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The local Langlands conjecture

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Abstract. We formulate the local Langlands conjecture for connected reductive groups over local fields, including the internal parametrization of L-packets using endoscopy.

1. Introduction

The first goal of these notes is to state the local Langlands conjecture for connected reductive groups G over a local field F, that is the existence of a map LL with finite fibers associating Langlands parameters to irreducible smooth representations ($(\mathfrak{g}, K)$-modules in the case where F is Archimedean) over an algebraically closed field C of characteristic zero (the field of complex numbers $\mathbb{C}$ in the Archimedean case). To be useful this map should satisfy certain properties, and we list some of them in Conjecture 6.1, after recalling in Sections 3 and 5 parallel features of the classification of representations of $G(F)$ and Langlands parameters. For representations of $G(F)$ we put the emphasis on the case of complex coefficients ($C = \mathbb{C}$), sometimes using notions relying on the topology of $\mathbb{C}$, because of the relatively simple (partial) classification of representations using parabolically induced representations of essentially discrete representations of Levi subgroups. We do point out however that the map LL should be "algebraic" (in particular, functorial in C) and formulate the (purely algebraic) semi-simplified Langlands correspondence (Conjecture 6.2). Unfortunately neither conjecture includes a characterization of the map LL, and proofs of cases of these conjectures use different characterizations.

We then formulate refined versions of the local Langlands correspondence, describing the fibers ("L-packets") of the maps LL using centralizers of Langlands parameters. In the case where G is quasi-split this is fairly straightforward (Conjectures 6.4 and 6.8) and includes Shahidi's conjecture (Conjecture 6.5). Formulating the refined correspondence in the non-quasi-split case (Conjectures 6.12 and 6.13) is surprisingly difficult in general, and was only relatively recently achieved by Kaletha, generalizing an idea of Vogan from the case of pure inner forms of quasi-split groups. This entails employing *Galois gerbs* instead of Galois groups, thus generalizing Galois cohomology sets. In this setting where explicit computations seem to be unavoidable it is favorable to work with a down-to-earth definition

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of such gerbs as extensions of absolute Galois groups by a group of multiplicative type.

We conclude in Section 7 with a short explanation of the relation between this definition of gerbs with the more conceptual one. This is motivated by the fact that the same Galois gerb, corresponding to the Tannakian category of isocrytals, appears in the study of Shimura varieties, in the internal structure of certain L-packets, and geometrically in the construction by Fargues and Scholze [**FS**] of a semi-simplified local Langlands correspondence.

2. Notations

Let F be a local field. We denote by $||\cdot||$ the normalized absolute value on F. In the non-Archimedean case it maps a uniformizer to q^{-1} where q is the cardinality of the residue field. If $F \simeq \mathbb{R}$ it is the usual absolute value, if $F \simeq \mathbb{C}$ it is given by $z \mapsto z\overline{z}$. We will denote by $\overline{F}$ a separable closure of F and $\Gamma = \mathrm{Gal}(\overline{F}/F)$. For a group scheme u of multiplicative type of finite type over F we denote $X^*(u) = \mathrm{Hom}(u_{\overline{F}}, \mathbb{G}_{m,\overline{F}})$, a finitely generated $\mathbb{Z}$-module with smooth action of Γ. For a torus T over F we also have $X_*(T) = \mathrm{Hom}(\mathbb{G}_{m,\overline{F}}, T_{\overline{F}}) = \mathrm{Hom}(X^*(T), \mathbb{Z})$.

3. Representations of reductive groups

3.1. Setup. In this section we focus on the case where F is non-Archimedean and occasionally indicate the differences for the Archimedean case.

Let G be a connected reductive group over F. We refer to [**Bor91**] [**Spr98**] [**BT65**] and [**DGA$^+$11**] for fundamental results about reductive groups. Let C be an algebraically closed field of characteristic zero, for example $\mathbb{C}$ or $\overline{\mathbb{Q}_\ell}$. If F is Archimedean we always take $C = \mathbb{C}$. We consider *smooth* representations of $G(F)$ with coefficients in C, i.e. pairs (V, π) where V is a vector space over C and $\pi : G(F) \to \mathrm{GL}(V)$ is a morphism of groups such the map

$$\begin{aligned} G \times V &\longrightarrow V \\ (g, v) &\longmapsto \pi(g)v \end{aligned}$$

is continuous for the natural topology on G and the discrete topology on V. If π is implicit we will also denote $g \cdot v$ for $\pi(g)v$. Recall that such a representation is called *admissible* if for any compact open subgroup K of $G(F)$ the subspace

$$V^K = \{v \in V \,|\, \forall k \in K,\, \pi(k)v = v\}$$

of V has finite dimension. It is a non-trivial but well-known fact that any irreducible representation is admissible. Denote by $Z(G)$ the center of G. By a suitable generalization of Schur's lemma, any irreducible representation has a central character $Z(G)(F) \to C^\times$. For a smooth representation V, π of $G(F)$ its contragredient $(\tilde{V}, \tilde{\pi})$ is the space of K-finite linear forms on V.

Remark 3.1. In the case of an Archimedean field F we only consider coefficients $C = \mathbb{C}$. The analogue of smooth representations are $(\mathfrak{g}, K)$-modules where $\mathfrak{g} = \mathbb{C} \otimes_{\mathbb{R}} \mathrm{Lie}\, G(F)$ and K is a maximal compact subgroup of $G(F)$. For many notions it is necessary to relate $(\mathfrak{g}, K)$-modules to continuous representations of $G(F)$ on topological vector spaces. See e.g. [**Wal88**, §3.4] for the relation between the two notions in the case of unitary irreducible representations.

3.2. Parabolic induction, the Jacquet functor and supercuspidal representations. Let P be a parabolic subgroup of G. Let N be the unipotent radical of P and $M = P/N$ its reductive quotient. Recall that there exists a section $M \to P$, unique up to conjugation by $N(F)$. Let

$$\delta_P(p) = ||\det(\mathrm{Ad}(p)|\,\mathrm{Lie}(N))|| : M(F) \longrightarrow q^{\mathbb{Z}}$$

be the modulus character for the action of $M(F)$ on $N(F)$). We choose a square root $\sqrt{q}$ of q in C, allowing us to define $\delta_P^{1/2}$. If $C = \mathbb{C}$ we naturally choose $\sqrt{q} \in \mathbb{R}_{>0}$.

Let (V, σ) be a smooth representation of $M(F)$, which we can see as a representation of $P(F)$ trivial on $N(F)$. The normalized parabolically induced representation $i_P^G \sigma$ is the space of locally constant functions $f : G(F) \to V$ such that for any $p \in P(F)$ and $g \in G(F)$ we have $f(pg) = \delta_P(p)^{1/2}\sigma(p)f(g)$, with left action of $G(F)$ by $(g \cdot f)(x) = f(xg)$. If σ is admissible (resp. has finite length) then $i_P^G \sigma$ is admissible (resp. has finite length). The introduction of $\delta_P^{1/2}$ in the definition are motivated by the fact that if $C = \mathbb{C}$ and (V, σ) is unitary, i.e. endowed with a $M(F)$-invariant Hermitian inner product, then $i_P^G \sigma$ has a natural $G(F)$-invariant Hermitian inner product. In particular if σ is admissible and unitarizable then $i_P^G \sigma$ is semi-simple.

For (π, V) a smooth representation of $G(F)$, denote by V_N the space of co-invariants for the action of $N(F)$, which is naturally a smooth representation π_N of $M(F)$. The normalized Jacquet functor applied to (π, V) is the smooth representation $r_P^G \pi = \delta_P^{-1/2} \otimes \pi_N$ of $M(F)$ on the space V_N. It also preserves admissibility and the property of being of finite length.

Recall that an irreducible smooth representation (V, π) of $G(F)$ is called supercuspidal if $V_N = 0$ for any parabolic $P = MN \subsetneq G$. This is equivalent to all "matrix coefficients"

$$\begin{aligned} G(F) &\longrightarrow C \\ g &\longmapsto \langle \pi(g)v, \tilde{v} \rangle \end{aligned}$$

for $v \in V$ and $\tilde{v} \in \tilde{V}$, being compactly supported modulo center. Note that if $\omega_\pi : Z(G(F)) \to C^\times$ is the central character of π then matrix coefficients of π are ω_π-equivariant.

We recall in the following theorem the notion of supercuspidal support.

THEOREM 3.2. *Let π be an irreducible representation of $G(F)$.*

(1) *There exists a parabolic subgroup $P = MN$ of G and a supercuspidal irreducible representation σ of $M(F)$ such that π embeds in $i_P^G \sigma$.*
(2) *If $P' = M'N'$ is a parabolic subgroup of G and σ' is a supercuspidal irreducible representation of $M'(F)$ then π is isomorphic to a subquotient of $i_{P'}^G \sigma'$ if and only if there exists an element of $G(F)$ conjugating (M, σ) and (M', σ').*

The conjugacy class of (M, σ) may be called the supercuspidal support of π.

PROOF. The first part is due to Jacquet: see [**Cas**, Theorem 5.1.2]. The second part seems to be due to Harish-Chandra: see Theorem 6.3.11 loc. cit. or [**Sil79**, Theorem 4.6.1, §5.3.1 and Theorem 5.4.4.1] for the "if" part. The "only if" part can be deduced from Bernstein center theory [**Ber84a**]. See also [**BZ77**]. □

The $G(F)$-conjugacy class of (M, σ) in the previous theorem is called the *supercuspidal support* of π.

3.3. Asymptotic properties. For the rest of this section we assume $C = \mathbb{C}$.

DEFINITION 3.3. Let (V, π) be a smooth irreducible representation of $G(F)$. Let $\omega_\pi : Z(G(F)) \to \mathbb{C}^\times$ be its central character. If ω_π is unitary then we say that π is essentially square-integrable if all of its matrix coefficients are square-integrable modulo center:

$$\forall v \in V \ \forall \tilde{v} \in \tilde{V} \ \int_{G(F)/Z(G(F))} |\langle \pi(g)v, \tilde{v} \rangle|^2 \, dg < \infty.$$

In general (without assuming that ω_π is unitary) there is a unique smooth character $\chi : G(F) \to \mathbb{R}_{>0}$ such that the central character of $\chi \otimes \pi$ is unitary [**Cas**, Lemma 5.2.5], and we say that π is essentially square-integrable if $\chi \otimes \pi$ is.

If π is an essentially square-integrable irreducible smooth representation of $G(F)$ and if ω_π is unitary then π is unitarizable.

Essential square-integrability can be checked on the Jacquet module of a representation, as recalled in Proposition 3.4 below. For a Levi subgroup M of G we denote by A_M the largest split torus in the centre of M. Denote $\mathfrak{a}_M^* = X^*(A_M) \otimes_{\mathbb{Z}} \mathbb{R}$. We have an isomorphism

$$\begin{aligned} \mathfrak{a}_M^* &\longrightarrow \mathrm{Hom}_{\mathrm{cont}}(A_M(F), \mathbb{R}_{>0}) \\ \chi \otimes s &\longmapsto (x \mapsto ||\chi(x)||^s). \end{aligned} \tag{3.1}$$

PROPOSITION 3.4 ([**Wal03**, Proposition III.1.1]). *Let (V, π) be an irreducible smooth representation of $G(F)$. Assume that the central character of π is unitary (we can reduce to this case by twisting). Then (V, π) is essentially square-integrable if and only if for every parabolic subgroup $P = MN$ of G, the absolute value of any character of $A_M(F)$ occurring in $r_P^G \pi$ is a linear combination with positive coefficients of the simple roots of A_M in N (via the isomorphism* (3.1)*).*

Replacing "positive" by "non-negative" in this characterization we get the notion of *tempered* representation. This is also equivalent to a growth condition on coefficients [**Wal03**, Proposition III.2.2].

We have the following implications, for an irreducible smooth representation of $G(F)$ having unitary central character:

supercuspidal $\Rightarrow$ essentially square-integrable $\Rightarrow$ tempered $\Rightarrow$ unitarizable.

For non-commutative G none of these implications is an equivalence.

PROPOSITION 3.5 ([**Wal03**, Proposition III.4.1]).
1. *Let $P = MN$ be a parabolic subgroup of G and σ an essentially square-integrable irreducible smooth representation of $M(F)$ having unitary central character. The induced representation $i_P^G \sigma$ is semi-simple, has finite length and any irreducible subrepresentation is tempered.*
2. *Let (P, σ) and (P', σ') be two pairs as in the previous point. Then $i_P^G \sigma$ and $i_{P'}^G \sigma'$ admit isomorphic irreducible subrepresentations if and only if the pairs (M, σ) and (M', σ') are conjugated by $G(F)$, and in this case the two induced representations are isomorphic.*

(3) *For any tempered irreducible smooth representation π of $G(F)$ there exists a pair (P, σ) as in the first point such that π is isomorphic to a subrepresentation of $i_P^G \sigma$.*

REMARK 3.6. For $G = \mathrm{GL}_n$, parabolically induced representations as in Proposition 3.5 are always irreducible [**Ber84b**, §0.2] and so the proposition completely classifies tempered representations in terms of essentially square-integrable representations of smaller general linear groups. Recall that for general linear groups essentially square-integrable representations can be explicitly classified in terms of supercuspidal representations of Levi subgroups [**Zel80**, Theorem 9.3].

For arbitrary G such induced representations are *generically* irreducible (see [**Wal03**, Proposition IV.2.2] for a precise statement), but decomposing such induced representations is a subtle problem in general.

The tempered representations are exactly the ones occurring in Harish-Chandra's Plancherel formula, expressing the values of any locally constant and compactly supported $f : G(F) \to \mathbb{C}$ in terms of the action of f in tempered representations (or expressing $f(1)$ in terms of the traces of f in tempered representations).

