X}, \sigma_X)$:

$$(X^{\sigma_X} \cap X^{\theta_X})/(G^{\sigma_G} \cap G^{\theta_G}) \simeq \ker\left(H^1(\sigma_G, (G^x)^{\sigma_G^c}) \rightarrow H^1(\sigma_G, G^{\sigma_G^c})\right).$$

By Corollary 4.7 we have the following commutative diagram:

$$\begin{tikzcd}
H^1(\sigma_G, G^x) \arrow[d] & H^1(\sigma_G, (G^x)^{\sigma_G^c}) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[d] & H^1(\theta_G, G^x) \arrow[d] \\
H^1(\sigma_G, G) & H^1(\sigma_G, G^{\sigma_G^c}) \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & H^1(\theta_G, G)
\end{tikzcd}$$

Note that σ_G and θ_G coincide on $G^{\sigma_G^c}$ so in the middle term we can replace $H^1(\sigma_G, *)$ with $H^1(\theta_G, *)$. This gives the two bijections of the Proposition.

These bijections (which involve the choice of x) agree with those of the Proposition (which are canonical). This comes down to: if $g \in G^{\sigma_G^c}$ then $g^{-1}\sigma_G(g) = g^{-1}\theta_G(g)$. This completes the proof. □

Remark 5.5 In Proposition 5.4, the hypothesis $X^\sigma \cap X^\theta \neq \emptyset$ is necessary. Consider for example $G = X = \mathbb{C}^\times$, with G acting by multiplication, and $\sigma_G(z) = 1/\overline{z}, \sigma_X(z) = -1/\overline{z}, \theta_G(z) = \theta_X(z) = z$. Then $X^{\sigma_X} = \emptyset$ but $X^{\theta_X} = X$.

To apply the result it would be good to know that $X^{\sigma_X} \neq \emptyset$ or $X^{\theta_X} \neq \emptyset$ implies that $X^{\sigma_X} \cap X^{\theta_X} \neq \emptyset$. As the Remark shows, this isn't always the case, but it holds under a weak additional assumption.

Lemma 5.6 *In the setting of Proposition 5.4, assume that $X^{\sigma_X\theta_X} \neq \emptyset$. Then the following conditions are equivalent: $X^{\sigma_X} \neq \emptyset$, $X^{\theta_X} \neq \emptyset$, and $X^{\sigma_X} \cap X^{\theta_X} \neq \emptyset$.*

Proof. If $x \in X^{\sigma_X\theta_X}$ then G^x is σ^c-stable so G^x is reductive by Lemma 3.11. Since these groups are all conjugate this holds for all $x \in X$.

Let us now show that if $X^{\sigma_X} \neq \emptyset$ then $X^{\sigma_X} \cap X^{\theta_X} \neq \emptyset$. Fix $x \in X^{\sigma_X}$. Then G^x is a reductive group stable under σ_G, and thus it admits a Cartan involution θ'_x. By Theorem 3.13 it extends to a Cartan involution θ'_G of G, and there exists $g \in G^{\sigma_G}$ such that $\theta_G = \mathrm{int}(g) \circ \theta'_G \circ \mathrm{int}(g^{-1})$, so that $g \cdot x \in X^{\sigma_X}$ has the property that $G^{g\cdot x}$ is θ_G-stable. In other words, after replacing x by $g \cdot x$, we may assume G^x is σ^c-stable, and $\sigma_c|_{G^x}$ is a compact real form of G^x. By Lemma 5.3 we can find $y \in X^{\sigma_X\theta_X}$ so that $G^y = G^x$.

Let $N_y = \mathrm{Norm}_G(G^y)$, and set $M_y = N_y/G^y$, By [25, Proposition 5.5.10] M_y is a linear algebraic group. Both N_y and M_y are σ^c-stable, and therefore reductive by Lemma 3.11 again.

Since $G^{\sigma_X(y)} = \sigma_G(G^y) = G^y$ there exists unique $m \in M_y$ such that

$$\sigma_X(y) = m \cdot y. \tag{5.7)(a}$$

Similarly since $G^x = G^y$ there exists unique $n \in M_y$ such that

$$x = n \cdot y \tag{5.7)(b}$$

Since $\sigma_X\theta_X$ fixes both y and $\sigma_X(y)$, applying this to both sides of (a) gives $\sigma_X(y) = \sigma^c(m)\cdot y$, and comparing this with (a) gives $m \in (M_y)^{\sigma^c}$. On the other hand applying σ_X to both sides of (a) gives $y = \sigma_G(m) \cdot \sigma_X(y) = \sigma_G(m)m \cdot y$, so $\sigma_G(m)m = 1$. Finally apply σ_X to both sides of (b) to give $\sigma_X(x) = \sigma_G(n) \cdot \sigma_X(y)$. Using $\sigma_X(x) = x$ and (a) gives $x = \sigma_G(n)m \cdot y$, and comparing this with (b) gives $\sigma_G(n)^{-1}n = m$.

These three facts imply that m defines an element of

$$\ker\Big(H^1(\sigma_G, (M_y)^{\sigma^c}) \to H^1(\sigma_G, M_y)\Big).$$

By Corollary 4.7 this kernel is trivial, so there exists $u \in (M_y)^{\sigma^c}$ such that $m = \sigma_G(u)^{-1}u$. Then $u \cdot y \in X^{\sigma_X\theta_X} \cap X^{\sigma_X} = X^{\sigma_X} \cap X^{\theta_X}$.

A similar argument, substituting θ for σ, shows that $X^{\theta_X} \neq \emptyset$ implies that $X^{\sigma_X} \cap X^{\theta_X} \neq \emptyset$. $\square$

We can now formulate our result in its most useful form.

Theorem 5.8 *Suppose G is a complex, reductive algebraic group, possibly disconnected, and X is a complex algebraic variety, equipped with an action of G. Suppose we are given:*

(1) a pair (σ_G, θ_G) consisting of a real form and a corresponding Cartan involution of G.

(2) a pair (σ_X, θ_X) of commuting involutions, with σ_X antiholomorphic and θ_X holomorphic.

Assume (σ_G, σ_X) are compatible, as are (θ_G, θ_X) (5.1)(b).

Assume that for all $x \in X^{\sigma_X} \cup X^{\theta_X}$ the G-orbit of x intersects $X^{\sigma_X \theta_X}$. Then the two natural maps

$$X^{\sigma_X}/G^{\sigma_G} \leftarrow (X^{\sigma_X} \cap X^{\theta_X})/(G^{\sigma_G} \cap G^{\theta_G}) \to X^{\theta_X}/G^{\theta_G}$$

are bijective.

Proof. It is enough to prove this with X replaced by the G-orbit $G \cdot x$ of any $x \in X^{\sigma_X} \cup X^{\theta_X}$. By Lemma 5.6 we can apply Proposition 5.4 to $G \cdot x$, which gives the conclusion. $\square$

6 Applications

Throughout this section we fix a complex reductive group G, a real form σ of G, and a corresponding Cartan involution θ. Set $G(\mathbb{R}) = G^\sigma$ and $K = G^\theta$.

6.1 Kostant-Sekiguchi correspondence

Let $\mathfrak{g} = \mathrm{Lie}(G)$. We say that $x \in \mathfrak{g}$ is nilpotent if $x \in [\mathfrak{g}, \mathfrak{g}]$ and $\mathrm{ad}\, x$ is nilpotent. The Jacobson-Morozov theorem (see [13, ch. VIII, §11]) gives a bijection between the nilpotent orbits of G on $\mathfrak{g}$ and G-conjugacy classes of homomorphisms from $\mathfrak{sl}(2,\mathbb{C})$ to $\mathfrak{g}$:

(6.1.1)(a) $$\{\phi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}\}/G.$$

Let $\mathfrak{g}_0 = \mathrm{Lie}(G(\mathbb{R})) = \mathfrak{g}^\sigma$. Then the same result applies to $G(\mathbb{R})$, and gives a bijection between the $G(\mathbb{R})$ conjugacy classes of nilpotent elements of $\mathfrak{g}_0$ and

(6.1.1)(b) $$\{\phi : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{g}_0\}/G(\mathbb{R}).$$

Equivalently if σ_0 denotes complex conjugation on $\mathfrak{sl}(2,\mathbb{C})$ with respect to $\mathfrak{sl}(2,\mathbb{R})$, then (b) is naturally in bijection with

(6.1.1)(c) $$\{\phi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g} \mid \phi(\sigma_0 X) = \sigma(\phi(X))\}/G(\mathbb{R}).$$

Now write $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k} = \mathfrak{g}^\theta = \mathrm{Lie}(K)$ and $\mathfrak{p} = \mathfrak{g}^{-\theta}$. For $X \in \mathfrak{sl}(2,\mathbb{C})$ define $\theta_0(X) = -{}^tX$; this is a Cartan involution for σ_0. Kostant and Rallis [17, Prop. 4 on p. 767] showed that the nilpotent K-orbits on $\mathfrak{p}$ are in bijection with

(6.1.1)(d) $$\{\phi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g} \mid \phi(\theta_0(X)) = \theta(\phi(X))\}/K.$$

To be precise, in [17] this bijection is stated for K^0-orbits, as well as for K-orbits under the assumption that G connected and adjoint, but the general case follows in the same way.