Finally the "Langlands classification", that we recall below, classifies irreducible smooth representations of $G(F)$ in terms of tempered representations of Levi subgroups. For a connected reductive group M denote by $X^*(M)^\Gamma$ the abelian group of morphisms $M \to \mathrm{GL}_1$ (defined over F). The restriction morphism $X^*(M)^\Gamma \to X^*(A_M)$ is an isogeny (it is injective with finite cokernel) and so it induces an isomorphism $\mathrm{res}^M_{A_M} : X^*(M)^\Gamma \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathfrak{a}_M^*$. We have an isomorphism

$$
\begin{aligned}
X^*(M)^\Gamma \otimes_{\mathbb{Z}} \mathbb{R} &\longrightarrow \mathrm{Hom}_{\mathrm{cont}}(M(F), \mathbb{R}_{>0}) \\
\chi \otimes s &\longmapsto (x \mapsto |\chi(x)|^s).
\end{aligned}
\tag{3.2}
$$

Fix a minimal parabolic subgroup P_0 of G and a Levi factor M_0 of P_0. Let $Y \subset X^*(A_{M_0})$ be the subgroup of characters which are trivial on $A_{M_0} \cap G_{\mathrm{der}}$. Recall from [**BT65**, Corollaire 5.8] that the set of roots of A_{M_0} in G is a root system in $(X^*(A_{M_0}), Y)$ (in the sense of §2.1 loc. cit.). Let $\Delta \subset X^*(A_{M_0})$ be the set of simple roots for the order corresponding to P_0. The rational Weyl group $N(A_{M_0}, G(F))/M_0(F)$ acts on $\mathfrak{a}^*_{M_0}$; fix an invariant inner product $(\cdot, \cdot)$ on $\mathfrak{a}^*_{M_0}$. For M a standard Levi subgroup of G the restriction map $X^*(A_{M_0}) \to X^*(A_M)$ induces a surjective map $\mathrm{res}^{A_{M_0}}_{A_M} : \mathfrak{a}^*_{M_0} \to \mathfrak{a}^*_M$. We also have a composite map in the other direction

$$
j^M_{M_0} : \mathfrak{a}^*_M \xrightarrow{(\mathrm{res}^M_{A_M})^{-1}} X^*(M)^\Gamma \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\mathrm{res}^M_{M_0}} X^*(M_0)^\Gamma \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\mathrm{res}^{M_0}_{A_{M_0}}} \mathfrak{a}^*_{M_0}
$$

and the composition $\mathrm{res}^{A_{M_0}}_{A_M} \circ j^M_{M_0}$ is $\mathrm{id}_{\mathfrak{a}^*_M}$. In fact one can check that $j^M_{M_0} \circ \mathrm{res}^{A_{M_0}}_{A_M}$ is the orthogonal projection $\mathfrak{a}^*_{M_0} \to j^M_{M_0}(\mathfrak{a}^*_M)$.

THEOREM 3.7 ([**Sil78**, Theorem 4.1]). (1) *Let P be a standard Levi subgroup of G (with respect to P_0) and M its Levi factor containing M_0. Let σ be a tempered irreducible smooth representation of $M(F)$ (in particular its central character is unitary). Let $\nu \in X^*(M)^\Gamma \otimes_{\mathbb{Z}} \mathbb{R}$ be such that for any $\alpha \in \Delta$ not occurring in M we have $(\mathrm{res}^M_{A_{M_0}} \nu, \alpha) > 0$. Consider ν as a character of $M(F)$ via (3.2), and denote by σ_ν the twist of σ by this character. Then the induced representation $i_P^G(\sigma_\nu)$ admits a unique irreducible quotient $J(P, \sigma, \nu)$. Let $\overline{P}$ be a parabolic subgroup of G which* *is opposite to P. We have $\dim_{\mathbb{C}} \operatorname{Hom}_G(i_P^G(\sigma_\nu), i_{\overline{P}}^G(\sigma_\nu)) = 1$ and any non-zero element in this line identifies $J(P,\sigma,\nu)$ with the unique irreducible subrepresentation of $i_{\overline{P}}^G(\sigma_\nu)$.*

(2) *Let π be an irreducible smooth representation of $G(F)$. There exists a unique triple (P,σ,ν) as above such that π is isomorphic to $J(P,\sigma,\nu)$.*

REMARK 3.8. It will be useful to reformulate the positivity condition on ν in terms of the absolute root system of G. First note that the condition does not depend on the choice of an admissible inner product on $\mathfrak{a}_{M_0}^*$. Let T be a maximal torus in $M_{0,F^{\mathrm{sep}}}$ and choose a Borel subgroup B of $G_{F^{\mathrm{sep}}}$ containing T and contained in $P_{0,F^{\mathrm{sep}}}$. Choose an admissible inner product $(\cdot,\cdot)_T$ on $X^*(T)\otimes_{\mathbb{Z}}\mathbb{R}$, i.e. one invariant under the absolute Weyl group. Consider the restriction map $X^*(T)\to X^*(A_{M_0})$, inducing a surjective map $\operatorname{res}^T_{A_{M_0}} : X^*(T)\otimes_{\mathbb{Z}}\mathbb{R}\to\mathfrak{a}_{M_0}^*$. It identifies $\mathfrak{a}_{M_0}^*$ with $\ker(\operatorname{res}^T_{A_{M_0}})^\perp$, and we can endow $\mathfrak{a}_{M_0}^*$ with the restriction of $(\cdot,\cdot)_T$. It turns out that this restriction is also an admissible inner product on $\mathfrak{a}_{M_0}^*$ for the relative Weyl group [**BT65**, §6.10]. The roots of A_{M_0} on $\operatorname{Lie} N$ are the restrictions of the roots of T on $\operatorname{Lie} N$. So the positivity condition in Theorem 3.7 is equivalent to $\langle \operatorname{res}^M_T \nu, \alpha^\vee\rangle > 0$ for any simple root $\alpha\in X^*(T)$ which does not occur in M.

For analogous results in the case where F is Archimedean see [**Lan89**] and [**Wal88**, Chapter 5].

In these notes we say nothing of the natural question of classifying unitary representations of connected reductive groups.

3.4. Harish-Chandra characters. Denote by $C_c^\infty(G(F))$ the space of locally constant and compactly supported functions $G(F)\to\mathbb{C}$. Recall that any such function is bi-invariant under some compact open subgroup of $G(F)$. Fix a Haar measure on $G(F)$. Let (V,π) be an admissible representation of $G(F)$. Any $f\in C_c^\infty(G(F))$ gives a linear map

$$\begin{aligned} V &\longrightarrow V \\ v &\longmapsto \int_{G(F)} f(g)\pi(g)v\, dg \end{aligned}$$

and its image is contained in V^K for some compact open subgroup K of $G(F)$. In particular $\pi(f)$ has finite range and we may consider $\Theta_\pi(f) = \operatorname{tr}\pi(f)$. The linear form $\Theta_\pi : C_c^\infty(G(F))\to\mathbb{C}$ is called the Harish-Chandra character of π. A standard result in representation theory of finite-dimensional associative algebras implies that the Harish-Chandra characters Θ_π of the irreducible smooth representations of $G(F)$ (up to isomorphism) are linearly independent. In particular a smooth representation of finite length is characterized up to semi-simplification by its Harish-Chandra character.

Denote by G_{rs} the regular semi-simple locus in G, an open dense subscheme. Recall that $G(F)\smallsetminus G_{\mathrm{rs}}(F)$ has measure zero.

THEOREM 3.9 ([**HC99**, Theorem 16.3]). *Assume that F is a non-Archimedean local field of characteristic zero. Let (V,π) be an irreducible smooth representation of $G(F)$. There exists a unique element of $L^1_{\mathrm{loc}}(G(F))$, also denoted Θ_π, such that* *for any* $f \in C_c^\infty(G(F))$ *we have*

$$\operatorname{tr} \pi(f) = \int_{G(F)} \Theta_\pi(g) f(g) dg.$$

Moreover Θ_π *is represented by a unique locally constant function on* $G_{\mathrm{rs}}(F)$*.*

To our knowledge this result is unfortunately not known in full generality in positive characteristic, but see [**CGH14**]. Harish-Chandra characters behave well with respect to parabolic induction [**vD72**] and Jacquet functors [**Cas77**].

See [**Wal88**, Chapter 8] for the Archimedean case.

4. Langlands dual groups

We recall the definition of Langlands dual groups. We refer to [**Bor79**, §I.2] for details not recalled below. In this section F could be any field, $\overline{F}$ is a separable closure of F and we denote $\Gamma = \mathrm{Gal}(\overline{F}/F)$.

4.1. Based root data. Let G be a connected reductive group over F. There exists a finite separable extension E/F such that G_E admits a Killing pair (also called Borel pair) (B, T) [**DGA$^+$11**, Exposé XXII Corollaire 2.4 and Proposition 5.5.1]. We may and do assume that E/F is a subextension of $\overline{F}/F$. Associated to (G_E, B, T) we have a based (reduced) root datum $(X, R, R^\vee, \Delta)$ where X is the group of characters of T, $R \subset X$ the set of roots of T in G_E, $R^\vee$ the set of coroots (a subset of $X^\vee = \mathrm{Hom}(X, \mathbb{Z})$, the group of cocharacters of T) and $\Delta \subset R$ the set of simple roots corresponding to B[1]. The group $G(E)$ acts (by conjugation) transitively on the set of Killing pairs in G_E (Exposé XXVI Corollaire 5.7 (ii) and Corollaire 1.8 loc. cit.) and the (scheme-theoretic) stabilizer of (B, T) is T (Exposé XXII Cor 5.3.12 and Proposition 5.6.1 loc. cit.), which centralizes T. It follows that other choices of Killing pair in G_E yield based root data *canonically* isomorphic to $(X, R, R^\vee, \Delta)$, and so do other choices for E.

We also obtain a continuous action of Γ on this based root datum, that we now recall. The group $\mathrm{Gal}(E/F)$ acts on the set of closed subgroups of G_E: if $G = \operatorname{Spec} A$ for a Hopf algebra A over F and a closed subgroup H corresponds to an ideal I of $A \otimes_F E$, then for $\sigma \in \mathrm{Gal}(E/F)$ we let $\sigma(H)$ be the closed subgroup corresponding to $\sigma(I)$. If $K = \operatorname{Spec} B$ is a linear algebraic group over F and $\lambda : H \to K_E$ is a morphism, dual to a morphism of Hopf algebras $\lambda^\sharp : B \otimes_F E \to (A \otimes_F E)/I$, define $\sigma(\lambda) : \sigma(H) \to K_E$ as dual to

$$\sigma \circ \lambda^\sharp \circ \sigma^{-1} : B \otimes_F E \longrightarrow (A \otimes_F E)/\sigma(I).$$

Now for $\sigma \in \mathrm{Gal}(E/F)$ there is a unique $T(E)g_\sigma \in T(E)\backslash G(E)$ such that we have $\sigma(B, T) = \mathrm{Ad}(g_\sigma^{-1})(B, T)$, and we get a well-defined isomorphism $\mathrm{Ad}(g_\sigma) : \sigma(T) \simeq T$. We obtain an action of Γ on $X = X^*(T)$ such that $\sigma \in \mathrm{Gal}(E/F)$ maps $\lambda : T \to \mathrm{GL}_{1,E}$ to $\sigma(\lambda) \circ \mathrm{Ad}(g_\sigma)^{-1}$. It is straightforward to check that this action preserves R and Δ and that the dual action on $X^\vee$ preserves $R^\vee$. It is routine to check that if we choose another triple (E', B', T') instead of (E, B, T) to obtain a based root datum $(X', R', R^{',\vee}, \Delta')$ with continuous action of Γ, the canonical isomorphism between $(X, R, R^\vee, \Delta$ and $(X', R', R^{',\vee}, \Delta')$ is Γ-equivariant.

[1]Strictly speaking we should also include in the datum the bijection $R \to R^\vee$ as in [**DGA$^+$11**, Exposé XXI], or include the orthogonal of $R^\vee$ in X as in [**BT65**, §2.1].

We denote by brd_F the resulting functor from the groupoid of connected reductive groups over F to the groupoid of based root data with continuous action of Γ.

Definition 4.1. Let G be a connected reductive group over F. Define a groupoid $\mathcal{IT}(G)$ as follows.

- The objects of $\mathcal{IT}(G)$ are the inner twists of G, i.e. pairs (G', ψ) consisting of a connected reductive group G' over F and an isomorphism $\psi : G_{\overline{F}} \simeq G'_{\overline{F}}$ such that for any $\sigma \in \Gamma$ the automorphism $\psi^{-1}\sigma(\psi)$ of $G_{\overline{F}}$ is inner.
- A morphism between two inner twists (G_1, ψ_1) and (G_2, ψ_2) of G is an element $g \in G_{\mathrm{ad}}(\overline{F})$ such that for any $\sigma \in \Gamma$ we have

$$\psi_2^{-1}\sigma(\psi_2) = \mathrm{Ad}(g)\psi_1^{-1}\sigma(\psi_1)\,\mathrm{Ad}(\sigma(g))^{-1}. \tag{4.1}$$

Remark 4.2.

(1) One can check that any inner twist $\psi : G_{\overline{F}} \to G'_{\overline{F}}$ yields a canonical isomorphism $\mathrm{brd}_F(G) \simeq \mathrm{brd}_F(G')$.

(2) For an inner twist $\psi : G_{\overline{F}} \to G'_{\overline{F}}$ the map

$$\begin{aligned} \Gamma &\longrightarrow G_{\mathrm{ad}}(\overline{F}) \\ \sigma &\longmapsto \psi^{-1}\sigma(\psi) \end{aligned}$$

is a 1-cocycle, i.e. an element of $Z^1_{\mathrm{cont}}(\Gamma, G_{\mathrm{ad}}) = Z^1(F, G_{\mathrm{ad}})$.

(3) The relations (4.1) imply that the isomorphism

$$\psi_2\,\mathrm{Ad}(g)\psi_1^{-1} : G_{1,\overline{F}} \longrightarrow G_{2,\overline{F}}$$

is defined over F, i.e. descends to an isomorphism $G_1 \simeq G_2$.

(4) For an inner twist (G', ψ) of G we have an isomorphism

$$\begin{aligned} \mathrm{Aut}_{\mathcal{IT}(G)}(G', \psi) &\longrightarrow G'_{\mathrm{ad}}(F) \\ g &\longmapsto \psi(g). \end{aligned}$$

Proposition 4.3. *Let b be a based root datum with continuous action of Γ. Let $\mathcal{CRG}_b$ be the groupoid of pairs (G, α) where G is a connected reductive group over F and $\alpha : b \simeq \mathrm{brd}_F(G)$ is an isomorphism of based root data with action of Γ, with obvious morphisms. (In other words $\mathcal{CRG}_b$ is the groupoid fiber of b for brd_F)*

(1) *There exists an object (G^*, α^*) of $\mathcal{CRG}_b$ such that G^* is quasi-split. Two such objects are isomorphic.*

(2) *Any object (G, α) of $\mathcal{CRG}_b$ yields equivalences of groupoids*

$$Z^1(F, G_{\mathrm{ad}}) \xleftarrow{\sim} \mathcal{IT}(G) \xrightarrow{\sim} \mathcal{CRG}_b.$$

This gives in particular a bijection between $H^1(F, G_{\mathrm{ad}})$ and the set of isomorphism classes in $\mathcal{CRG}_b$.

Proof. This is a reformulation of [**DGA**$^+$**11**, Exposé XXIV Théorème 3.11] in the case where the base is the spectrum of a field, also using 3.9.1 loc. cit. to prove uniqueness in the first point. □

To sum up, we can "classify" connected reductive groups over F as follows:

- fix a representative in each isomorphism class of based root datum with continuous action of Γ,
- for each such representative b, fix a quasi-split connected reductive group G^* over F together with an isomorphism $\mathrm{brd}_F(G^*) \simeq b$,

- for each element of $H^1(F, G^*_{\mathrm{ad}})$ choose an inner twist (G, ψ) of G^* representing it.

Up to isomorphism each connected reductive group G over F arises in this way. It can happen that an isomorphism class of connected reductive groups arises more than once, because $H^1(F, G_{\mathrm{ad}}) \to H^1(F, \mathrm{Aut}(G))$ is not injective in general.

4.2. Langlands dual groups. Let C be an algebraically closed field of characteristic zero. Let G be a connected reductive group over F and let $\mathrm{brd}_F(G) = (X, R, R^\vee, \Delta)$ be its associated based root datum endowed with a continuous action of Γ. Let $(\widehat{G}, \mathcal{B}, \mathcal{T}, (X_\alpha)_{\alpha\in\Delta^\vee})$ be the pinned connected reductive group over C with associated based root datum $(X^\vee, R^\vee, R, \Delta^\vee)$, i.e. the *dual* of $\mathrm{brd}_F(G)$ (ignoring the action of Γ for now). The choice of a pinning induces a splitting of the extension

$$1 \to \widehat{G}_{\mathrm{ad}} \to \mathrm{Aut}(\widehat{G}) \to \mathrm{Out}(\widehat{G}) \to 1$$

because the subgroup $\mathrm{Aut}(\widehat{G}, \mathcal{B}, \mathcal{T}, (X_\alpha)_{\alpha\in\Delta^\vee})$ of $\mathrm{Aut}(\widehat{G})$ maps bijectively onto $\mathrm{Out}(\widehat{G})$ [**DGA$^+$11**, Exposé XXIV Théorème 1.3]. As explained loc. cit. we also have an isomorphism

$$\mathrm{Out}(\widehat{G}) \simeq \mathrm{Aut}(X^\vee, R^\vee, R, \Delta^\vee) \simeq \mathrm{Aut}(X, R, R^\vee, \Delta)$$

and so we have an action of Γ on $\widehat{G}$ (preserving the pinning and factoring through a finite Galois group). Denote ${}^LG = \widehat{G} \rtimes \Gamma$ the Langlands dual group, also called L-group. It is sometimes useful (or just convenient) to replace Γ by a finite Galois group or by the Weil group in this semi-direct product.

One can give a more pedantic definition of Langlands dual groups in order to avoid the inelegant choice of pinning. Namely, define an L-group for G as an extension ${}^LG$ of Γ by $\widehat{G}$, where $\widehat{G}$ is a split connected reductive group endowed with an isomorphism of its base root datum with the dual of that of G, such that the induced morphism $\Gamma \to \mathrm{Out}(\widehat{G})$ is as above, and endowed with a $\widehat{G}$-conjugacy class of splittings $\Gamma \to {}^LG$, called distinguished splittings, such that any (equivalently, one) of these splittings s preserves a pinning of $\widehat{G}$. It is not necessary to specify the pinning, since for a distinguished splitting s we have that $\widehat{G}^{s(\Gamma)}$ acts transitively on the set of such pinnings: see [**Kot84**, Corollary 1.7]. In the other direction, for a pinning of $\widehat{G}$ fixed by a distinguished splitting s, the set of distinguished splittings fixing this pinning is parametrized by

$$\ker\left(Z^1(\Gamma, Z(\widehat{G})) \to H^1(\Gamma, s, \widehat{G})\right) \tag{4.2}$$

where the notation $H^1(\Gamma, s, \widehat{G})$ means the first cohomology set for the action of Γ on $\widehat{G}$ via s and conjugation in ${}^LG$. Note that all distinguished splittings induce the same action of Γ on $Z(\widehat{G})$. By Lemma 1.6 loc. cit. the kernel (4.2) is simply the group of coboundaries $B^1(\Gamma, Z(\widehat{G}))$, and so the distinguished splittings fixing a given pinning of $\widehat{G}$ form a single conjugacy class by $Z(\widehat{G})$.

By Proposition 4.3 for two connected reductive groups G_1 and G_2 their Langlands dual groups ${}^LG_1$ and ${}^LG_2$ are isomorphic as extensions of Γ if and only if G_1 and G_2 are inner forms of each other, and in this case they are even isomorphic as extensions endowed with conjugacy classes of distinguished splittings.

The construction of the Langlands dual group is not functorial for arbitrary morphisms between connected reductive groups, however in the following cases functoriality is straightforward.

- Let G be a quasi-split connected reductive group and (B, T) a Borel pair (defined over F). Choose a distinguished splitting $s_G : \Gamma \to {}^L G$ preserving a pinning $(\mathcal{B}, \mathcal{T}, (X_\alpha)_\alpha)$ of $\widehat{G}$ and a distinguished splitting $s_T : \Gamma \to {}^L T$. Then the canonical isomorphism $\widehat{T} \simeq \mathcal{T}$ extends to an embedding ${}^L T \hookrightarrow {}^L G$ whose composition with s_T is s_G.
- For $G = G_1 \times_F G_2$ we can identify ${}^L G$ with ${}^L G_1 \times_\Gamma {}^L G_2$.
- A central isogeny [**DGA+11**, Exposé XXII Définition 4.2.9] $G \to H$ induces a surjective morphism with finite kernel ${}^L H \to {}^L G$.
- There are weaker forms of functoriality. Let G be a connected reductive group and T a maximal torus of G defined over F. Choose a Borel subgroup B of $G_{\overline{F}}$ containing $T_{\overline{F}}$ and a splitting $s : \Gamma \to {}^L G$ preserving a pinning $(\mathcal{B}, \mathcal{T}, (X_\alpha)_\alpha)$ of $\widehat{G}$. We have a canonical isomorphism $\widehat{T} \simeq \mathcal{T}$, but the Galois actions differ by a 1-cocycle taking values in the Weyl group. In general we don't have a canonical embedding ${}^L T \hookrightarrow {}^L G$ (see [**LS87**, §2.6] and [**Kal**] however), but note that the induced embedding $Z(\widehat{G}) \hookrightarrow \widehat{T}$ is Γ-equivariant.

In the next section we recall how the first case generalizes to parabolic subgroups in arbitrary connected reductive groups.

4.3. Parabolic subgroups and L-embeddings. A parabolic subgroup $\mathcal{P}$ of ${}^L G$ is a closed subgroup mapping onto Γ and such that $\mathcal{P}^0 := \mathcal{P} \cap \widehat{G}$ is a parabolic subgroup of $\widehat{G}$. The set of parabolic subgroups is clearly stable under conjugation by $\widehat{G}$. If $\mathcal{P}$ is a parabolic subgroup of ${}^L G$ then $\mathcal{P}$ is the normalizer of $\mathcal{P}^0$ in ${}^L G$.

Choose a Killing pair $(\mathcal{B}, \mathcal{T})$ of $\widehat{G}$. Recall that a parabolic subgroup of $\widehat{G}$ is conjugated to a unique one containing $\mathcal{B}$, and that parabolic subgroups of $\widehat{G}$ containing $\mathcal{B}$ correspond bijectively to subsets of $\Delta^\vee$ (or Δ, using the bijection $\alpha \mapsto \alpha^\vee$), by associating to $\mathcal{P}^0$ the set of $\alpha \in \Delta^\vee$ (seen as characters of $\mathcal{T}$) such that $-\alpha$ is a root of $\mathcal{T}$ in $\mathcal{P}^0$. Embed $\mathcal{B}$ in a pinning $(\mathcal{B}, \mathcal{T}, (X_\alpha)_{\alpha \in \Delta^\vee})$ of $\widehat{G}$, and let $s : \Gamma \to {}^L G$ be a distinguished section fixing this pinning. Then $\mathcal{B}s(\Gamma)$ is a (minimal) parabolic subgroup of ${}^L G$, and any parabolic subgroup of ${}^L G$ is conjugated under $\widehat{G}$ to one containing $\mathcal{B}s(\Gamma)$. A parabolic subgroup $\mathcal{P}^0$ of $\widehat{G}$ containing $\mathcal{B}$ is such that its normalizer $\mathcal{P}$ in ${}^L G$ maps onto Γ (i.e. $\mathcal{P}$ is a parabolic subgroup of ${}^L G$) if and only if the corresponding subset of $\Delta^\vee$ is stable under Γ. Therefore $\widehat{G}$-conjugacy classes of parabolic subgroups of ${}^L G$ also correspond bijectively to Γ-stable subsets of $\Delta^\vee$.

Using the bijection between Δ and $\Delta^\vee$ we obtain a bijection between the set of Γ-stable $G(\overline{F})$-conjugacy classes of parabolic subgroups of $G_{\overline{F}}$ and the set of $\widehat{G}$-conjugacy classes of parabolic subgroups of ${}^L G$. The obvious map from the set of $G(F)$-conjugacy classes of parabolic subgroups of G to the set of Γ-stable $G(\overline{F})$-conjugacy classes of parabolic subgroups of $G_{\overline{F}}$ is injective, and it is surjective if and only if G is quasi-split.

Recall from [**Bor79**, §3.4] that if $\mathcal{P}$ is a parabolic subgroup of ${}^L G$ and $\mathcal{M}^0$ is a Levi factor of $\mathcal{P}^0$ then the normalizer $\mathcal{M}$ of $\mathcal{M}^0$ in $\mathcal{P}$ maps onto Γ and $\mathcal{P}$ is the semi-direct product of its unipotent radical and $\mathcal{M}$. In this situation we say that

$\mathcal{M}$ is a Levi factor of $\mathcal{P}$, and a Levi subgroup of ${}^LG$. See Lemma 3.5 loc. cit. for another characterization of Levi subgroups.

Let P be a parabolic subgroup of G. Choose a distinguished splitting $s : \Gamma \to {}^LG$ stabilizing a pinning $\mathcal{E} = (\mathcal{B}, \mathcal{T}, (X_\alpha)_{\alpha \in \Delta^\vee})$ of $\widehat{G}$, and let $\mathcal{P}$ be the parabolic subgroup of ${}^LG$ corresponding to P and containing $\mathcal{B}$. Let $M = P/N$ be the reductive quotient of P. Taking Killing pairs inside P in the definition of brd_F we obtain an isomorphism between $\mathrm{brd}_F(M)$ and $(X, R_P, R_P^\vee, \Delta_P)$ where Δ_P is the set of simple roots $\alpha \in \Delta$ such that $-\alpha$ also occurs in P, $R_P = R \cap \mathrm{span}(\Delta_P)$, $\Delta_P^\vee = \{\alpha^\vee \,|\, \alpha \in \Delta_P\}$ and $R_P^\vee = R^\vee \cap \mathrm{span}(\Delta_P^\vee)$. Let $\mathcal{E}_M = (\mathcal{B}_M, \mathcal{T}_M, (Y_\alpha)_\alpha)$ be a pinning of $\widehat{M}$ and $s_M : \Gamma \to {}^LM$ a corresponding distinguished splitting. These choices determine and embedding

$$\iota[P, \mathcal{E}, s, \mathcal{E}_M, s_M] : {}^LM \longrightarrow {}^LG$$

characterized by the following properties.

- It maps $(\mathcal{B}_M, \mathcal{T}_M)$ to $(\mathcal{B}, \mathcal{T})$, and on $\mathcal{T}_M$ it is the isomorphism $\mathcal{T}_M \simeq \mathcal{T}$ induced by the above embedding $\mathrm{brd}_F(M) \hookrightarrow \mathrm{brd}_F(G)$,
- it maps $\mathcal{E}_M$ to $\mathcal{E}$, and
- we have $\iota[P, \mathcal{E}, s, \mathcal{E}_M, s_M] \circ s_M = s$.

The image of $\iota[P, \mathcal{E}, s, \mathcal{E}_M, s_m]$ is clearly a Levi subgroup of ${}^LG$. The formation of $\iota[P, \mathcal{E}, s, \mathcal{E}_M, s_M]$ satisfies obvious equivariance properties with respect to conjugation by $\widehat{M}$ and $\widehat{G}$. In particular we have an embedding $\iota_P : {}^LM \to {}^LG$ well-defined up to conjugation by $\widehat{G}$.

Lemma 4.4. *Let M be a Levi subgroup of G. Let P and P' be parabolic subgroups of G admitting M as a Levi factor. Then ι_P and $\iota_{P'}$ are conjugated by $\widehat{G}$.*

This statement is contained in [**Lan89**, Lemma 2.5] but we give a self-contained proof.