The Kostant-Sekiguchi correspondence is a bijection between the nilpotent orbits of $G(\mathbb{R})$ on $\mathfrak{g}_0$ and the nilpotent K-orbits on $\mathfrak{p}$ [23].

Let X be the set of morphisms $\mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}$. This has a natural structure of complex algebraic variety. Define an antiholomorphic involution σ_X of X by

(6.1.2)(a) $$\sigma_X(\psi)(A) = \sigma(\psi(\sigma_0(A))) \quad (A \in \mathfrak{sl}(2,\mathbb{C}), \psi \in X).$$

Also define a holomorphic involution θ_X by

(6.1.3) $$\theta_X(\psi)(A) = \theta(\psi(\theta_0(A))) \quad (A \in \mathfrak{sl}(2,\mathbb{C}), \psi \in X).$$

It is straightforward to check that (σ_G, σ_X) and (θ_G, θ_X) are compatible.

Lemma 6.1.4 *Every orbit of G on X contains a $\sigma_X\theta_X$-invariant point. In particular, $\sigma_X\theta_X$ acts trivially on X/G, and an orbit of G on X is σ_X-stable if and only if it is θ_X-stable.*

Proof. We need to show that for any morphism $\phi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}$, there exists $g \in G$ such that the morphism $\mathrm{Ad}(g)\circ\phi$ is $\sigma\theta$-equivariant. Any such ϕ integrates to an algebraic morphism $\psi : \mathrm{SL}_2(\mathbb{C}) \to G^0$. Let $SU(2) = SL(2,\mathbb{C})^{\sigma_0\theta_0}$, with Lie algebra $\mathfrak{su}(2)$. Since $SU(2)$ is compact, so is its image in G^0, so by Theorem 3.8 there exists $g \in G^0$ such that $g\psi(\mathrm{SU}(2))g^{-1} \subset (G^0)^{\sigma\theta}$. Since $\mathfrak{sl}(2,\mathbb{C}) = \mathfrak{su}(2) \otimes_{\mathbb{R}} \mathbb{C}$ this implies that $\mathrm{Ad}(g)\cdot\phi$ is $\sigma\theta$-equivariant. $\square$

The Kostant-Sekiguchi correspondence is now an immediate consequence of Theorem 5.8.

Proposition 6.1.5 *For any nilpotent orbit $\mathcal{O}$ of G on $\mathfrak{g}$, there is a canonical bijection between $(\mathcal{O} \cap \mathfrak{g}_0)/G(\mathbb{R})$ and $(\mathcal{O} \cap \mathfrak{p})/K$.*

Proof. Let $\phi : \mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}$ be a morphism corresponding to an element of $\mathcal{O}$ as in (a). Let $Y \subset X$ be the G-orbit of ϕ, which only depends on $\mathcal{O}$ and not on the choice of a particular morphism. By Lemma 6.1.4, Y is σ_X-stable if and only if it is θ_X-stable. If it it not the case, both quotient sets are empty.

If it is the case we can apply Theorem 5.8 to Y, and by the Jacobson-Morozov theorem over $\mathbb{R}$ and the result of Kostant and Rallis recalled above, we obtain:

$$(\mathcal{O} \cap \mathfrak{g}_0)/G(\mathbb{R}) \simeq X^{\sigma_X}/G^{\sigma} \simeq X^{\theta_X}/G^{\theta} \simeq (\mathcal{O} \cap \mathfrak{p})/K.$$

$\square$

Remark 6.1.6 The set of orbits $(X^{\sigma_X} \cap X^{\theta_X})/(G^{\sigma} \cap G^{\theta})$ that appears as a middle term in Theorem 5.8, that is the set of $K(\mathbb{R})$-conjugacy classes of morphisms $\mathfrak{sl}(2,\mathbb{C}) \to \mathfrak{g}$ equivariant under σ and θ, does *not* have an obvious link to nilpotent orbits, since $\mathfrak{p}_0$ has no non-zero nilpotent elements.

6.2 Matsuki Duality

Matsuki duality is a bijection between the $G(\mathbb{R})$ and K orbits on the space $\mathcal{B}$ of Borel subgroups of G^0 [20].

Unlike in the case of Kostant-Sekiguchi duality, $G(\mathbb{R})$ and K are acting on the same space $\mathcal{B}$. So to derive this from Theorem 5.8 we need to find (X, σ_X, θ_X) so that $X^{\sigma_X} \simeq X^{\theta_X} \simeq \mathcal{B}$. This holds if we take $X = \mathcal{B} \times \mathcal{B}$, and define $\sigma_X(B_1, B_2) = (\sigma(B_2), \sigma(B_1))$, $\theta_X(B_1, B_2) = (\theta(B_2), \theta(B_1))$. However with this definition the condition $X^{\sigma_X} \cap X^{\theta_X} \neq \emptyset$ of Theorem 5.8 does not hold. Also note that the stabilizer of a point in $\mathcal{B}$ is the intersection of two Borel subgroups, which is typically not reductive. Instead we use a variant of X.

Write $\sigma_G = \sigma, \theta_G = \theta$.

Definition 6.2.1 *Let*

(6.2.2) $X = \{(B_1, B_2, T) \mid B_1, B_2 \in \mathcal{B}, T \subset B_1 \cap B_2 \textit{ is a maximal torus of } G^0\}$

Let G act on X by conjugation on each factor. Define involutive automorphisms σ_X and θ_X of X as follows:

$$\sigma_X(B_1, B_2, T) = (\sigma_G(B_2\text{-}opp), \sigma_G(B_1\text{-}opp), \sigma_G(T)) \tag{6.2.3}$$

where -opp denotes the opposite Borel with respect to T, and

$$\theta_X(B_1, B_2, T) = (\theta_G(B_2), \theta_G(B_1), \theta_G(T)). \tag{6.2.4}$$

Thanks to the Bruhat decomposition [6, §14.12], for any $(B_1, B_2) \in \mathcal{B} \times \mathcal{B}$ the algebraic subgroup $B_1 \cap B_2$ of G^0 is connected and solvable and contains a maximal torus of G^0. In particular the natural map $X \to \mathcal{B} \times \mathcal{B}$ is surjective.

Lemma 6.2.5 *The conditions of Theorem 5.8 hold.*

Proof. The fact that σ_X, θ_X commute, and the facts that (σ_G, σ_X) and (θ_G, θ_X) are compatible is immediate. Let us check that each G-orbit in X contains a $\sigma_X\theta_X$-fixed point. Let $(B_1, B_2, T) \in X$. Since the connected real reductive group $(G^0, \sigma_G\theta_G)$ has a maximal torus defined over $\mathbb{R}$ [6, Theorem 18.2], up to conjugating by an element of G^0 we can assume that T is $\sigma_G\theta_G$-stable. Since $(T, \sigma_G\theta_G)$ is anisotropic we have $\sigma_G\theta_G(B_i) = B_i$-opp for $i \in \{1, 2\}$, and (B_1, B_2, T) is automatically fixed by $\sigma_X\theta_X$. □

Theorem 5.8 now applies to give a bijection

$$X^{\sigma_X}/G(\mathbb{R}) \longleftrightarrow X^{\theta_X}/K. \tag{6.2.6}$$

Lemma 6.2.7 *Consider the projection p on the first factor, taking X to $\mathcal{B}$.*

(1) p restricted to X^{σ_X} is equivariant with respect to $G(\mathbb{R})$ and induces a bijection $X^{\sigma_X}/G(\mathbb{R}) \simeq \mathcal{B}/G(\mathbb{R})$.

(2) p restricted to X^{θ_X} is equivariant with respect to K and induces a bijection $X^{\theta_X}/K \simeq \mathcal{B}/K$.

Proof. The fact that p is G-equivariant, and $p|_{X^{\sigma_X}}$ is $G(\mathbb{R})$-equivariant, are immediate. Let B be a Borel subgroup of G^0. Then $B \cap \sigma_G(B)$ is an algebraic subgroup of G^0 defined over $\mathbb{R}$, and so it contains a maximal torus T which is defined over $\mathbb{R}$. The Bruhat decomposition implies that T is also a maximal torus of G^0. This shows that $B \in p(X^{\sigma_X})$.

Moreover the unipotent radical U of B acts transitively on the set of maximal tori of B [6, Theorem 10.6], and since G is reductive this action is also free. Therefore U^{σ_G} acts simply transitively on the set of σ_G-stable maximal tori in B. This implies that p induces a bijection $X^{\sigma_X}/G(\mathbb{R}) \simeq \mathcal{B}/G(\mathbb{R})$.

The proof of (2) is similar, except for the fact that $B \cap \theta_G(B)$ contains a maximal torus which is θ_G-stable, which follows from [26, 7.6] applied to θ_G acting on $B \cap \theta_G(B)$. ☐

Together with (6.2.6) this proves:

Proposition 6.2.8 *There is a canonical bijection $\mathcal{B}/G(\mathbb{R}) \leftrightarrow \mathcal{B}/K$.*

6.3 Weyl groups and conjugacy of Cartan subgroups

We next give short proofs of two well-known facts about Weyl groups and conjugacy of Cartan subgroups.

Let X be the set of Cartan subgroups (i.e. maximal tori) of G^0. This is a homogeneous space for the conjugation action of G, with σ_X, θ_X coming from σ and θ. It is well-known [6, Theorem 18.2] that G^0 has a σ-stable Cartan subgroup, that is $X^{\sigma_X} \neq \emptyset$. This also applies to G^0 equipped with its real form $\sigma\theta$, so that $X^{\sigma_X\theta_X} \neq \emptyset$.

Matsuki's result on Cartan subgroups ([20],[5, Proposition 6.18]) now follows from Theorem 5.8.

Proposition 6.3.1 *There are canonical bijections between*

- *$G(\mathbb{R})$-conjugacy classes of σ-stable Cartan subgroups of G^0,*
- *$K(\mathbb{R})$-conjugacy classes of σ- and θ-stable Cartan subgroups of G^0,*
- *K-conjugacy classes of θ-stable Cartan subgroups of G^0.*

In particular we recover the fact that G admits a θ-stable Cartan subgroup H in every $G(\mathbb{R})$-conjugacy class of σ-stable Cartan subgroups.

Next, we recover the following description of the *real* or *rational* Weyl group of H. See also [30, Proposition 1.4.2.1], [27, Definition 0.2.6].

Proposition 6.3.2 *Let H be a Cartan subgroup of G^0 which is stable under both σ and θ. Then the two natural morphisms*

$$\mathrm{Norm}_{G(\mathbb{R})}(H(\mathbb{R}))/H(\mathbb{R}) \leftarrow \mathrm{Norm}_{K(\mathbb{R})}(H(\mathbb{R}))/H(\mathbb{R})^{\theta} \rightarrow \mathrm{Norm}_K(H)/(H \cap K)$$

are isomorphisms.

Proof. We want to apply Theorem 5.8 with $G = H$ and $X = N = \mathrm{Norm}_G(H)$, with H acting by multiplication, and σ and θ acting naturally on N and $W = N/H$. The three quotients in the conclusion of the Theorem are precisely the three quotients appearing in the Proposition.

The compatibility conditions of the Theorem are clear. For the final condition, by Lemma 3.11, $N^{\sigma\theta}$ meets every connected component of N (this also follows from Remark 3.16). This says that every H-orbit on N contains a $\sigma\theta$-fixed point, so the final condition holds.

□

Remark 6.3.3 *In the setting of Proposition 6.3.2, if A is a distinguished subgroup of N containing H which is σ-stable (equivalently, θ-stable), then the conclusion also holds with H replaced by A, with the same proof.*

7 Relation with Cohomology of Cartan subgroups

In this section we assume G is a connected complex reductive group. Suppose σ is a real form of G, and θ is a Cartan involution for σ.

We say a σ-stable Cartan subgroup H_f of G is fundamental if $H_f(\mathbb{R})$ is of minimal split rank. Borovoi computes $H^1(\sigma, G)$ in terms of $H^1(\sigma, H_f)$ as follows. Before stating his result we make a few remarks about Weyl groups.

Lemma 7.1 *Suppose H is a σ-stable Cartan subgroup. There is an action of W^σ on $H^1(\sigma, H)$ defined as follows. Suppose $w \in W^\sigma$ and $h \in H^{-\sigma}$. Choose $n \in N$ mapping to w. Then the action of w on $H^1(\sigma, H)$ is $w : cl(h) \to cl(nh\sigma(n^{-1}))$; this is well defined, independent of the choices involved.*

The image of $H^1(\sigma, H)$ in $H^1(\sigma, N)$ is isomorphic to $H^1(\sigma, H)/W^\sigma$.

This is immediate. See [24, I.5.5, Corollary 1].

Suppose a Cartan subgroup H is σ-stable. Then σ acts on the roots of H in G. We say a root α of H in G is *imaginary*, *real*, or *complex* if $\sigma(\alpha) = -\alpha$, $\sigma(\alpha) = \alpha$, or $\sigma(\alpha) \neq \pm\alpha$, respectively. The set of imaginary roots is a root system. Let W_i denote its Weyl group.

Lemma 7.2 $H^1(\sigma, H)/W^\sigma = H^1(\sigma, H)/W_i$.

Proof. Let W_r (respectively W_i) be the Weyl group of the set of real (resp. imaginary) roots. Let Δ^C be the root system of [28, Proposition 3.12], and let W^C be its Weyl group. Then σ acts on W^C and defines a complex root datum.

For example suppose $G = G_1 \times G_1$ and $\sigma(g, h) = (\overline{h}, \overline{g})$, so that $G^\sigma \simeq G(\mathbb{C})$ (viewed as a real group by restriction of scalars). Then $W_i = W_r = 0$ and $W = W^C = W_1 \times W_1$, and $(W^C)^\sigma \simeq W_1$.

Now $(W^C)^\sigma$ acts naturally on W_i and W_r, and there is an isomorphism ([28, Proposition 3.12])

$$W^\sigma = (W_C)^\sigma \ltimes [W_i \times W_r].$$

By a calculation in $SL(2,\mathbb{R})$ it is easy to see that s_α (for α any real root) acts trivially on $H^1(\sigma, H)$, hence all of W_r acts trivially. The same holds for $(W_C)^\sigma$, by a calculation in $SL(2,\mathbb{C})$. See [4, Proposition 12.16]. □

Proposition 7.3 (Borovoi [9]**,** [10]**)** *Suppose H_f is a fundamental σ-stable Cartan subgroup. The natural map $H^1(\sigma, H_f) \to H^1(\sigma, G)$ induces an isomorphism $H^1(\sigma, H_f)/W_i \simeq H^1(\sigma, G)$.*

The Theorem in [9] is stated in terms of W^σ, so we have used Lemma 7.2 to replace this with W_i.

Proposition 7.4 *There is a canonical bijection of pointed sets $\phi : H^1(\theta, G) \simeq H^1(\theta, H_f)/W_i$ making the following diagram commute:*

$$\begin{tikzcd} H^1(\sigma, G) \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & H^1(\sigma, H_f)/W_i \arrow[d, "\simeq"] \\ H^1(\theta, G) \arrow[r, "\phi", "\simeq"'] & H^1(\theta, H_f)/W_i \end{tikzcd}$$

The top arrow is Borovoi's result and the two vertical arrows are from Theorem 1.1 applied to G and H, respectively.

This is immediate.

Remark 7.5 In an earlier version of this paper we proved the isomorphism $H^1(\sigma, G) \simeq H^1(\theta, G)$ using this diagram. It is simpler to prove this isomorphism directly as we have done in Section 4 and deduce this as a consequence.

For later use we note that, in the unequal rank case (see Proposition 8.16 for the definition of "being of equal rank"), the cohomology is captured by a proper subgroup.

Suppose H is a θ-stable Cartan subgroup. Then $H = TA$ where T and A are connected complex tori, T is the identity component of H^θ, and A is the identity component of $H^{-\theta}$.