Proof. First we recall a general construction. Fix a pinning $\mathcal{E} = (\mathcal{B}, \mathcal{T}, (X_\alpha)_\alpha)$ in $\widehat{G}$ and a distinguished splitting $s : \Gamma \to {}^LG$ fixing it. For a Killing pair (B, T) in $G_{\overline{F}}$ we denote by $\gamma[(B, T), (\mathcal{B}, \mathcal{T})]$ the isomorphism $X^*(\mathcal{T}) \simeq X_*(T)$. Considering Weyl groups inside automorphism groups of tori this also induces an isomorphism

$$\omega[(B, T), (\mathcal{B}, \mathcal{T})] : W(T, G_{\overline{F}}) \simeq W(\mathcal{T}, \widehat{G}).$$

We have an action of Γ on $W(T, G_{\overline{F}})$: for $\sigma \in \Gamma$ let $T(\overline{F})g_\sigma \in T(\overline{F})\backslash G(\overline{F})$ be the class for which $\sigma(B, T) = \mathrm{Ad}(g_\sigma^{-1})(B, T)$, then $x \mapsto \mathrm{Ad}(g_\sigma)(\sigma(x))$ induces an automorphism of $W(T, G_{\overline{F}})$. One can check that the isomorphism $\omega[(B, T), (\mathcal{B}, \mathcal{T})]$ is Γ-equivariant for this action on $W(T, G_{\overline{F}})$ and the action via s on $W(\mathcal{T}, \widehat{G})$.

Fix $\mathcal{E}$, s, $\mathcal{E}_M$ and s_M as above. Fix a Borel pair (B_M, T) in $M_{\overline{F}}$. This determines two Borel subgroups B and B' in $G_{\overline{F}}$, characterized by the properties $B \cap M_{\overline{F}} = B_M$ and $N_{\overline{F}} \subset B$ and similarly for B'. There is a unique $x \in W(T, G_{\overline{F}})$ for which $\mathrm{Ad}(x)(B, T) = (B', T)$. Let $n : W(\mathcal{T}, \widehat{G}) \to N(\mathcal{T}, \widehat{G})$ be the set-theoretic splitting determined by $\mathcal{E}$ [**Spr98**, §9.3.3]. Denote $w = n(\omega[(B, T), (\mathcal{B}, \mathcal{T})](x))$. We claim that we have

$$\mathrm{Ad}(w) \circ \iota[P', \mathcal{E}, s, \mathcal{E}_M, s_M] = \iota[P, \mathcal{E}, s, \mathcal{E}_M, s_M]. \tag{4.3}$$

To simplify notation in the rest of the proof we abbreviate $\iota = \iota[P, \mathcal{E}, s, \mathcal{E}_M, s_M]$ and $\iota' = \iota[P', \mathcal{E}, s, \mathcal{E}_M, s_M]$.

First we check that ι and ι' coincide on $\mathcal{T}_M$. Denote $T' = T$ for clarity. We have $(B', T') = \mathrm{Ad}(x)(B, T)$ so if we also denote by $\mathrm{Ad}(x)$ the induced isomorphism $X_*(T) \simeq X_*(T')$ we have $\mathrm{Ad}(x)\gamma[(B, T), (\mathcal{B}, \mathcal{T})] = \gamma[(B', T'), (\mathcal{B}, \mathcal{T})]$. Here because $T' = T$ we obtain

$$\gamma[(B', T), (\mathcal{B}, \mathcal{T})] = \gamma[(B, T), (\mathcal{B}, \mathcal{T})] \circ \omega[(B, T), (\mathcal{B}, \mathcal{T})](x).$$

The isomorphism $\iota|_{\mathcal{T}_M} : \mathcal{T}_M \simeq \mathcal{T}$ is dual to the isomorphism

$$\gamma[(B_M, T), (\mathcal{B}_M, \mathcal{T}_M)]^{-1} \circ \gamma[(B, T), (\mathcal{B}, \mathcal{T})] : X^*(\mathcal{T}) \simeq X^*(\mathcal{T}_M).$$

Similarly $\iota'|_{\mathcal{T}_M} : \mathcal{T}_M \simeq \mathcal{T}$ is dual to the isomorphism

$$\begin{aligned} &\gamma[(B_M, T), (\mathcal{B}_M, \mathcal{T}_M)]^{-1} \circ \gamma[(B', T), (\mathcal{B}, \mathcal{T})] \\ =&\gamma[(B_M, T), (\mathcal{B}_M, \mathcal{T}_M)]^{-1} \circ \gamma[(B, T), (\mathcal{B}, \mathcal{T})] \circ \omega[(B, T), (\mathcal{B}, \mathcal{T})](x) \end{aligned}$$

and the equality

$$\iota'|_{\mathcal{T}_M} = \omega[(B, T), (\mathcal{B}, \mathcal{T})](x)^{-1} \circ \iota|_{\mathcal{T}_M}$$

follows.

To check that the equality 4.3 holds on $\widehat{M}$ it is enough to check that we have $\mathrm{Ad}(w)\iota(Y_\alpha) = \iota'(Y_\alpha)$ for any $\alpha \in \Delta(\mathcal{T}_M, \mathcal{B}_M)$. We have

$$\iota(Y_\alpha) = X_\beta \text{ and } \iota'(Y_\alpha) = X_{\beta'}$$

where

$$\begin{aligned} \beta &= \gamma[(B, T), (\mathcal{B}, \mathcal{T})]^{-1}\gamma[(B_M, T), (\mathcal{B}_M, \mathcal{T}_M)](\alpha), \\ \beta' &= \gamma[(B', T), (\mathcal{B}, \mathcal{T})]^{-1}\gamma[(B_M, T), (\mathcal{B}_M, \mathcal{T}_M)](\alpha) \\ &= w^{-1}(\beta) \end{aligned}$$

both belong to $\Delta(\mathcal{T}, \mathcal{B})$. By [**Spr98**, Proposition 9.3.5] we have $X_\beta = \mathrm{Ad}(w)(X_{\beta'})$.

Finally we need to check $\mathrm{Ad}(w) \circ s = s$, i.e. that w commutes with $s(\Gamma)$. For $\sigma \in \Gamma$ and $y \in W(\mathcal{T}, \widehat{G})$ we have $s(\sigma)n(y)s(\sigma)^{-1} = n(\sigma(y))$ and so it is enough to check that $w\mathcal{T} \in W(\mathcal{T}, \widehat{G})$ is fixed by Γ. For any $\sigma \in \Gamma$ there exists $g_\sigma \in M(\overline{F})$ such that $\sigma(B_M, T) = \mathrm{Ad}(g_\sigma^{-1})(B_M, T)$ and this implies $\sigma(B, T) = \mathrm{Ad}(g_\sigma^{-1})(B, T)$ and $\sigma(B', T) = \mathrm{Ad}(g_\sigma^{-1})(B', T)$ because N and N' are both defined over F. A simple computation shows that we have $\mathrm{Ad}(g_\sigma)(\sigma(x)) = x$ in $W(T, G_{\overline{F}})$, i.e. x is Γ-invariant. □

The lemma shows that for a Levi subgroup M of G we have an embedding $\iota_M : {}^L M \to {}^L G$, well-defined up to conjugation by $\widehat{G}$. We call the image of such an embedding a *relevant* Levi subgroup of ${}^L G$.

5. Langlands parameters

In this section F is a local field.

5.1. Weil-Deligne groups. We briefly recall the definition of Weil-Deligne groups of local fields. We refer the reader to [**Tat79**] for more details.

If $F \simeq \mathbb{C}$ define $\mathrm{W}_F = F^\times$. If $F \simeq \mathbb{R}$ define W_F as the unique non-split central extension

$$1 \to \overline{F}^\times \to \mathrm{W}_F \to \mathrm{Gal}(\overline{F}/F) \to 1$$

where $\mathrm{Gal}(\overline{F}/F)$ acts on $\overline{F}^\times$ in the natural way. Explicitly, $\mathrm{W}_F = \overline{F}^\times \sqcup j\overline{F}^\times$ with $j^2 = -1$.

If F is a non-Archimedean local field, we have a short exact sequence of topological groups

$$1 \to I_F \to \mathrm{Gal}(\overline{F}/F) \to \mathrm{Gal}(\overline{k}/k) \to 1$$

where k is the residue field of F and I_F is called the inertia subgroup of $\mathrm{Gal}(\overline{F}/F)$. Since k is finite, say of cardinality q, $\mathrm{Gal}(\overline{k}/k)$ is isomorphic to $\widehat{\mathbb{Z}}$ and topologically generated by the Frobenius automorphism $x \mapsto x^q$. This automorphism generates a natural subgroup $\mathbb{Z}$ of $\mathrm{Gal}(\overline{k}/k)$, and the Weil group W_F is defined as its preimage, a dense subgroup of $\mathrm{Gal}(\overline{F}/F)$. Instead of the induced topology, we endow W_F with the topology making I_F an open subgroup, with its topology induced from that of $\mathrm{Gal}(\overline{F}/F)$.

Recall that the Artin reciprocity map is an isomorphism $\mathrm{W}_F^{\mathrm{ab}} \simeq F^\times$. Composing with the norm $||\cdot|| : F^\times \to \mathbb{R}_{>0}$ we get a continuous morphism still denoted $||\cdot|| : \mathrm{W}_F \to \mathbb{R}_{>0}$.

For non-Archimedean F, we now recall three possible definitions for the Weil-Deligne group.

(1) $\mathrm{W}'_F := \mathbb{G}_a \rtimes \mathrm{W}_F$, where the action of W_F on $\mathbb{G}_a$ is by $w(x) = ||w||x$.
(2) $\mathrm{WD}_F := \mathrm{W}_F \times \mathrm{SL}_2$, where the second factor is the algebraic group over $\mathbb{Q}$.
(3) the (unnamed) locally compact topological group $\mathrm{W}_F \times \mathrm{SU}(2)$.

For Archimedean F it will be convenient to denote $\mathrm{WD}_F = \mathrm{W}_F$.

5.2. Langlands parameters. First assume that F is non-Archimedean.

For the first version of the Weil-Deligne group, a Weil-Deligne Langlands parameter[2] is a pair (ρ, N) such that

- $\rho : \mathrm{W}_F \to {}^L G$ is a continuous representation, i.e. there exists an open subgroup U of I_F which acts trivially on $\widehat{G}$ and is mapped to $1 \times U \subset \widehat{G} \rtimes \Gamma$, such that the composition with the projection ${}^L G \to \Gamma$ is the usual map,
- $N \in \mathrm{Lie}\,\widehat{G}$ satisfies $\rho(w) N \rho(w)^{-1} = ||w|| N$ for all $w \in \mathrm{W}_F$ (this forces N to be nilpotent),
- for any $w \in \mathrm{W}_F$ (equivalently, for some $w \in \mathrm{W}_F \smallsetminus \mathrm{I}_F$) we have that $\rho(w)$ is semi-simple.

One of the motivations for using the first version of the Weil-Deligne group, rather than the other two, is the ℓ-adic monodromy theorem [**Tat79**, Theorem 4.2.1]. This roughly says that for a prime ℓ not equal to the residual characteristic of F and for $C = \overline{\mathbb{Q}_\ell}$[3], any continuous morphism $\mathrm{W}_F \to {}^L G$ *for the natural topology on* $\widehat{G}$ compatible with ${}^L G \to \Gamma$ is given by a pair (ρ, N) satisfying the first two conditions above. Continuous ℓ-adic Galois representations occur naturally in algebraic geometry (Tate modules of elliptic curves over F, or more generally in the étale cohomology of varieties defined over F). Another reason for preferring W'_F is that this version requires fewer "choices of a square root of q" in the local Langlands correspondence, and is more obviously compatible with parabolic induction (property (10) in Conjecture 6.1 below).

For the second version WD_F, over any algebraically closed field C of characteristic zero, Langlands parameters are defined as morphisms $\phi : \mathrm{W}_F \times \mathrm{SL}_2(C) \to {}^L G$

[2] This terminology is not standard.
[3] One could work with a finite extension of $\mathbb{Q}_\ell$ instead.

which are compatible with ${}^LG \to \Gamma$, continuous and semi-simple on the first factor and algebraic on the second factor.

For the third version, we need to assume $C = \mathbb{C}$ and we consider continuous (for the natural topology on $\widehat{G}$) semi-simple morphisms $\phi : \mathrm{W}_F \times \mathrm{SU}(2) \to {}^LG$ which are compatible with ${}^LG \to \Gamma$. By restriction via $\mathrm{SU}(2) \subset \mathrm{SL}_2(\mathbb{C})$ we obtain exactly the same morphisms as in the second version, essentially because $\mathrm{SL}_2(\mathbb{C})$ is the complexification of the compact Lie group $\mathrm{SU}(2)$.

Recall that we have already chosen a square root of q in C in order to normalize parabolic induction. We have a natural map from Langlands parameters to Weil-Deligne Langlands parameters:

$$\phi \mapsto \left(\phi \circ \iota_W, \mathrm{d}\phi|_{\mathrm{SL}_2}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) \tag{5.1}$$

where $\iota_W(w) = (w, \mathrm{diag}(||w||^{1/2}, ||w||^{-1/2})$. By a refinement of the Jacobson-Morozov theorem (see [**GR10**, Lemma 2.1]) this induces a bijection between sets of $\widehat{G}$-conjugacy classes of parameters.

If F is Archimedean we assume $C = \mathbb{C}$ and define Langlands parameters as semi-simple continuous morphisms $\phi : \mathrm{W}_F \to {}^LG$ which are compatible with ${}^LG \to \Gamma$.

We will denote by $\Phi(G)$ the set of $\widehat{G}$-conjugacy classes of Langlands parameters taking values in ${}^LG$. As explained above all versions of the Weil-Deligne group give equivalent sets of $\widehat{G}$-conjugacy classes.

5.3. Reductions. We briefly recall from [**SZ**] the Langlands classification for parameters. Assume $C = \mathbb{C}$ and let $\mathrm{cl}(\phi) \in \Phi(G)$. Applying the polar decomposition to $\phi(w)$ for any $w \in W_F$ with positive valuation, we obtain a canonical tuple $(\mathcal{P}, \mathcal{M}, \phi_0, \chi)$ giving a decomposition

$$\phi = \phi_0 \chi \tag{5.2}$$

subject to the following conditions.