Corollary 7.6 *Suppose H_f is a σ and θ-stable fundamental Cartan subgroup. Let A_f be the identity component of $H_f^{-\theta}$, and let $M_f = \mathrm{Cent}_G(A_f)$. Then*

$$H^1(\sigma, G) \simeq H^1(\sigma, M_f) \simeq H^1(\theta, M_f) \simeq H^1(\theta, G).$$

Note that $A_f \subset Z \Leftrightarrow M_f = G \Leftrightarrow$ the derived group of G is of equal rank.

This follows from Proposition 7.4, and the fact that the imaginary Weyl groups of H_f in G and M_f are the same.

8 Strong real forms

We continue to assume G is a connected complex reductive group.

Lemma 8.1 *Fix a real form σ of G. The set of equivalence classes of real forms in the inner class of σ is parametrized by $H^1(\sigma, G_{\mathrm{ad}})$.*

Explicitly the map is $cl(h) \mapsto [\mathrm{int}(h) \circ \sigma]$ where $h\sigma(h) = 1$.

Recall (Definition 2.5) that we have a well-defined pointed set $H^1([\sigma], G_{\mathrm{ad}}) = H^1(\sigma, G_{\mathrm{ad}})$.

Remark 8.2 Our definition of equivalence of real forms (Definition 3.2) is by conjugation by an inner automorphism of G. The standard definition, for example see [24, III.1], allows conjugation by $\mathrm{Aut}(G)$. With the standard definition Lemma 8.1 would hold with $H^1(\sigma, G_{\mathrm{ad}})$ replaced by the image of the map to $H^1(\sigma, \mathrm{Aut}(G))$.

For example suppose $G = \mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$. In the inner class of the split real form of G, there are four equivalence classes of real forms according to our definition: split, compact, split $\times$ compact and compact $\times$ split. If one allows conjugation by outer automorphisms there are only three real forms, since the last two are equivalent.

For simple groups these two notions of equivalence agree except in type D_{2n}. See [4, Section 3], [2, Example 3.3] and Section 10.3.

We now recall how inner classes are parametrized, and make a natural choice of base point in each inner class.

Recall that if $T \subset B$ are Cartan and Borel subgroups of G, then the corresponding based root datum is $(X^*(T), \Delta, X_*(T), \Delta^\vee)$, where $X^*(T)$ (resp. $X_*(T)$) is the group of characters (resp. cocharacters) of T and Δ (resp. $\Delta^\vee$) is the basis of the set of roots $\Phi(G,T)$ (resp. coroots $\Phi^\vee(G,T)$) corresponding to B. Furthermore *the* based root datum $\phi_0(G)$ is the (projective or injective) limit over all such pairs $T \subset B$. It gives rise to a short exact sequence

$$1 \to \mathrm{Int}(G) \to \mathrm{Aut}(G) \to \mathrm{Aut}(\phi_0(G)) \to 1$$

identifying $\mathrm{Out}(G)$ with $\mathrm{Aut}(\phi_0(G))$.

The exact sequence splits. A splitting is obtained by choosing a *pinning* for G, that is a triple $\mathcal{P} = (B, T, \{X_\alpha\}_{\alpha\in\Delta})$ where $T \subset B$ are Cartan and Borel subgroups of G, respectively, and for α a simple root, X_α is a basis vector for the Lie algebra of the α-root space of T in $\mathrm{Lie}(B)$. Then G acts transitively on the set of pinnings by conjugation, the stabilizer of any pinning is $Z(G)$, and the subgroup of $\mathrm{Aut}(G)$ preserving $\mathcal{P}$ is isomorphic to $\mathrm{Out}(G)$.

If σ is a real form of G and θ is a Cartan involution for (G, σ), then both σ and θ naturally act on $\phi_0(G)$ (see [7] for the case of σ), giving rise to involutions $\overline{\sigma}, \overline{\theta} \in \mathrm{Aut}(\phi_0(G))$, i.e. of elements of the set $\mathrm{Out}(G)_2$ of involutions in $\mathrm{Out}(G)$. They are related by $\overline{\sigma}\overline{\theta} = -w_0$, where w_0 is the longest element of the Weyl group of $\phi_0(G)$ and -1 is the inversion automorphism of T. Note that w_0 is invariant under $\mathrm{Aut}(\phi_0(G))$, and so $\iota := -w_0$ is a central involution in $\mathrm{Out}(G)$.

If σ and σ' are real forms of G, they are inner to each other if and only if $\overline{\sigma} = \overline{\sigma'}$ in $\mathrm{Out}(G)$. Of course this is equivalent to $\overline{\theta} = \overline{\theta'}$, where θ (resp. θ') is a Cartan involution for (G,σ) (resp. (G,σ')). In other words, inner classes of real forms of G are parametrized by involutions in $\mathrm{Out}(G)$, in two ways related by ι. Although in this section we will break the symmetry between Galois and Cartan and favor the former, it is more convenient to use $\overline{\theta}$ rather than $\overline{\sigma}$ since the definition of $\overline{\theta}$ is simpler. For this reason we will say that σ is in the inner class defined by $\delta \in \mathrm{Out}(G)_2$ when $\iota\delta = \overline{\sigma}$.

Recall that a real form σ for G is called *quasi-split* if it preserves a Borel subgroup of G. There is a unique equivalence class of quasi-split real forms in a given inner class, providing a base point. It is constructed explicitly as follows.

Definition 8.3 *Suppose $\delta \in \mathrm{Out}(G)_2$. For a pinning $\mathcal{P}$ of G, there is a unique real form $\sigma_{qs}(\delta,\mathcal{P})$ of G preserving $\mathcal{P}$ and such that $\overline{\sigma_{qs}(\delta,\mathcal{P})} = \iota\delta$, and it is clearly quasi-split.*

For $g \in G_{\mathrm{ad}}$ we have $\sigma_{qs}(\delta, \mathrm{int}(g)(\mathcal{P})) = \mathrm{int}(g) \circ \sigma_{qs}(\delta,\mathcal{P}) \circ \mathrm{int}(g)^{-1}$ and so the equivalence class $[\sigma_{qs}(\delta,\mathcal{P})]$ does not depend on the choice of $\mathcal{P}$ and is simply denoted $[\sigma_{qs}(\delta)]$. The corresponding equivalence class of Cartan involutions is denoted $[\theta_{qs}(\delta)]$.

If σ is any quasi-split real form of G, then there exists a pinning of G fixed by σ, so that the above construction yields all quasi-split real forms of G.

Let us reformulate Corollary 3.17 cohomologically. This is just a restatement of Corollary 4.7.

Lemma 8.4 *For $\delta \in \mathrm{Out}(G)_2$, there is a canonical bijection of pointed sets*

$$H^1([\sigma_{qs}(\delta)], G_{\mathrm{ad}}) \simeq H^1([\theta_{qs}(\delta)], G_{\mathrm{ad}}). \tag{8.5}$$

Let $Z = Z(G)$. The action of $\mathrm{Aut}(G)$ on Z factors to an action of $\mathrm{Out}(G)$ on Z. Let Z_{tor} be the subgroup of Z consisting of all elements of finite order.

Lemma 8.6 *Fix $\delta \in \mathrm{Out}(G)_2$, and suppose σ is a real form in the inner class defined by δ. Let θ be a Cartan involution for σ. Note that the actions of θ and δ on Z coincide.*

Then $Z_{\mathrm{tor}}^{\sigma} = Z_{\mathrm{tor}}^{\theta}$ and there is a canonical isomorphism

$$Z^{\sigma}/(1+\sigma)Z \simeq Z^{\delta}/(1+\delta)Z.$$

Proof. The closure $\overline{Z_{\mathrm{tor}}}$ of Z_{tor} is compact, so by Theorem 3.8 Z_{tor} is a subgroup of every compact real form of G. Therefore $\sigma^c = \theta\sigma$ acts trivially on Z_{tor}, i.e. θ, σ and δ all have the same action on Z_{tor}. Observe that Z is a quotient of $Z^0 \times Z(G_{\mathrm{der}})$ where the second factor is finite, so that Z_{tor} surjects to Z/Z^0. Thus multiplication by $2 = \mathrm{card}(\mathbb{Z}/2\mathbb{Z})$ is an automorphism of Z/Z_{tor}, which implies that this module is cohomologically trivial for both actions, and so

$Z_{\rm tor} \hookrightarrow Z$ induces isomorphisms in Tate cohomology in all degrees. We can conclude $Z^\sigma/(1+\sigma)Z \simeq Z^\sigma_{\rm tor}/(1+\sigma)Z_{\rm tor} \simeq Z^\delta_{\rm tor}/(1+\delta)Z_{\rm tor} \simeq Z^\delta/(1+\delta)Z$. □

Definition 8.7 *Fix $\delta \in \mathrm{Out}(G)_2$ and a real form σ in the inner class defined by δ. Identify $[\sigma]$ with a class in $H^1([\sigma_{qs}(\delta)], G_{\rm ad})$, and define the central invariant*

$$\mathrm{inv}([\sigma]) \in Z^\delta/(1+\delta)Z \tag{8.8}$$

by the composition of maps:

$$H^1([\sigma_{qs}(\delta)], G_{\rm ad}) \to H^2(\sigma_{qs}(\delta), Z) \xrightarrow{\simeq} \widehat{H}^0(\sigma, Z) \xrightarrow{\simeq} Z^\sigma/(1+\sigma)Z \xrightarrow{\simeq} Z^\delta/(1+\delta)Z$$

The first map is from the connecting homomorphism in (2.1) *coming from the exact sequence $1 \to Z \to G \to G_{\rm ad} \to 1$. The second and third arrows are from properties of Tate cohomology (see Section 2), and the last one is from Lemma 8.6.*

Remark 8.9 Alternatively we could define $\mathrm{inv} : H^1([\theta_{\rm qs}(\delta)], G_{\rm ad}) \to Z^\delta/(1+\delta)Z$ similarly, with $\theta, \theta_{\rm qs}$ in place of $\sigma, \sigma_{\rm qs}$. It is clear from Corollary 4.7, Lemma 8.6 and Definition 8.7 that the following diagram commutes:

$$\begin{tikzcd}
H^1([\sigma_{\rm qs}(\delta)], G_{\rm ad}) \arrow[r] \arrow[d, "\simeq"'] & Z^\delta/(1+\delta)Z \\
H^1([\theta_{\rm qs}(\delta)], G_{\rm ad}) \arrow[ur] &
\end{tikzcd}$$

The central invariant allows us to see how $H^1(\sigma, G)$ varies in a given inner class, as in Example 4.9. See [24, Section I.5.7, Remark 1].

Lemma 8.10 *Suppose σ_1, σ_2 are real forms of G in the same inner class. If $\mathrm{inv}([\sigma_1]) = \mathrm{inv}([\sigma_2])$ then $H^1(\sigma_1, G) \simeq H^1(\sigma_2, G)$.*

Proof. Fix a quasi-split real form $\sigma_{\rm qs}$ in the inner class of σ_1 and σ_2. Write $\sigma_i = \mathrm{int}(g_i) \circ \sigma_{\rm qs}$, where $g_i\sigma_{\rm qs}(g_i) \in Z$ $(i = 1, 2)$. A straightforward calculation shows that the map $h \mapsto hg_1g_2^{-1}$ induces the desired isomorphism, provided $g_1\sigma_{\rm qs}(g_1) = g_2\sigma_{\rm qs}(g_2)$. Unwinding Definition 8.7 we see this condition is equivalent to $\mathrm{inv}(\sigma_1) = \mathrm{inv}(\sigma_2)$. We leave the details to the reader. □

This bijection is not canonical in general.

The map $H^1(\sigma, G) \to H^1(\sigma, G_{\rm ad})$ is not necessarily surjective. This failure of surjectivity causes some difficulties in precise statements of the Langlands classification. See [3], [29], and for the p-adic case [15]. This leads to the notion of *strong real form* of G.

Definition 8.11 *Fix $\delta \in \mathrm{Out}(G)_2$ and a quasisplit real form σ_{qs} in the inner class defined by δ. A strong real form in the inner class of σ_{qs} is an element of* $\mathrm{SRF}_{\sigma_{qs}}(G) := Z^1(\sigma_{qs}, G; Z_{\mathrm{tor}})/(1+\sigma_{qs})Z$. *Two strong real forms g, h are said to be equivalent if they map to the same element of $H^1(\sigma_{qs}, G; Z_{\mathrm{tor}})$. We will also write* $[\mathrm{SRF}_{\sigma_{qs}}(G)]$ *for $H^1(\sigma_{qs}, G; Z_{\mathrm{tor}})$.*

If g is a strong real form define $\mathrm{inv}(g) = g\sigma_{qs}(g) \in Z^\delta_{\mathrm{tor}}$ *(Lemma 8.6). We refer to* inv *as the central invariant of a strong real form. This factors to a well defined map* $\mathrm{inv} : [\mathrm{SRF}_{\sigma_{qs}}(G)] \to Z^\delta_{\mathrm{tor}}$.

Since the notion of real form does not depend on a choice of base points, we want to eliminate the dependence of $\mathrm{SRF}_{\sigma_{\mathrm{qs}}}(G)$ on the choice of σ_{qs}.

Consider quasi-split real forms σ_{qs} and σ'_{qs} of G in the inner class defined by δ. There are pinnings $\mathcal{P}$ and $\mathcal{P}'$ of G such that $\sigma_{\mathrm{qs}} = \sigma_{\mathrm{qs}}(\delta, \mathcal{P})$ and $\sigma'_{\mathrm{qs}} = \sigma_{\mathrm{qs}}(\delta, \mathcal{P}')$. There exists $h \in G$ such that $\mathcal{P}' = \mathrm{int}(h)(\mathcal{P})$, and so we obtain a bijection

$$\begin{aligned} \mathrm{SRF}_{\sigma'_{\mathrm{qs}}}(G) &\longrightarrow \mathrm{SRF}_{\sigma_{\mathrm{qs}}}(G) \\ g &\longmapsto gh\sigma_{\mathrm{qs}}(h)^{-1} \end{aligned}$$

which does not depend of the choice of h, is compatible with inv and induces a bijection of pointed sets $H^1(\sigma'_{\mathrm{qs}}, G; Z_{\mathrm{tor}}) \simeq H^1(\sigma_{\mathrm{qs}}, G; Z_{\mathrm{tor}})$. Note however that this bijection depends on the pinnings and not just on the real forms. This is due to the fact that in general $G^{\sigma_{\mathrm{qs}}}$ does not act transitively on the set of pinnings of G fixed by σ_{qs}.

Definition 8.12 *Fix $\delta \in \mathrm{Out}(G)_2$. We define the set of strong real forms in the inner class defined by δ as*

$$\mathrm{SRF}_\delta(G) = \lim_{\mathcal{P}} \mathrm{SRF}_{\sigma_{qs}(\delta,\mathcal{P})}(G)$$

where the (projective or injective) limit is taken over all pinnings of G. Define the set $[\mathrm{SRF}_\delta(G)]$ *of equivalence classes of strong real forms in the inner class defined by δ similarly (or as a quotient of* $\mathrm{SRF}_\delta(G)$*). We have a well-defined map* $\mathrm{inv} : [\mathrm{SRF}_\delta(G)] \to Z^\delta_{\mathrm{tor}}$.

As the pinning varies the maps $g \in \mathrm{SRF}_{\sigma_{qs}(\delta,\mathcal{P})} \mapsto \mathrm{int}(g) \circ \sigma_{qs}(\delta, \mathcal{P})$ *are compatible and induce a* surjection *from* $\mathrm{SRF}_\delta(G)$ *to the set of real forms in the inner class defined by δ. This induces a surjection from* $[\mathrm{SRF}_\delta(G)]$ *to the set* $H^1([\sigma_{qs}(\delta)], G_{\mathrm{ad}})$ *of equivalences classes of real forms in the inner class defined by δ. Moreover this surjection is compatible with the two definitions (8.7 and 8.11) of central invariant.*

Remark 8.13 We say that an algebraic automorphism of G is *distinguished* if it fixes a pinning of G. We say a real form is *quasicompact* if one (equivalently, any) of its Cartan involutions is distinguished. In [4] we prefer θ over σ. Consequently strong involutions are defined in [4, Definition 5.5] with respect to a distinguished involution θ_{qc} of G. Write $[\mathcal{SI}_\delta]$ for the set of G-conjugacy classes of strong involutions (as in the Galois case we take a limit to make this definition only depend on δ).