- The subgroup $\mathcal{P}$ of ${}^LG$ is a parabolic subgroup and $\mathcal{M}$ is a Levi subgroup of $\mathcal{P}$. We denote by $\mathcal{N}$ the unipotent radical of $\mathcal{P}$.
- ϕ_0 is a Langlands parameter taking values in $\mathcal{M}$ and bounded on W_F.
- $\chi \in Z^1(\mathrm{W}_F, X_*(Z(\mathcal{M})^0) \otimes_{\mathbb{Z}} \mathbb{R}_{>0})$ where $X_*(Z(\mathcal{M})^0) \otimes_{\mathbb{Z}} \mathbb{R}_{>0}$ is seen as a subgroup of the torus $X_*(Z(\mathcal{M})^0) \otimes_{\mathbb{Z}} \mathbb{C}^\times = Z(\mathcal{M})^0$.
- The eigenvalues of $\chi(\mathrm{Frob})$ if F is non-Archimedean (resp. $\chi(x)$ for any real $x > 1$ if F is Archimedean) on $\mathrm{Lie}\,\mathcal{N}$ are all greater than 1.

This corresponds to the Langlands classification (Theorem 3.7) using Remark 3.8. This reduction explains why we are mainly interested in bounded parameters ϕ. We will also call such parameters tempered. A nice property of tempered (or more generally, essentially tempered, allowing twists by cocycles $\mathrm{W}_F \to Z(\widehat{G})$) parameters is that the restriction of the map (5.1), associating Weil-Deligne parameters to Langlands parameters, to the set of tempered parameters is injective: see [**?**, Corollary 3.15]. This may also be proved by observing that for a tempered Langlands parameter ϕ with associated Weil-Deligne Langlands parameter (ρ, N) we can recover $\phi|_{\mathrm{W}_F}$ and the semi-simple element in the $\mathfrak{sl}_2$-triple corresponding to $\phi|_{\mathrm{SL}_2}$ from the polar decomposition 5.2 of ρ, and using [**?**, Corollary 3.5]. In particular the centralizers in both versions coincide in the tempered case. This is not true in general, see Example 3.8 loc. cit.

The following proposition does not assume $C = \mathbb{C}$.

PROPOSITION 5.1 ([**Bor79**, Proposition 3.6]). *Let $\phi : \mathrm{WD}_F \to {}^L G$ be a Langlands parameter. The Levi subgroups of ${}^L G$ which are minimal among those containing $\phi(\mathrm{WD}_F)$ are all conjugated under the centralizer of ϕ in $\widehat{G}$.*

This proposition may be seen as a generalization of the isotypical decomposition of a semi-simple linear group representation. A Langlands parameter ϕ is called *essentially discrete* if this Levi subgroup is ${}^L G$, i.e. if ϕ is "${}^L G$-irreducible". This condition is equivalent to $\mathrm{Cent}(\phi, \widehat{G})/Z(\widehat{G})^\Gamma$ being finite. A Langlands parameter ϕ is called *relevant* if this Levi subgroup is relevant (see Section 4.3).

5.4. Weil restriction. Let E be a finite subextension E of $\overline{F}/F$ and let $\Gamma_E = \mathrm{Gal}(\overline{F}/E)$ be the corresponding open subgroup of Γ. Let G_0 be a connected reductive group G_0 over E. Let $G = \mathrm{Res}_{E/F}\, G_0$ be the Weil restriction, a connected reductive group over F such that the topological groups $G(F)$ and $G_0(E)$ are isomorphic. Recall from [**Bor79**, §5] that we may identify $\widehat{G}$ endowed with its action of Γ with the induction from Γ_E to Γ of $\widehat{G_0}$. By Shapiro's lemma we have a bijection $\Phi(G) \simeq \Phi(G_0)$.

6. The local Langlands conjecture

6.1. Crude local Langlands correspondence. Denote by $\Pi(G)$ be the set of isomorphism classes of irreducible admissible representations of $G(F)$ over $\mathbb{C}$ (in the Archimedean case, $(\mathfrak{g}, K)$-modules).

CONJECTURE 6.1. *There should exist maps* $\mathrm{LL} : \Pi(G) \to \Phi(G)$ *for all connected reductive groups G over F, satisfying the following properties. Denote* $\Pi_\phi(G) = \mathrm{LL}^{-1}(\phi)$.

1. *If G is a torus then* LL *should be the bijection that Langlands deduced from class field theory* [**Bor79**, §9].
2. *For any G all fibers of* LL *should be finite and the image of* LL *should contain all essentially discrete parameters.*
3. *If $G = G_1 \times G_2$ then, using the identification of ${}^L G$ with ${}^L G_1 \times_\Gamma {}^L G_2$, for any irreducible admissible representation $\pi \simeq \pi_1 \otimes \pi_2$ of $G(F)$ we should have* $\mathrm{LL}(\pi) = (\mathrm{LL}(\pi_1), \mathrm{LL}(\pi_2))$.
4. *If $\theta : G \to H$ is a central isogeny with dual $\widehat{\theta} : {}^L H \to {}^L G$ then for $\pi \in \Pi(H)$ and any constituent π' of the restriction $\pi|_{G(F)}$ (which is semi-simple of finite length) we should have* $\mathrm{LL}(\pi') = \widehat{\theta} \circ \mathrm{LL}(\pi)$.
5. *In the setup of Section 5.4 (Weil restriction) we should have a commutative diagram*
$$\begin{array}{ccc} \Pi(G) & \xrightarrow{\mathrm{LL}} & \Phi(G) \\ \downarrow{\scriptstyle\sim} & & \downarrow{\scriptstyle\sim} \\ \Pi(G_0) & \xrightarrow{\mathrm{LL}} & \Phi(G_0) \end{array}$$
where the left vertical map is induced by the isomorphism $G(F) \simeq G_0(E)$ and the right vertical map is Shapiro's lemma.
6. *For an irreducible smooth representation π of $G(F)$ we should have that π is essentially square-integrable if and only if* $\mathrm{LL}(\pi)$ *is essentially discrete.*

(7) *Let M be a Levi subgroup of G. Recall from Lemma 4.4 that we have an embedding $\iota_M : {}^L M \hookrightarrow {}^L G$, well-defined up to $\widehat{G}$-conjugacy. If σ is an irreducible smooth representation of $M(F)$ which is essentially square-integrable and has unitary central character then for any constituent π of $i_P^G \sigma$ we should have $\mathrm{LL}(\pi) = \iota_M \circ \mathrm{LL}(\sigma)$.*

(8) *In the situation of Theorem 3.7 we should have*

$$\mathrm{LL}(J(P, \sigma, \nu)) = \iota_P \circ \mathrm{LL}(\sigma \otimes \nu).$$

(9) *For Archimedean F the maps* LL *should be compatible with infinitesimal characters in the following sense. Assume $F \simeq \mathbb{R}$ (we may reduce to this case if $F \simeq \mathbb{C}$ by* (5) *above) and choose an isomorphism $\overline{F} \simeq \mathbb{C}$, so that we use the same field $\mathbb{C}$ for the coefficients, to construct the Weil group of F and to define $\mathfrak{g} = \mathbb{C} \otimes_F \mathrm{Lie}\, G$. Let π be an irreducible $(\mathfrak{g}, K)$-module. The restriction of $\mathrm{LL}(\pi)$ to $\mathbb{C}^\times$ is conjugated to a morphism of the form $z \mapsto z^\lambda \overline{z}^\mu$ where $\lambda, \mu \in X_*(\mathcal{T}) \otimes_{\mathbb{Z}} \mathbb{C}$ satisfy $\lambda - \mu \in X_*(\mathcal{T})$ and $z^\lambda \overline{z}^\mu$ is a suggestive notation for $(z\overline{z})^{(\lambda+\mu)/2}(z/|z|)^{\lambda-\mu}$. The infinitesimal character of π should be identified to λ by the Harish-Chandra isomorphism.*

(10) *Assume that F is non-Archimedean. If $P = MN$ is a parabolic subgroup of G and σ is an irreducible smooth representation of $M(F)$, then for any irreducible subquotient π of $i_P^G \sigma$ we should have $\mathrm{LL}(\pi) \circ \iota_W = \iota_M \circ \mathrm{LL}(\sigma) \circ \iota_W$. Equivalently, the same but just for supercuspidal σ.*

The list of properties in Conjecture 6.1 is not exhaustive, in particular we did not discuss the relation with L-functions, ϵ-factors and γ-factors. This list is certainly not enough to characterize the map LL, and if we omit (10) it may be possible to prove the conjecture in the non-Archimedean case using rather formal arguments (essentially by comparing, for simple and simply connected G, the cardinality of the set of essentially square-integrable irreducible representations of $G(F)$ with that of the set of essentially discrete parameters), but this would not give a lot of insight. So it is desirable to have constructions and characterizations of the map LL rather than just a proof of Conjecture 6.1. We refer the interested reader to [**Har**] for a survey of the possible characterizations.

We also warn the reader that there are actually two versions of the conjecture, corresponding to the two possible normalizations of the Artin reciprocity map in local class field theory. According to [**KS**, §4] these should be related by a certain automorphism of ${}^L G$, which according to [**Kal13**], [**AV16**] and [**Pra19**] is itself related to taking contragredient representations. Thus another way to state the relation between the two normalizations is to say that we should obtain one from the other by composing with the involution $\pi \mapsto \tilde{\pi}$.

Cases for which the conjecture is known (with a "natural" construction or characterization) include the Archimedean case [**Lan89**], general linear groups over non-Archimedean fields [**LRS93**] [**Hen00**] [**HT01**] [**Sch13**], GSp_4 over finite extensions of $\mathbb{Q}_p$ [**GT11**], inner forms of special linear groups over finite extensions of $\mathbb{Q}_p$ [**HS12**], and quasi-split classical groups over finite extensions of $\mathbb{Q}_p$ [**Art13**] [**Mok15**]. More cases will be discussed later.

The rest of this section is devoted to remarks on the properties in the conjecture.

6.1.1. *Compatibility with the case of tori.* The functoriality assumptions (3) and (4) imply the following compatibilities with the case of tori.

- The map LL should be compatible with central characters in the following sense. Let Z be the maximal central torus in G so that we have a surjective morphism ${}^L G \to {}^L Z$. Then all elements of $\Pi_\phi(G)$ should have central character determined by composing ϕ with this surjection and applying LL^{-1}.
- Langlands defined (see [**Bor79**, §10.2]) a morphism
$$H^1_{\mathrm{cont}}(\mathrm{W}_F, Z(\widehat{G})) \to \mathrm{Hom}_{\mathrm{cont}}(G(F), \mathbb{C}^\times).$$
For a continuous 1-cocycle $c : \mathrm{W}_F \to Z(\widehat{G})$ with corresponding character $\chi : G(F) \to \mathbb{C}^\times$ we should have $\mathrm{LL}(\pi \otimes \chi) = c\,\mathrm{LL}(\pi)$.

6.1.2. *Reduction to the discrete case.* Using Proposition 3.5, the Langlands classification (Theorem 3.7, including Remark 3.8) and the "Langlands classification for parameters" (see Section 5.3), properties (7) and (8) imply that π is tempered if and only if $\mathrm{LL}(\pi)$ is tempered. In fact we see that these parallel results for smooth representations of reductive groups and Langlands parameters reduce the construction of LL to the essentially square-integrable case, and with property (2) we see that the image of LL should be the set of relevant Langlands parameters.

6.1.3. *The unramified case.* From properties (1), (7) and (8) it follows that if G is unramified and K is a hyperspecial compact open subgroup of $G(F)$ then on K-unramified irreducible representations of $G(F)$ (i.e. representations having non-zero K-invariants) the map LL is given by the Satake isomorphism. More precisely in this case the minimal Levi subgroup M_0 is an unramified torus and unramified representations of $G(F)$ are parametrized by orbits under the rational Weyl group of continuous characters $\chi : M_0(F) \to \mathbb{C}^\times$. The unramified representation π corresponding to the orbit of χ is the unique unramified constituent of $i_B^G \chi$, for any Borel subgroup B of G containing M_0. We have $\mathrm{LL}(\pi) = \iota_{M_0} \circ \mathrm{LL}(\chi)$. In the tempered case, that is when χ is unitary, this follows immediately from property (7). The general case is more subtle, and can be deduced from the Gindikin-Karpelevich formula [**Cas80**, Theorem 3.1] (see [**CS80**, p. 219] for the values of the constants in the case of an unramified group)[4].

6.1.4. *The semisimplified correspondence and algebraicity.* For non-Archimedean F property (10) says that the map $\mathrm{LL}^{\mathrm{ss}} : \pi \mapsto \mathrm{LL}(\pi) \circ \iota_W$ is compatible with the notion of supercuspidal support (Theorem 3.2). This suggests the following conjecture.

Conjecture 6.2. *Assume that F is non-Archimedean. Let C be any algebraically closed field of characteristic zero and choose a square root $\sqrt{q} \in C$. There should exist for each connected reductive group G over F a map $\mathrm{LL}^{\mathrm{ss}}$ from the set of isomorphism classes of smooth irreducible representations of $G(F)$ over C to the set of $\widehat{G}$-conjugacy classes of continous semi-simple morphisms $\mathrm{W}_F \to {}^L G$ which are compatible with ${}^L G \to \Gamma$, satisfying the obvious analogue of* (1), (3), (4) *in Conjecture 6.1, as well as the following analogue of property* (10) *in Conjecture 6.1.*

If $P = MN$ is a parabolic subgroup of G and σ is an irreducible smooth representation of $M(F)$ then for any irreducible subquotient π of $i_P^G \sigma$ we should have $\mathrm{LL}^{\mathrm{ss}}(\pi) = \iota_M \circ \mathrm{LL}^{\mathrm{ss}}(\sigma)$.

[4] To be honest the arguments in [**Cas80**] assume that χ is regular but similar arguments work using only partial regularity.

Moreover these maps $\mathrm{LL}^{\mathrm{ss}}$ *should be functorial in* $(C, \sqrt{q})$[5].