There is a natural bijection between $[\mathrm{SRF}_\delta(G)]$ and $[\mathcal{SI}_\delta(G)]$. To state this, fix a pinning $\mathcal{P}$ and $g \in \mathrm{SRF}_{\sigma_{\mathrm{qs}}(\delta,\mathcal{P})}(G)$ such that $\sigma_{\mathrm{qc}} := \mathrm{int}(g) \circ \sigma_{\mathrm{qs}}(\delta,\mathcal{P})$ is a quasicompact real form. Choose a Cartan involution θ_{qc} for σ_{qc}. Then

$$[\mathrm{SRF}_\delta(G)] = [\mathrm{SRF}_{\sigma_{\mathrm{qs}}(\delta,\mathcal{P})}(G)] \simeq H^1(\sigma_{\mathrm{qc}}, G; Z_{\mathrm{tor}}) \simeq H^1(\theta_{\mathrm{qc}}, G; Z_{\mathrm{tor}}) \simeq [\mathcal{SI}_\delta(G)]$$

by twisting (Lemma 2.4) and Corollary 4.7. Also see [4, Remark 5.17] and [5, (9.7)].

We now describe equivalence classes of strong real forms in terms of the usual Galois cohomology pointed sets $H^1(\sigma, G)$.

Proposition 8.14 *Suppose σ is a real form of G, in the inner class defined by δ. Choose a representative $z \in Z^\delta_{\mathrm{tor}}$ of $\mathrm{inv}([\sigma]) \in Z^\delta/(1+\delta)Z$ (see Lemma 8.6). Then there is a bijection*

$$H^1(\sigma, G) \longleftrightarrow \textit{equivalence classes of strong real forms of central invariant } z.$$

Proof. Choose a strong real form $\tilde{\sigma}$ lifting σ and having central invariant z, and use twisting (Lemma 2.4). □

Note that the bijection not only depends on the choice of representative $z \in Z^\delta_{\mathrm{tor}}$ of $\mathrm{inv}([\sigma]) \in Z^\delta_{\mathrm{tor}}/(1+\delta)Z_{\mathrm{tor}}$, but also on the choice of $\tilde{\sigma}$ in the proof, which is only unique up to $H^1(\sigma, Z)$.

Corollary 8.15 *Suppose $\delta \in \mathrm{Out}(G)_2$. Choose representatives $\{z_i \mid i \in I\}$ for the image of $\mathrm{inv} : [\mathrm{SRF}_\delta(G)] \to Z^\delta_{\mathrm{tor}}$. For each $i \in I$ choose a real form σ_i of G such that $\mathrm{inv}([\sigma_i]) = z_i \mod (1+\delta)Z_{\mathrm{tor}}$. Then there is a bijection*

$$[\mathrm{SRF}_\delta(G)] \longleftrightarrow \bigcup_i H^1(\sigma_i, G).$$

This gives an interpretation of $[\mathrm{SRF}_\delta(G)]$ in classical cohomological terms. A similar statement holds in the p-adic case [15].

The set I is finite if and only if the identity component of the center of G is split (this condition only depends on δ). As in [15] or [4, Section 13] the theory can be modified to replace this with a finite set even when this condition is not satisfied. In any case the group $Z^\delta_{\mathrm{tor}}/(1+\delta)Z_{\mathrm{tor}}$ is finite, and for $z \in Z^\delta_{\mathrm{tor}}$ and $x \in Z_{\mathrm{tor}}$ there is an obvious bijection of pointed sets

$$H^1(\sigma, G; \{z\}) \simeq H^1(\sigma, G; \{zx\delta(x)\}).$$

Proposition 8.16 *Suppose σ is an equal rank real form of G, i.e. that it belongs to the inner class defined by $\delta = 1$. Choose $x \in G$ so that $\mathrm{int}(x)$ is a Cartan involution for σ, and let $z = x^2 \in Z$. Then we have an explicit bijection*

$$H^1(\sigma, G) \longleftrightarrow S(z)$$

where $S(z)$ is the set of conjugacy classes of G with square equal to z. If H is a Cartan subgroup of G, with Weyl group W, then $S(z)$ is equal to

$$\{h \in H \mid h^2 = z\}/W. \tag{8.17}$$

This bijection is not canonical in general.

Proof. The assertion does not depend on the choice of x: if $x' = tgxg^{-1}$ with $t \in Z$ and $g \in G^\sigma$, then $(x')^2 = t^2 z$ and $h \mapsto th$ is a bijection from $S(z)$ to $S(t^2 z)$.

Now fix θ, and note that $\sigma(x) \in xZ$. The compact Lie group $(G_{\mathrm{ad}})^{\sigma\theta}$ is connected, and so $G^{\sigma\theta} \to (G_{\mathrm{ad}})^{\sigma\theta}$ is surjective. Therefore we can take $x \in G^{\sigma\theta}$, equivalently $x \in G^\sigma$. Let $\sigma_c = \mathrm{int}(x) \circ \sigma$. Note that $z = x^2 \in Z^{\sigma_c}$. By twisting (Lemma 2.4) and Corollary 4.7 applied to (G, σ_c),

$$H^1(\sigma, G) \simeq H^1(\sigma_c, G; z^{-1}) \simeq H^1(1, G; z^{-1}) = S(z^{-1}) \simeq S(z).$$

The identification of $S(z)$ with (8.17) follows from the well-known fact that H/W parametrizes semisimple conjugacy classes in G. □

Example 8.18 Taking $x = z = I$ gives $G(\mathbb{R})$ compact and recovers [24, III.4.5]: $H^1(\sigma, G)$ is the set of conjugacy classes of involutions in G. See Example 4.8.

Example 8.19 Let $G(\mathbb{R}) = Sp(2n, \mathbb{R})$. We can take $x = \mathrm{diag}(iI_n, -iI_n)$, $z = -I$. It is easy to see that every element of G whose square is $-I$ is conjugate to x. This gives the classical result $H^1(\sigma, G) = 1$, which is equivalent to the classification of nondegenerate symplectic forms [22, Chapter 2].

Example 8.20 Suppose $G(\mathbb{R}) = SO(Q)$, the isometry group of a nondegenerate real quadratic form. Suppose Q has signature (p, q). If pq is even we can assume that q is even (up to replacing Q by $-Q$) and take $x = \mathrm{diag}(I_p, -I_q)$ in Proposition 8.16, and the set (8.17) is equal to $\{\mathrm{diag}(I_r, -I_s) \mid r + s = p + q;\ s \text{ even}\}$.

Suppose p and q are odd. Apply Corollary 7.6 with $M_f(\mathbb{R}) = SO(p-1, q-1) \times GL(1, \mathbb{R})$. By the previous case we conclude $H^1(\sigma, G)$ is parametrized by $\{\mathrm{diag}(I_r, I_s) \mid r + s = p + q - 2;\ r, s \text{ even}\}$. Adding $(1, 1)$ this is the same as $\{\mathrm{diag}(I_r, -I_s) \mid r + s = p + q;\ s \text{ odd}\}$.

In all cases we recover the classical fact that $H^1(\sigma, G)$ parametrizes the set of equivalence classes of quadratic forms of the same dimension and discriminant as Q [22, Chapter 2], [24, III.3.2].

Example 8.21 Now suppose $G(\mathbb{R}) = \mathrm{Spin}(p, q)$, which is a (connected) two-fold cover of the identity component of $\mathrm{SO}(p, q)$. A calculation similar to that in the previous example shows that $|H^1(\sigma, \mathrm{Spin}(p, q))| = \lfloor \frac{p+q}{4} \rfloor + \delta(p, q)$ where $0 \le \delta(p, q) \le 3$ depends on $p, q \pmod 4$. See Section 10.2.

Skip Garibaldi pointed out this result can also be derived from the exact cohomology sequence associated to the exact sequence $1 \to \mathbb{Z}_2 \to \mathrm{Spin}(n, \mathbb{C}) \to$

$SO(n,\mathbb{C}) \to 1$; the preceding result; the fact that $SO(p,q)$ is connected if $pq=0$ and otherwise has two connected components; and a calculation of the image of the map from $H^1(\sigma,\mathrm{Spin}(n,\mathbb{C})) \to H^1(\sigma, SO(n,\mathbb{C}))$. See [16, after (31.41)], [24, III.3.2] and also section 9. The result is:

$|H^1(\sigma,\mathrm{Spin}(Q))|$ equals the number of quadratic forms having the same dimension, discriminant, and Hasse invariant as Q with each (positive or negative) definite form counted twice.

Remark 8.22 Kottwitz relates $H^1(\sigma,G)$ to the center of the dual group [19, Theorem 1.2]. This is a somewhat different type of result. It describes a certain quotient $H^1_{\mathrm{sc}}(\sigma,G)$ of $H^1(\sigma,G)$ (see [15, 3.4]), but if G is simply connected this gives no information.

9 Fibers of $H^1(\sigma,G) \to H^1(\sigma,\overline{G})$

In this section G is a connected complex reductive group, and σ is a real form of G. Suppose $A \subset Z(G)$ is σ-stable and let $\overline{G} = G/A$. It is helpful to analyze the fibers of the map $\psi : H^1(\sigma,G) \to H^1(\sigma,\overline{G})$. In particular taking $G = G_{\mathrm{sc}}, \overline{G} = G_{\mathrm{ad}}$, and summing over $H^1(\sigma,G_{\mathrm{ad}})$, we obtain a description of $H^1(\sigma,G_{\mathrm{sc}})$, complementary to that of Proposition 8.14.

Write $G(\mathbb{R},\sigma) = G^\sigma$ and $\overline{G}(\mathbb{R},\sigma) = \overline{G}^\sigma$. Write p for the projection map $G \to \overline{G}$. This restricts to a map $G(\mathbb{R},\sigma) \to \overline{G}(\mathbb{R},\sigma)$, taking the identity component of $G(\mathbb{R},\sigma)$ to that of $\overline{G}(\mathbb{R},\sigma)$. Therefore p factors to a map (not necessarily an injection):

(9.1)(a) $$p^* : \pi_0(G(\mathbb{R},\sigma)) \to \pi_0(\overline{G}(\mathbb{R},\sigma)).$$

Define

(9.1)(b) $$\pi_0(G,\overline{G},\sigma) = \pi_0(\overline{G}(\mathbb{R},\sigma))/p^*(\pi_0(G(\mathbb{R},\sigma))).$$

There is a natural action of $\overline{G}(\mathbb{R},\sigma)$ on $H^1(\sigma,A)$ defined as follows. Suppose $g \in \overline{G}(\mathbb{R},\sigma)$. Choose $h \in G$ satisfying $p(h) = g$. Then $g : a \to ha\sigma(h^{-1})$ factors to a well defined action of $\overline{G}(\mathbb{R},\sigma)$ on $H^1(\sigma,A)$. Furthermore the image of $G(\mathbb{R},\sigma)$, which includes the identity component, acts trivially, so this factors to an action of $\pi_0(G,\overline{G},\sigma)$.

Proposition 9.2 *Suppose $\gamma \in H^1(\sigma,G)$, and write $\gamma = cl(g)$ ($g \in G^{-\sigma}$). Let $\sigma_\gamma = \mathrm{int}(g)\circ\sigma$. Then there is a bijection*

$$H^1(\sigma,G) \supset \psi^{-1}(\psi(\gamma)) \longleftrightarrow H^1(\sigma,A)/\pi_0(G,\overline{G},\sigma_\gamma).$$

Proof. First assume γ is trivial, and take $g=1$. Consider the exact sequence

$$H^0(\sigma,G) \to H^0(\sigma,\overline{G}) \to H^1(\sigma,A) \xrightarrow{\phi} H^1(\sigma,G) \xrightarrow{\psi} H^1(\sigma,\overline{G}).$$

This says $\psi^{-1}(\psi((\gamma))=\phi(H^1(\sigma,A))$, i.e. the orbit of the group $H^1(\sigma,A)$ acting on the identity coset. This is $H^1(\sigma,A)$, modulo the action of $H^0(\sigma,\overline{G})$, and this action factors through the image of $H^0(\sigma,G)$. The general case follows from an easy twisting argument. □

We specialize to the case $G=G_{\mathrm{sc}}$ is simply connected and $\overline{G}=G_{\mathrm{ad}}=G_{\mathrm{sc}}/Z_{\mathrm{sc}}$ is the adjoint group.

Corollary 9.3 *Suppose σ is a real form of G_{sc} and consider the map $\psi: H^1(\sigma,G_{\mathrm{sc}})\to H^1(\sigma,G_{\mathrm{ad}})$.*

Suppose $\gamma\in H^1(\sigma,G_{\mathrm{ad}})$, and write $\gamma=cl(g)$ ($g\in G_{\mathrm{ad}}^{-\sigma}$). Let $\sigma_\gamma=\mathrm{int}(g)\circ\sigma$, viewed as an involution of G_{sc}.

$$\text{(9.4)(a)}\qquad \gamma \textit{ is in the image of } \psi \Leftrightarrow \mathrm{inv}([\sigma_\gamma])=\mathrm{inv}([\sigma]),$$

in which case

$$\text{(9.4)(b)}\qquad |\psi^{-1}(\gamma)|=|H^1(\sigma,Z_{\mathrm{sc}})|/|\pi_0(G_{\mathrm{ad}}(\mathbb{R},\sigma_\gamma))|.$$

Furthermore

$$\text{(9.4)(c)}\qquad |H^1(\sigma,G_{\mathrm{sc}})|=|H^1(\sigma,Z_{\mathrm{sc}})|\sum_{\substack{\gamma\in H^1(\sigma,G_{\mathrm{ad}})\\ \mathrm{inv}([\sigma_\gamma])=\mathrm{inv}([\sigma])}}|\pi_0(G_{\mathrm{ad}}(\mathbb{R}),\sigma_\gamma)|^{-1}$$

Proof. Statements (b) and (c) follow from Proposition 9.2. For (a), when σ is quasi-split the proof is immediate, and the general case is similar using twisting. We leave the details to the reader. □

10 Tables

Most of these results can be computed by hand from Theorem 1.2, or using Proposition 9.2 and the classification of real forms (i.e. the adjoint case).

By Theorem 1.2 the computation of $H^1(\Gamma,G)$ reduces to calculating the strong real forms of G and their central invariants. The Atlas of Lie Groups and Representations does this computation as part of its parametrization of (strong) real forms. This comes down to calculating the orbits of a finite group (a subgroup of the Weyl group) on a finite set (related to elements of order 2 in a Cartan subgroup). See [4, Proposition 12.9] and `www.liegroups.org/tables/galois`.

10.1 Classical groups

Group	$\lvert H^1(\sigma, G)\rvert$	
$SL(n,\mathbb{R}), GL(n,\mathbb{R})$	1	
$SU(p,q)$	$\lfloor \frac{p}{2} \rfloor + \lfloor \frac{q}{2} \rfloor + 1$	Hermitian forms of rank $p+q$ and discriminant $(-1)^q$
$SL(n,\mathbb{H})$	2	$\mathbb{R}^*/\mathrm{Nrd}_{\mathbb{H}/\mathbb{R}}(\mathbb{H}^*)$
$Sp(2n,\mathbb{R})$	1	real symplectic forms of rank $2n$
$Sp(p,q)$	$p+q+1$	quaternionic Hermitian forms of rank $p+q$
$SO(p,q)$	$\lfloor \frac{p}{2} \rfloor + \lfloor \frac{q}{2} \rfloor + 1$	real symmetric bilinear forms of rank n and discriminant $(-1)^q$
$SO^*(2n)$	2	

Here $\mathbb{H}$ is the quaternions, and $\mathrm{Nrd}_{\mathbb{H}/\mathbb{R}}$ is the reduced norm map from $\mathbb{H}^*$ to $\mathbb{R}^*$ (see [22, Lemma 2.9]). Also $Sp(p,q)$ (respectively $SO^*(2n)$) is the isometry group of a Hermitian (resp. skew-Hermitian) form on a quaternionic vector space. For more information on Galois cohomology of classical groups see [24], [22, Sections 2.3 and 6.6] and [16, Chapter VII].

10.2 Simply connected groups

The only simply connected groups with classical root system, which are not in the table in Section 10.1 are $\mathrm{Spin}(p,q)$ and $\mathrm{Spin}^*(2n)$.

Define $\delta(p,q)$ by the following table, depending on $p, q \pmod 4$.

	0	1	2	3
0	3	2	2	2
1	2	1	1	0
2	2	1	0	0
3	2	0	0	0

See Example 8.21 for an explanation of these numbers.

Group	$\lvert H^1(\sigma, G)\rvert$
$\mathrm{Spin}(p,q)$	$\lfloor \frac{p+q}{4} \rfloor + \delta(p,q)$
$\mathrm{Spin}^*(2n)$	2

Simply connected exceptional groups					
inner class	group	K	real rank	name	$\|H^1(\sigma, G)\|$
compact	E_6	A_5A_1	4	quasisplit quaternionic	3
	E_6	D_5T	2	Hermitian	3
	E_6	E_6	0	compact	3
split	E_6	C_4	6	split	2
	E_6	F_4	2	quasicompact	2
compact	E_7	A_7	7	split	2
	E_7	D_6A_1	4	quaternionic	4
	E_7	E_6T	3	Hermitian	2
	E_7	E_7	0	compact	4
compact	E_8	D_8	8	split	3
	E_8	E_7A_1	4	quaternionic	3
	E_8	E_8	0	compact	3
compact	F_4	C_3A_1	4	split	3
	F_4	B_4	1		3
	F_4	F_4	0	compact	3
compact	G_2	A_1A_1	2	split	2
	G_2	G_2	0	compact	2

10.3 Adjoint groups

If G is adjoint $|H^1(\sigma, G)|$ is the number of real forms in the given inner class, which is well-known. We also include the component group, which is useful in connection with Corollary 9.3.