For $C = \overline{\mathbb{Q}_\ell}$, where ℓ does not equal the residue characteristic of F, Genestier-Lafforgue [**GL**] (in positive characteristic) and Fargues-Scholze [**FS**] have constructed maps $\mathrm{LL}^{\mathrm{ss}}$ satisfying all properties in Conjecture 6.2 except for functoriality with respect to the coefficient field, which seems to remain open.

Conjecture 6.1 implies the case $C = \mathbb{C}$ of Conjecture 6.2, again excluding functoriality in $(C, \sqrt{q})$. Assuming Conjecture 6.1 one can also show that the map $\mathrm{LL} \circ \iota_W$ determines the map LL, by considering first the case of tempered representations and using the decomposition (5.2) and the fact that an $\mathfrak{sl}_2$ triple (here, in the connected centralizer of ϕ_0 in $\widehat{G}$) is determined by its semi-simple element up to conjugation. In the case of general linear groups the construction of the map LL was reduced to the supercuspidal case by Zelevinsky [**Zel80**]. In general however Conjecture 6.2 does not immediately imply Conjecture 6.1. What is missing is the fact that for any essentially square-integrable irreducible smooth representation π of $G(F)$, the semi-simplified parameter $\mathrm{LL}^{\mathrm{ss}}(\pi)$ comes from an *essentially discrete* Langlands parameter (which as above is automatically unique up to conjugation by the centralizer of $\mathrm{LL}^{\mathrm{ss}}(\pi)$ in $\widehat{G}$). In this direction Gan, Harris, Sawin and Beuzart-Plessis have recently shown [**?**, Theorem 1.2] that the Genestier-Lafforgue semi-simplified Langlands parameter of an essentially square-integrable representation, say with central character having finite order, comes from a (again, unique) *tempered* Langlands parameter.

Note that properties (6), (7) and (8) in Conjecture 6.1 make essential use of the topology on the coefficient field $\mathbb{C}$. The notion of essentially discrete Langlands parameter is purely algebraic (it does not rely on the topology of the coefficient field) so there ought to be a purely algebraic characterization of essentially square-integrable representations. We check the validity of this intuition in the following proposition.

PROPOSITION 6.3. *Let F be a non-Archimedean local field. Assume Conjecture 6.1. Let π be an irreducible smooth representation of $G(F)$. Assume that its central character ω_π has finite order (we may reduce to this case by twisting by a continuous character $G(F) \to C^\times$). Then π is essentially square-integrable if and only if for any parabolic subgroup $P = MN$ of G and for any character χ of $A_M(F)$ occurring in $r_P^G \pi$ there exists an integer $N \geq 1$ such that χ^N is equal to $\prod_\alpha ||\alpha||^{n_\alpha}$ for some integers $n_\alpha > 0$, where the product ranges over the simple roots of A_M in* $\mathrm{Lie}\, N$.

PROOF. Let $P = MN$ and χ be as in the proposition. There exists an irreducible quotient σ of the representation $r_P^G \pi$ of $M(F)$ whose central character ω_σ satisfies $\omega_\sigma|_{A_M(F)} = \chi$, and by Frobenius reciprocity we have an embedding $\pi \hookrightarrow i_P^G \sigma$. Embedding σ in a parabolically induced representation from a supercuspidal one (see Theorem 3.2), we may reduce to the case where σ is supercuspidal. The details of this reduction are left to the reader. Let $\phi_M = \mathrm{LL}(\sigma)$ and $\phi = \mathrm{LL}(\pi)$. By property (10) in Conjecture 6.1 we have $\iota_M \circ \phi_M \circ \iota_W = \phi \circ \iota_W$ (up to conjugacy by $\widehat{G}$). The character χ corresponds by local class field theory to the composition

$$\mathrm{WD}_F \xrightarrow{\phi_M} {}^L M \to {}^L A_M$$

[5]One could certainly avoid the choice of a square root of q by modifying Langlands dual groups. We do not attempt to explain this here, see [**BG14**, §5.3] and [**?**, §2.2].

which may be seen as a continuous morphism $\mathrm{W}_F \to \widehat{A_M}$ because the torus A_M is split. Note the pre-composing with ι_W does not change this morphism. Because we already know Proposition 3.4 it is enough to prove that some integral power of χ is of the form $\prod_\alpha ||\alpha||^{n_\alpha}$ for some integers n_α. By assumption some integral power of χ is trivial on $A_G(F)$. The simple roots of A_M on N give an isogeny $A_M/A_G \to \mathbb{G}_m^n$ for some integer $n \geq 0$, so it is enough to prove that some integral power of χ takes values in $q^{\mathbb{Z}}$.

There exists an open subgroup U of the inertia subgroup $I_F \subset \mathrm{W}_F$ such that the action of U on $\widehat{G}$ is trivial and ϕ is trivial on U. Up to taking a smaller subgroup we may assume that U is normalized by W_F (this follows from a simple Galois-theoretic argument). There exists an integer $N \geq 1$ such that for any $w \in \mathrm{W}_F$ the action of w^N by conjugation on W_F/U is trivial (this can be proved in two steps, first for I_F/U and then for a lift of a generator of W_F/I_F). It will be convenient to see the Langlands dual group ${}^LG$ as $\widehat{G} \rtimes \mathrm{Gal}(E/F)$ for some finite Galois extension E/F. Thus for any $w \in \mathrm{W}_F$ we have that $\phi(w)^N$ centralizes $\phi(\mathrm{WD}_F)$. Up to replacing N by $|\,\mathrm{Gal}(E/F)| \times |\,\mathrm{Cent}(\phi, \widehat{G})/Z(\widehat{G})^\Gamma|$ we obtain $\phi(w)^N \in Z(\widehat{G})^\Gamma$ for all $w \in \mathrm{W}_F$. The natural map $Z(\widehat{G})^\Gamma \to \widehat{A_G}$ has finite kernel. Because we have assumed that ω_π has finite order, up to replacing N by a non-zero multiple we even have $\phi(w)^N = 1$ for all $w \in \mathrm{W}_F$. Up to replacing N by $2N$, this implies that for any finite-dimensional algebraic representation $r : {}^LG \to \mathrm{GL}(V)$ and for any $w \in \mathrm{W}_F$, any eigenvalue $\lambda \in \mathbb{C}^\times$ of $r(\phi(\iota_W(w)))$ satisfies $\lambda^N \in q^{\mathbb{Z}}$. Taking for r a closed embedding ${}^LG \hookrightarrow \mathrm{SL}(V)$, we obtain the claim because any irreducible representation of ${}^LM$, and in particular any character of $\widehat{A_M}$, occurs in the restriction of $V^{\otimes a}$ for some integer $a \geq 0$. □

6.1.5. *Cuspidality and parameters.* Assume that F is non-Archimedean. Property (10) implies that for any irreducible smooth representation π of $G(F)$, if $\mathrm{LL}(\pi)$ is essentially discrete and trivial on SL_2 then π is supercuspidal. Contrary to the case of GL_n, in general the converse is not true, i.e. there exists supercuspidal representations π whose Langlands parameter $\mathrm{LL}(\pi)$ is not trivial on SL_2. A related matter is that the classification of essentially square-integrable representations in terms of supercuspidal representations (of Levi subgroups) is much more complicated in general than in the case of GL_n. See [**?**] and [**?**] for the case of classical groups.

6.2. Refined local Langlands for quasi-split groups. In some applications having just the map LL is too crude, e.g. to formulate the global multiplicity formula for the automorphic spectrum of a connected reductive group over a global field, and so we would like to understand the fibers $\Pi_\phi(G)$.

In this section we assume that G is quasi-split. For a Langlands parameter $\phi : \mathrm{WD}_F \to {}^LG$ denote $S_\phi = \mathrm{Cent}(\phi, \widehat{G})$ (a reductive subgroup of $\widehat{G}$), and define $\overline{S}_\phi = S_\phi/Z(\widehat{G})^\Gamma$. Recall that a parameter ϕ is essentially discrete if and only if $\overline{S}_\phi$ is finite. It can happen that $\pi_0(\overline{S}_\phi)$ is non-abelian (even in the principal series case, that is if ϕ factors through $\iota_T : {}^LT \hookrightarrow {}^LG$ where T is part of a Borel pair (B, T) defined over F!). For $F = \mathbb{R}$ however, it is always abelian, in fact there is a maximal torus $\mathcal{T}$ of $\widehat{G}$ such that $S_\phi \cap \mathcal{T}$ meets every connected component of S_ϕ. For a finite group A denote by $\mathrm{Irr}(A)$ the set of isomorphism classes of irreducible representations of A over $\mathbb{C}$.

Conjecture 6.4. *For each Langlands parameter* ϕ *there should exist an embedding* $\Pi_\phi(G) \to \mathrm{Irr}(\pi_0(\overline{S}_\phi))$. *For non-Archimedean* F *this should be a bijection.*

Langlands's classification again reduces the construction of this bijection to the tempered case. So we assume from now on that ϕ is tempered. The bijection in Conjecture 6.4 is not canonical in general: it depends on the choice of a Whittaker datum (up to conjugation by $G(F)$).

We briefly recall the notions of Whittaker datum and generic representation for a quasi-split connected reductive group G. Choose a Borel subgroup B with unipotent radical U. For a Galois orbit $\mathcal{O}$ on the set of simple roots, the group $U_{\mathcal{O}} = \left(\prod_{\alpha\in\mathcal{O}} U_\alpha(\overline{F})\right)^{\mathrm{Gal}_F}$ is isomorphic to a finite separable extension $F_{\mathcal{O}}$ of F. We have a natural surjective morphism from $U(F)$ to $\prod_{\mathcal{O}} U_{\mathcal{O}}$. Choosing a non-trivial morphism $U_{\mathcal{O}} \to \mathbb{C}^\times$ for each orbit $\mathcal{O}$ yields a morphism $\theta : U(F) \to \mathbb{C}^\times$, called a generic character. A Whittaker datum $\mathfrak{w}$ for G is such a pair (U, θ). The adjoint group $G_{\mathrm{ad}}(F)$ acts transitively on the set of such pairs, and so their are only finitely many $G(F)$-conjugacy classes of Whittaker data. If F is non-Archimedean an irreducible smooth representation (π, V) of $G(F)$ is called $\mathfrak{w}$-generic if there is a non-zero linear functional $V \to \mathbb{C}$ such that $\lambda(\pi(u)v) = \theta(u)\lambda(v)$ for all $u \in U(F)$ and $v \in V$. For Archimedean F the notion is more subtle because it requires a topology on the representation.

Conjecture 6.5 (Shahidi). *There should be a unique* $\mathfrak{w}$*-generic representation in each* $\Pi_\phi(G)$. *The conjectural bijection* $\iota_{\mathfrak{w}} : \Pi_\phi(G) \to \mathrm{Irr}(\pi_0(\overline{S}_\phi))$ *(which depends on* $\mathfrak{w}$*) should map this* $\mathfrak{w}$*-generic representation to the trivial representation of* $\overline{S}_\phi$.

In order to characterize the bijections $\iota_{\mathfrak{w}}$ we have to introduce endoscopic data. Let $s \in S_\phi$ be a semi-simple element. From the pair (s, ϕ) one can construct the following objects. For $\pi \in \Pi_\phi(G)$ denote $\langle s, \pi\rangle_{\mathfrak{w}} = \mathrm{tr}(\iota_{\mathfrak{w}}(\pi))(s)$. On the one hand we have

$$\Theta^{\mathfrak{w}}_{\phi,s} = \sum_{\pi\in\Pi_\phi(G)} \langle s, \pi\rangle_{\mathfrak{w}} \Theta_\pi.$$

This is a virtual character on $G(F)$. In the case $s = 1$ we introduce the special notation

$$S\Theta_\phi = \Theta^{\mathfrak{w}}_{\phi,1}.$$

The reason for not recording $\mathfrak{w}$ in the notation in this case will be explained below.

On the other hand we consider the complex connected reductive subgroup $\mathcal{H}^0 = \mathrm{Cent}(s, \widehat{G})^0$ of $\widehat{G}$. It contains $\phi(1 \times \mathrm{SL}_2)$ and is normalized by $\phi(\mathrm{W}_F)$. Thus $\mathcal{H} = \mathcal{H}^0 \cdot \phi(\mathrm{W}_F)$ is a subgroup of ${}^LG$, which is an extension $1 \to \mathcal{H}^0 \to \mathcal{H} \to \mathrm{W}_F \to 1$. The resulting morphism $\mathrm{W}_F \to \mathrm{Out}(\mathcal{H}^0)$ factors through the Galois group of a finite extension of F. By Proposition 4.3 there exists a quasi-split connected reductive group H over F together with an inner class of isomorphisms $\eta : \mathcal{H}^0 \simeq \widehat{H}$ such that the above morphism $\mathrm{W}_F \to \mathrm{Out}(\mathcal{H}^0)$ and the morphism $\mathrm{W}_F \to \mathrm{Out}(\widehat{H})$ used to define ${}^LH = \widehat{H} \rtimes \mathrm{W}_F$ correspond to each other via η, and for any two such groups H_1 and H_2 we have an isomorphism $H_1 \simeq H_2$, well-defined up to $H_{1,\mathrm{ad}}(F)$. It may unfortunately happen that the two extensions $\mathcal{H}$ and $\widehat{H}$ of W_F are not isomorphic. We shall ignore this difficulty, as its resolution is not terribly exciting (see [**KS99**, Lemma 2.2.A]). So let's assume there exists an isomorphism of extensions ${}^L\eta : \mathcal{H} \to {}^LH$. Then $\mathfrak{e} = (H, s, {}^L\eta)$ is called an extended endoscopic triple. By construction we have a unique Langlands parameter $\phi_H : \mathrm{WD}_F \to {}^L H$ such that we have ${}^L\eta \circ \phi_H = \phi$. We have the virtual character $S\Theta_{\phi_H}$ on $H(F)$.