One technical point arises in the case of $PSO^*(2n)$. If n is even there are two real forms which are related by an outer, but not an inner, automorphism. See Remark 8.2.

Adjoint classical groups		
Group	$\lvert\pi_0(G(\mathbb{R}))\rvert$	$\lvert H^1(\sigma, G)\rvert$
$PSL(n,\mathbb{R})$	$\begin{cases} 2 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$	$\begin{cases} 2 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$
$PSL(n,\mathbb{H})$	1	2
$PSU(p,q)$	$\begin{cases} 2 & p=q \\ 1 & \text{otherwise} \end{cases}$	$\lfloor \frac{p+q}{2} \rfloor + 1$
$PSO(p,q)$	$\begin{cases} 1 & pq=0 \\ 1 & p,q \text{ odd and } p \neq q \\ 4 & p=q \text{ even} \\ 2 & \text{otherwise} \end{cases}$	$\begin{cases} \lfloor \frac{p+q+2}{4} \rfloor & p,q \text{ odd} \\ \frac{p+q}{4}+3 & p,q \text{ even}, p+q=0 \pmod 4 \\ \frac{p+q-2}{4}+2 & p,q \text{ even}, p+q=2 \pmod 4 \\ \frac{p+q+1}{2} & p,q \text{ opposite parity} \end{cases}$
$PSO^*(2n)$	$\begin{cases} 2 & n \text{ even} \\ 1 & n \text{ odd} \end{cases}$	$\begin{cases} \frac{n}{2}+3 & n \text{ even} \\ \frac{n-1}{2}+2 & n \text{ odd} \end{cases}$
$PSp(2n,\mathbb{R})$	2	$\lfloor \frac{n}{2} \rfloor + 2$
$PSp(p,q)$	$\begin{cases} 2 & p=q \\ 1 & \textit{else} \end{cases}$	$\lfloor \frac{p+q}{2} \rfloor + 2$

The groups E_8, F_4 and G_2 are both simply connected and adjoint. Furthermore in type E_6 the center of the simply connected group G_{sc} has order 3, and it follows that $H^1(\sigma, G_{\mathrm{ad}}) = H^1(\sigma, G_{\mathrm{sc}})$ in these cases. So the only groups not covered by the table in Section 10.2 are adjoint groups of type E_7.

Adjoint exceptional groups						
inner class	group	K	real rank	name	$\pi_0(G(\mathbb{R}))$	$\lvert H^1(G)\rvert$
compact	E_7	A_7	7	split	2	4
	E_7	D_6A_1	4	quaternionic	1	4
	E_7	E_6T	3	Hermitian	2	4
	E_7	E_7	0	compact	1	4

References

[1] Jeffrey Adams, *Lifting of characters*, Harmonic analysis on reductive groups (Brunswick, ME, 1989), 1991, pp. 1–50. MR1168475 (93d:22014)

[2] ———, *Guide to the Atlas software: computational representation theory of real reductive groups*, Representation theory of real reductive Lie groups, 2008, pp. 1–37. MR2454331

[3] Jeffrey Adams, Dan Barbasch, and David A. Vogan Jr., *The Langlands classification and irreducible characters for real reductive groups*, Progress in Mathematics, vol. 104, Birkhäuser Boston Inc., Boston, MA, 1992. MR1162533 (93j:22001)

[4] Jeffrey Adams and Fokko du Cloux, *Algorithms for representation theory of real reductive groups*, J. Inst. Math. Jussieu **8** (2009), no. 2, 209–259. MR2485793

[5] Jeffrey Adams and David A. Vogan Jr., *L-groups, projective representations, and the Langlands classification*, Amer. J. Math. **114** (1992), no. 1, 45–138. MR1147719 (93c:22021)

[6] Armand Borel, *Linear algebraic groups*, Second, Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991. MR1102012 (92d:20001)

[7] A. Borel, *Automorphic L-functions*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, 1979, pp. 27–61.

[8] Armand Borel and Jean-Pierre Serre, *Théorèmes de finitude en cohomologie galoisienne*, Comment. Math. Helv. **39** (1964), 111–164. MR0181643

[9] Mikhail Borovoi, *Galois cohomology of real reductive groups and real forms of simple lie algebras*, Funct. Anal. Appl. **22** (1988), no. 2, 135–136.

[10] ———, *Galois cohomology of reductive algebraic groups over the field of real numbers* (2014). arXiv:1401.5913.

[11] Mikhail Borovoi and Dmitry A. Timashev, *Galois cohomology of real semisimple groups* (2015). arXiv:1506.06252.

[12] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 21, Springer-Verlag, Berlin, 1990. MR1045822

[13] Nicolas Bourbaki, *Lie groups and Lie algebras. Chapters 7–9*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2005. Translated from the 1975 and 1982 French originals by Andrew Pressley. MR2109105

[14] Gerhard Hochschild, *The structure of Lie groups*, Holden-Day, Inc., San Francisco-London-Amsterdam, 1965. MR0207883

[15] Tasho Kaletha, *Rigid inner forms of real and p-adic groups*, Ann. of Math. (2) **184** (2016), no. 2, 559–632.

[16] Max-Albert Knus, Alexander Merkurjev, Markus Rost, and Jean-Pierre Tignol, *The book of involutions*, American Mathematical Society Colloquium Publications, vol. 44, American Mathematical Society, Providence, RI, 1998. With a preface in French by J. Tits. MR1632779 (2000a:16031)

[17] Bertram Kostant and Stephen Rallis, *Orbits and representations associated with symmetric spaces*, Amer. J. Math. **93** (1971), 753–809. MR0311837

[18] Robert E. Kottwitz, *Stable trace formula: cuspidal tempered terms*, Duke Math. J. **51** (1984), no. 3, 611–650. MR757954

[19] ———, *Stable trace formula: elliptic singular terms*, Math. Ann. **275** (1986), no. 3, 365–399. MR858284 (88d:22027)

[20] Toshihiko Matsuki, *The orbits of affine symmetric spaces under the action of minimal parabolic subgroups*, J. Math. Soc. Japan **31** (1979), no. 2, 331–357. MR527548 (81a:53049)

[21] George D. Mostow, *Self-adjoint groups*, Ann. of Math. (2) **62** (1955), 44–55. MR0069830

[22] Vladimir Platonov and Andrei Rapinchuk, *Algebraic groups and number theory*, Pure and Applied Mathematics, vol. 139, Academic Press Inc., Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen. MR1278263 (95b:11039)

[23] Jirō Sekiguchi, *Remarks on real nilpotent orbits of a symmetric pair*, J. Math. Soc. Japan **39** (1987), no. 1, 127–138. MR867991

[24] Jean-Pierre Serre, *Galois cohomology*, English, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002. Translated from the French by Patrick Ion and revised by the author. MR1867431 (2002i:12004)

[25] Tonny A. Springer, *Linear algebraic groups*, Second, Progress in Mathematics, vol. 9, Birkhäuser Boston Inc., Boston, MA, 1998. MR1642713 (99h:20075)

[26] Robert Steinberg, *Générateurs, relations et revêtements de groupes algébriques*, Colloq. Théorie des Groupes Algébriques (Bruxelles, 1962), 1962, pp. 113–127. MR0153677 (27 #3638)

[27] David A. Vogan Jr., *Representations of real reductive Lie groups*, Progress in Mathematics, vol. 15, Birkhäuser Boston, Mass., 1981. MR632407 (83c:22022)

[28] ______, *Irreducible characters of semisimple Lie groups. IV. Character-multiplicity duality*, Duke Math. J. **49** (1982), no. 4, 943–1073. MR683010 (84h:22037)

[29] ______, *The local Langlands conjecture*, Representation theory of groups and algebras, 1993, pp. 305–379. MR1216197 (94e:22031)

[30] Garth Warner, *Harmonic analysis on semi-simple Lie groups. II*, Springer-Verlag, New York, 1972. Die Grundlehren der mathematischen Wissenschaften, Band 189. MR0499000 (58 #16980)