The two virtual characters $\Theta^{\mathfrak{w}}_{\phi,s}$ and $S\Theta_{\phi_H}$ are expected to be related by a certain kernel function. This function, called the Langlands-Shelstad transfer factor, is itself non-conjectural and explicit. It is a function

$$\Delta[\mathfrak{w}, \mathfrak{e}] : H(F)_{G-\mathrm{sr}} \times G(F)_{\mathrm{sr}} \to \mathbb{C}$$

whose construction depends on the Whittaker datum and the extended endoscopic triple. We will not recall the definition of $\Delta[\mathfrak{w}, \mathfrak{e}]$ (which is rather technical, see [**LS87**] [**KS99**] [**KS**]), but let us recall what its support is (a correspondence between strongly regular semisimple conjugacy classes in $G(F)$ and G-strongly regular semisimple stable conjugacy classes in $H(F)$), and recall a meaningful variance property.

Definition 6.6. Recall that an element of $G(\overline{F})$ is called strongly regular if its centralizer is a torus. Two semisimple strongly regular elements δ, δ' in $G(F)$ are called stably conjugate if there exists $g \in G(\overline{F})$ such that $g\delta g^{-1} = \delta'$.

Using maximal tori and identifications of Weyl groups one can define [**KS99**, Theorem 3.3.A] a canonical map m from semisimple conjugacy classes in $H(\overline{F})$ to semisimple conjugacy classes in $G(\overline{F})$. A conjugacy class in $H(\overline{F})$ is called G-strongly regular if its image under m is strongly regular. We denote by $H(F)_{G-\mathrm{sr}}$ the set of G-strongly regular elements of $H(F)$. The map m enjoys the following properties.

(1) The map m is Γ-equivariant.
(2) If $\gamma \in H(F)$ is semisimple G-strongly regular then $m([\gamma]) \cap G(F)$ is a non-empty[6] finite union of $G(F)$-conjugacy classes. In this situation we say that (the stable conjugacy class of) γ and (the conjugacy class) of $\delta \in m([\gamma]) \cap G(F)$ *match*. Given a strongly regular stable conjugacy class for G, there are finitely many stable conjugacy classes for H in its preimage by m.
(3) For any matching pair $(\gamma, \delta) \in H(F)_{G-\mathrm{sr}} \times G(F)_{\mathrm{sr}}$, denoting $T_H = \mathrm{Cent}(\gamma, H)$ and $T = \mathrm{Cent}(\delta, g)$ (maximal tori of H and G), there is a canonical isomorphism $T_H \simeq T$ identifying γ and δ.

The fact that m is defined at the level of conjugacy classes over $\overline{F}$ rather than F is one justification for introducing the notion of stable conjugacy.

Let δ be a strongly regular element of $G(F)$, and denote $T = \mathrm{Cent}(\delta, G)$. The set of $G(F)$-conjugacy classes $[\delta']$ which are stably conjugate to δ is parametrized by $\ker\left(H^1(F,T) \to H^1(F,G)\right)$, by mapping δ' to $\mathrm{inv}(\delta,\delta') := (\sigma \mapsto \sigma(g)^{-1}g)$ where as above $g\delta g^{-1} = \delta'$. Because of this description of stable conjugacy it is desirable to better understand these Galois cohomology sets. Recall from [**Tat66**] that the Tate-Nakayama isomorphism for tori over F identifies $H^1(F,T)$ with

$$\widehat{H}^{-1}(E/F, X_*(T)) = X_*(T)^{N_{E/F}=0}/I_{E/F}X_*(T) \tag{6.1}$$

where E/F is any finite Galois subextension of $\overline{F}/F$ splitting T, $N_{E/F}$ is the norm map, and for a $\mathbb{Z}[\mathrm{Gal}(E/F)]$-module Y we denote by $I_{E/F}Y$ the submodule $\sum_{\sigma\in\mathrm{Gal}(E/F)}(\sigma-1)Y$. Note that the right-hand side of (6.1) can also be described as the torsion subgroup of the coinvariants $X_*(T)_\Gamma$. Kottwitz interpreted this

[6] For non-emptiness the fact that G is quasi-split is essential.

isomorphism in terms of Langlands dual groups and generalized it to connected reductive groups in [**Kot86**]. Recall that $\widehat{T}$ is a torus over $\mathbb{C}$ endowed with an isomorphism $X^*(\widehat{T}) \simeq X_*(T)$. Using the exactness of the functor mapping a finitely generated abelian group A to the diagonalizable group scheme Z with character group A (considered as a sheaf on the étale site of $\mathbb{C}$, say) we see that $X^*(\widehat{T})_\Gamma$ is identified with $X^*(\widehat{T}^\Gamma)$. It follows that the Tate-Nakayama isomorphism may be written as

$$\alpha_T : H^1(F,T) \simeq \mathrm{Irr}\left(\pi_0(\widehat{T}^\Gamma)\right). \tag{6.2}$$

It is formal to check that this identification is functorial in T. As for the Artin reciprocity map it would be just as natural to consider the same isomorphism composed with $x \mapsto x^{-1}$.

Theorem 6.7 ([**Kot86**, Theorem 1.2]). *There is a unique extension of the above family of isomorphisms to a family of maps of pointed sets*

$$\alpha_G : H^1(F,G) \to \mathrm{Irr}\left(\pi_0(Z(\widehat{G})^\Gamma)\right)$$

for connected reductive G, "functorial" in the following sense. For any morphism $H \to G$ which is either the embedding of a maximal torus in a connected reductive group G or a central isogeny between connected reductive groups we have a commutative diagram

$$\begin{array}{ccc} H^1(F,H) & \longrightarrow & H^1(F,G) \\ \downarrow{\scriptstyle \alpha_H} & & \downarrow{\scriptstyle \alpha_G} \\ \mathrm{Irr}\left(\pi_0(Z(\widehat{H})^\Gamma)\right) & \longrightarrow & \mathrm{Irr}\left(\pi_0(Z(\widehat{G})^\Gamma)\right) \end{array}$$

where the bottom horizontal map is the one induced by the Γ-equivariant map $Z(\widehat{G}) \to Z(\widehat{H})$ recalled (in both cases) at the end of Section 4.2.

For two connected reductive groups G_1 and G_2 we have $\alpha_{G_1 \times G_2} = \alpha_{G_1} \times \alpha_{G_2}$.

In [**Kot86**] this is proved in the case where F has characteristic zero but the same proof works for all local fields, using Bruhat and Tits' generalization of Kneser's theorem [**BT87**]. Kneser's theorem is the special case where G is semi-simple and simply connected over a p-adic field, in which case we have $Z(\widehat{G}) = 1$ and so the theorem says that $H^1(F,G)$ is trivial. If F is non-Archimedean then each α_G is a bijection, in particular $H^1(F,G)$ has a commutative group structure. In the Archimedean case the kernel and image of α_G are described loc. cit. We will also denote $\alpha_G(c)(s) = \langle c, s\rangle$.

We resume the above notation: $(H, s, {}^L\eta)$ is an extended endoscopic triple, $(\gamma,\delta) \in H(F)_{G-\mathrm{sr}} \times G(F)_{\mathrm{sr}}$ is a matching pair, $T_H = \mathrm{Cent}(\gamma, H)$ and $T = \mathrm{Cent}(\delta, G)$ and we have a canonical isomorphism $T_H \simeq T$. By Theorem 6.7 the kernel of $H^1(F,T) \to H^1(F,G)$ is identified with the group of characters of $\pi_0(\widehat{T}^{\mathrm{Gal}_F})$ which are trivial on $Z(\widehat{G})^{\mathrm{Gal}_F}$. The element ${}^L\eta(s) \in Z(\widehat{H})^{\mathrm{Gal}_F}$ defines an element $s_{\gamma,\delta}$ of $\widehat{T}_H^{\mathrm{Gal}_F} \simeq \widehat{T}^{\mathrm{Gal}_F}$. We can finally state the variance property of transfer factors: we have

$$\Delta[\mathfrak{w}, \mathfrak{e}](\gamma, \delta') = \Delta[\mathfrak{w}, \mathfrak{e}](\gamma,\delta)\langle \mathrm{inv}(\delta,\delta'), s_{\gamma,\delta}\rangle^{-1}. \tag{6.3}$$

As for the Artin reciprocity map and the pairing (6.2) there are several natural normalizations for the transfer factors [**KS**, §4], and for half of these normalizations the exponent -1 on the right-hand side should be removed. The relation (6.3) is far from characterizing $\Delta[\mathfrak{w}, \mathfrak{e}]$ because it does not compare the values at unrelated matching pairs.

CONJECTURE 6.8. *Let G be a quasi-split connected reductive group over F. Let $\phi : \mathrm{WD}_F \to {}^L G$ be a tempered Langlands parameter.*

(1) *The map $S\Theta_\phi : G_{\mathrm{rs}}(F) \to \mathbb{C}$ should be invariant under* stable *conjugacy*[7].
(2) *For any semi-simple $s \in S_\phi$ and any strongly regular semisimple $G(F)$-conjugacy class $[\delta]$ we should have*

$$\Theta^{\mathfrak{w}}_{\phi,s}(\delta) = \sum_{\gamma \in H(F)/st} \Delta[\mathfrak{w}, \mathfrak{e}](\gamma, \delta) S\Theta_{\phi_H}(\gamma)$$

where $((H, s, {}^L\eta), \phi_H)$ is an extended endoscopic triple and Langlands parameter $\phi_H : \mathrm{WD}_F \to {}^L H$ corresponding to (ϕ, s).

REMARK 6.9. (1) The equation uniquely determines $\iota_{\mathfrak{w}}$ provided it exists, due to the linear independence of characters. In particular, one can deduce how $\iota_{\mathfrak{w}}$ should depend on $\mathfrak{w}$. Namely, to each pair $\mathfrak{w}$ and $\mathfrak{w}'$ one can associate unconditionally a character $(\mathfrak{w}, \mathfrak{w}')$ of S_ϕ and then $\iota_{\mathfrak{w}'}(\pi) = \iota_{\mathfrak{w}}(\pi) \otimes (\mathfrak{w}, \mathfrak{w}')$. See [**Kal13**, §3] for details. In particular, $\dim(\iota_{\mathfrak{w}}(\pi))$ is independent of the choice of $\mathfrak{w}$, and hence $S\Theta_\phi$ is also independent.

(2) While Conjecture 6.1 readily reduces to the essentially discrete case using Harish-Chandra's work, the putative analogous reductions for Conjectures 6.4 and 6.8 appear to be more subtle, involving the study of intertwining operators. See [**KS88**] for character formulas in the case of principal series representations.
(3) Implicit in the conjecture is the fact that the choice of a semisimple s in its connected component in $\pi_0(\overline{S}_\phi)$ is irrelevant. One can reduce to the case where s is "generic" (implying that ϕ_H is essentially discrete) by parabolic induction (which behaves well with respect to $S\Theta$).
(4) This conjecture reduces the characterization of the local Langlands correspondence to a characterization of the stable functions $S\Theta_\phi$.

6.3. Refined Langlands correspondence for non-quasi-split groups. Recall from Proposition 4.3 that two connected reductive groups that are inner forms of each other have isomorphic Langlands dual groups, and thus the "same" Langlands parameters. Vogan's idea is to consider the L-packets $\Pi_\phi(G)$, for a given ϕ and G varying in an inner class, as one big L-packet Π_ϕ. It is natural to take the quasi-split group given in Proposition 4.3 as "base point" in the inner class because we already have a satisfying conjecture in this case, and for reasons explained below. So we fix a quasi-split group G^*. Recall that isomorphism classes of inner twists of G^* are parametrized by $H^1(F, G^*_{\mathrm{ad}})$. We may consider the groupoid of triples (G, ψ, π) where (G, ψ) is an inner twist of G^* and π is an irreducible smooth representation of $G(F)$, with the obvious notion of isomorphism. The problem with this definition is that for an inner twist (G, ψ) of G^* its automorphism group

[7]For convenience we only defined stable conjugacy in the strongly regular case, so strictly speaking one should say that the restriction of $S\Theta_\phi$ to the strongly regular locus should be stable. Note that the complement of the strongly regular locus still has measure zero.

in $\mathcal{IT}(G^*)$ is $G_{\mathrm{ad}}(F)$, which acts non-trivially on the set of isomorphism classes of irreducible smooth representations of $G(F)$. The motivates the introduction of *pure* inner twists: augment the datum (G,ψ) with a 1-cocycle $z:\Gamma\to G^*(\overline{F})$ lifting

$$\begin{aligned}\Gamma &\longrightarrow G_{\mathrm{ad}}(\overline{F})\\ \sigma &\longmapsto \psi^{-1}\sigma(\psi).\end{aligned}$$

This effectively solves the above problem but creates a new one because the map $H^1(F,G^*)\to H^1(F,G^*_{\mathrm{ad}})$ is not surjective in general. For $F=\mathbb{R}$ Adams, Barbasch and Vogan [**ABV92**] found an ad-hoc generalization of $Z^1(\mathbb{R},G^*)$, called strong real forms, that surjects onto $H^1(\mathbb{R},G^*_{\mathrm{ad}})$. Kottwitz suggested using his theory of isocrystals with additional structure [**Kot85**] [**Kot97**] in the case of non-Archimedean fields of characteristic zero as a generalization of $H^1(F,G^*)$. This suggestion was implemented completely by Kaletha and will be recalled below, but unfortunately it does not capture all inner forms of a given quasi-split group in general. Kaletha later introduced another generalization of inner forms, called *rigid* inner forms, for any local field F of characteristic zero and which captures all inner forms. Specializing to $F=\mathbb{R}$ recovers strong real forms. It turns out that all of these generalizations can be understood as replacing the Galois group Γ (or the étale site of $\operatorname{Spec} F$) by an appropriate *Galois gerb*. We summarize the three theories (pure, isocrystal and rigid) for a local field F of characteristic zero below and refer the interested reader to Dillery's paper [**Dil**] for the generalization to functions fields, which uses Čech cohomology instead of Galois cohomology and also provides a more conceptual point of view using actual gerbs.

In characteristic zero and for a commutative band, following [**LR87**] the above mentioned Galois gerbs may prosaically be defined as group extensions

$$1\to u(\overline{F})\to\mathcal{E}\to\Gamma\to 1$$

where u is a commutative group scheme over F and the action by conjugation of Γ on $u(\overline{F})$ coincides with the usual one. In practice u is a projective limit of groups $(u_i)_{i\geq 0}$ of multiplicative type and finite type over F with surjective morphisms between them, and the extension $\mathcal{E}$ is built from a class in $H^2_{\mathrm{cont}}(\Gamma,u(\overline{F}))$ where $u(\overline{F})$ is endowed with the topology induced by the discrete topology on each $u_i(\overline{F})$. Note that we have set-theoretic sections $\Gamma\to\mathcal{E}$, endowing $\mathcal{E}$ with a natural topology. Define $H^1_{\mathrm{alg}}(\mathcal{E},G)\subset H^1_{\mathrm{cont}}(\mathcal{E},G(\overline{F}))$ as the subset of classes of 1-cocycles $\mathcal{E}\to G(\overline{F})$ whose restriction to $u(\overline{F})$ is given by an algebraic morphism from $u_{\overline{F}}$ to $G_{\overline{F}}$. Define $H^1_{\mathrm{bas}}(\mathcal{E},G)\subset H^1_{\mathrm{alg}}(\mathcal{E},G)$ as the set of classes of cocycles for which the algebraic morphism $u_{\overline{F}}\to G_{\overline{F}}$ takes values in the center $Z(G)(\overline{F})$. By the cocycle condition it descends in this case to a morphism $u\to Z(G)$ defined over F. Note that such a morphism is induced from a morphism $u_i\to Z(G)$ for some index i because the center of G has finite type over F. We will also consider, for Z a subgroup scheme of $Z(G)$, the subset $H^1(u\to\mathcal{E},Z\to G)$ of $H^1_{\mathrm{bas}}(\mathcal{E},G)$ consisting of classes of cocycles whose associated map $u\to Z(G)$ factors through Z.

We consider three cases in parallel.

(1) If we take $u=1$ we obtain the trivial extension $\mathcal{E}^{\mathrm{pur}}=\Gamma$, recovering the usual Galois cohomology group $H^1(F,G)$.

(2) Consider the pro-torus u over F with character group

$$X^*(u) = \begin{cases} \mathbb{Q} & \text{if } F \text{ is non-Archimedean,} \\ \frac{1}{2}\mathbb{Z} & \text{if } F \simeq \mathbb{R}. \end{cases}$$

(We exclude the case $F \simeq \mathbb{C}$ here because it is essentially trivial.) We have

$$H^2_{\mathrm{cont}}(\Gamma, u(\overline{F})) \simeq \begin{cases} \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} & \text{if } F \text{ is non-Archimedean,} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } F \simeq \mathbb{R}. \end{cases}$$

Let $\mathcal{E}^{\mathrm{iso}}$ be the extension of Γ by $u(\overline{F})$ corresponding to the class of 1.

(3) Consider the pro-finite algebraic group u over F with character group $X^*(u)$ the set of locally constant functions $f : \Gamma \to \mathbb{Q}/\mathbb{Z}$ satisfying $\sum_{\sigma\in\Gamma} f(\sigma) = 0$ if F is Archimedean. We have

$$H^2_{\mathrm{cont}}(\Gamma, u(\overline{F})) \simeq \begin{cases} \widehat{\mathbb{Z}} & \text{if } F \text{ is non-Archimedean,} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } F \simeq \mathbb{R}. \end{cases}$$

(As above we exclude the case $F \simeq \mathbb{C}$.) Let $\mathcal{E}^{\mathrm{rig}}$ be the extension of Γ by $u(\overline{F})$ corresponding to the class -1.

We have the following generalizations of the Tate-Nakayama isomorphisms.

THEOREM 6.10. *We have natural maps*

$$\kappa_G : H^1_{\mathrm{bas}}(\mathcal{E}^{\mathrm{iso}}, G) \to X^*(Z(\widehat{G})^\Gamma)$$

extending the maps α_G *of Theorem 6.7, i.e. sitting in commutative diagrams*

$$\begin{array}{ccc} H^1(F,G) & \xrightarrow{\alpha_G} & \mathrm{Irr}(\pi_0(Z(\widehat{G})^\Gamma)) \\ \downarrow & & \downarrow \\ H^1_{\mathrm{bas}}(\mathcal{E}^{\mathrm{iso}}, G) & \xrightarrow{\kappa_G} & X^*(Z(\widehat{G})^\Gamma) \end{array}$$

and functorial in G *similarly to Theorem 6.7 (in the case of an inclusion of a maximal torus* $T \subset G$ *we have to restrict to elements of* $H^1_{\mathrm{bas}}(\mathcal{E}^{\mathrm{iso}}, T)$ *for which the induced map* $u \to T$ *factors through* $Z(G)$*).*

The map κ_G *is bijective if* F *is non-Archimedean.*

PROOF. See [**Kot**, Proposition 13.1 and Proposition 13.4] and [**Kal18**, §3.1]. □

For a connected reductive group G over F and a finite central subgroup scheme Z denote $\overline{G} = G/Z$. We have a dual map $\widehat{\overline{G}} \to \widehat{G}$; denote by $Z(\widehat{\overline{G}})^+$ be the preimage of $Z(\widehat{G})^\Gamma$ in $Z(\widehat{\overline{G}})$.

THEOREM 6.11 ([**Kal16**, Corollary 5.4]). *We have natural maps*

$$H^1(u \to \mathcal{E}^{\mathrm{rig}}, Z \to G) \to X^*(Z(\widehat{\overline{G}})^+)$$

extending the maps α_G *and functorial in* $Z \to G$ *as in Theorem 6.7.*

These maps are bijective in the non-Archimedean case.

We also have natural maps $H^1(u \to \mathcal{E}^?, Z \to G) \to H^1(F, G/Z)$, and the above generalizations of the Tate-Nakayama morphism are also compatible with $\alpha_{G/Z}$. One can deduce that the maps $H^1_{\mathrm{bas}}(\mathcal{E}^{\mathrm{iso}}, G) \to H^1(F, G/Z(G)^0)$ and

$$H^1(u \to \mathcal{E}^{\mathrm{rig}}, Z(G_{\mathrm{der}}) \to G) \to H^1(F, G_{\mathrm{ad}})$$

are both surjective. In particular all inner forms can be realized as rigid inner twists, or as isocrystal inner twists if the center of G is connected. In general not all inner forms can be realized as isocrystal inner twists, e.g. when G is split semisimple but not adjoint.

There is [**Kal18**, §3.3] a natural map of extensions $\mathcal{E}^{\mathrm{rig}} \to \mathcal{E}^{\mathrm{iso}}$, inducing $H^1_{\mathrm{bas}}(\mathcal{E}^{\mathrm{iso}}, G) \to H^1_{\mathrm{bas}}(\mathcal{E}^{\mathrm{rig}}, G)$ for any group G. The relation with Theorems 6.10 and 6.11 is not so obvious, see Proposition 3.3 loc. cit.

To simplify the notation for $z \in Z^1_{\mathrm{bas}}(\mathcal{E}, G)$ we denote by (G_z, ψ_z) the associated inner twist of G.

Conjecture 6.12. *Let G^* be a quasi-split connected reductive group over F. Let $\mathfrak{w}$ be a Whittaker datum for G^*. Let $\phi : \mathrm{WD}_F \to {}^L G^*$ be a tempered Langlands parameter. Let $? \in \{\mathrm{pur}, \mathrm{iso}, \mathrm{rig}\}$. Define $\Pi^?_\phi$ as the set of isomorphism classes of pairs (z, π) where $z \in Z^1_{\mathrm{bas}}(\mathcal{E}^?, G^*)$ and $\pi \in \Pi_\phi(G^*_z)$. Define*

(1) *$Z^{\mathrm{pur}} = 1$, $\mathcal{S}^{\mathrm{pur}}_\phi = \pi_0(S_\phi)$ and $\mathcal{Z}^{\mathrm{pur}} = \pi_0(Z(\widehat{G})^\Gamma)$,*
(2) *$Z^{\mathrm{iso}} = Z(G)^0$, $\mathcal{S}^{\mathrm{iso}}_\phi = S_\phi/(S_\phi \cap \widehat{G}_{\mathrm{der}})^0$ and $\mathcal{Z}^{\mathrm{iso}} = Z(\widehat{G})^\Gamma$,*
(3) *Z^{rig} is any finite subgroup scheme of $Z(G)$, $\mathcal{S}^{\mathrm{rig}}_\phi = \pi_0(S^+_\phi)$ where S^+_ϕ is the preimage of S_ϕ in $\widehat{\overline{G}}$ and $\mathcal{Z}^{\mathrm{rig}} = \pi_0(Z(\widehat{\overline{G}})^+)$.*

There should exist a bijection $\iota_{\mathfrak{w}}$ making the following diagram commutative.

$$\begin{array}{ccc} \Pi^?_\phi & \xrightarrow[\sim]{\iota_{\mathfrak{w}}} & \mathrm{Irr}(S^?_\phi) \\ \downarrow & & \downarrow \\ H^1(u \to \mathcal{E}^?, Z^? \to G^*) & \longrightarrow & X^*(\mathcal{Z}^?) \end{array}$$

Here the left vertical map is induced by the forgetful map $(z, \pi) \mapsto z$, the right vertical map is induced by the obvious map $\mathcal{Z}^? \to \mathcal{S}^?_\phi$ and the bottom horizontal map is given by Theorem 6.7 (resp. 6.10, resp. 6.11).

The relation with Conjecture 6.1 is that for any $z \in Z^1(u \to \mathcal{E}^?, Z^? \to G^*)$ we should have $\Pi_\phi(G^*_z) = \{\pi \mid (z, \pi) \in \Pi^?_\phi\}$.

As for Conjectures 6.4 and 6.8, the map $\iota_{\mathfrak{w}}$ in Conjecture 6.12 should be characterized by endoscopic character relations. In order to state these relations we need normalized transfer factors. Their definition was suggested by Kottwitz and established by Kaletha in the case of pure inner forms [**Kal11**, §2.2] and extended to the isocrystal and rigid case by Kaletha [**Kal14**] [**Kal16**].

Let (G, ψ, z) be a pure/isocrystal/rigid inner twist of G^* and $\phi : \mathrm{WD}_F \to {}^L G$ a tempered Langlands parameter. Consider a semi-simple $s \in S_\phi$ if $? \in \{\mathrm{pur}, \mathrm{iso}\}$ or $s \in S^+_\phi$ if $? = \mathrm{rig}$. As in Section 6.2 we obtain an extended endoscopic triple[8] $\mathfrak{e} = (H, s, {}^L\eta)$ and a tempered Langlands parameter $\phi_H : \mathrm{WD}_F \to {}^L H$. Consider matching strongly regular $\gamma \in H(F)$ and $\delta \in G(F)$. Using Steinberg's theorem [**Ste65**, Theorem I.7] we see that for any strongly regular $\delta \in G(F)$ there exists

[8]A refined one in the rigid case, i.e. s belongs to the cover $\widehat{\overline{G}}$ of $\widehat{G}$.

$\delta^* \in G^*(F)$ stably conjugate to δ, i.e. for which there exists $g \in G^*(\overline{F})$ satisfying $\psi(g^{-1}\delta^* g) = \delta$. Clearly δ^* is also strongly regular; denote its centralizer in G^* by T^*. In this situation let $\mathrm{inv}[\psi, z](\delta^*, \delta) \in H^1(u \to \mathcal{E}^?, Z^? \to T^*)$ be the class of $w \mapsto g z_w w(g)^{-1}$. This class does not depend on the choice of g. Similarly to the quasi-split case we can associate $s_{\gamma,\delta^*} \in \widehat{T^*}^\Gamma$ (resp. $\widehat{T^*}^\Gamma$, resp. $\widehat{\widehat{T^*}}^+$) to s and the matching pair (γ, δ^*), and pair it with $\mathrm{inv}(\delta^*, \delta)$ using Theorem 6.7 (resp. 6.10, resp. 6.11). In analogy with (6.3) define

$$\Delta[\mathfrak{w}, \mathfrak{e}, \psi, z](\gamma, \delta) = \Delta[\mathfrak{w}, \mathfrak{e}](\gamma, \delta^*) \langle \mathrm{inv}(\delta^*, \delta), s_{\gamma,\delta^*} \rangle^{-1}.$$

It turns out that this is well-defined, i.e. the right-hand side does not depend on the choice of δ^*, and this defines a normalization of transfer factors for $(H, s, {}^L\eta)$. Again there are several natural normalizations and in half of these normalizations the exponent -1 should be removed.

We can now formulate the generalization of Conjecture 6.8. As in Section 6.2 we abbreviate $\langle s, \pi \rangle_{\mathfrak{w},z} = \mathrm{tr}\, \iota_{\mathfrak{w}}(z, \pi)(s)$ and define

$$\Theta^{\mathfrak{w},z}_{\phi,s} = e(G_z) \sum_{\pi \in \Pi_\phi(G_z)} \langle s, \pi \rangle_{\mathfrak{w},z} \Theta_\pi$$

where $e(G_z)$ is the sign defined by Kottwitz [**Kot83**].

Conjecture 6.13. *In the setting of Conjecture 6.12, for any $z \in Z^1(u \to \mathcal{E}^?, Z^? \to G^*)$, any strongly regular $G_z(F)$-conjugacy class $[\delta]$ and any semi-simple $s \in S_\phi$ (resp. S_ϕ, resp. S_ϕ^+) we should have*

$$\Theta^{\mathfrak{w},z}_{\phi,s}(\delta) = \sum_{\gamma \in H(F)/st} \Delta[\mathfrak{w}, \mathfrak{e}, \psi, z](\gamma, \delta) S\Theta_{\phi_H}(\gamma).$$

By linear independence of characters the conjecture implies that packets $\Pi_{\phi_H}(H)$ for all endoscopic groups of G^* — all quasi-split groups — should determine the refined Langlands correspondence for all pure/isocrystal/rigid inner forms of G^*.

If we fix an inner twist G, ψ of G^* then it may be realized as a rigid inner twist in more than one way: one can multiply $z \in Z^1(u \to \mathcal{E}^{\mathrm{rig}}, Z \to G^*)$ by any $c \in Z^1(u \to \mathcal{E}^{\mathrm{rig}}, Z \to Z)$. By [**Kal18**, §6] Conjectures 6.12 6.13 for z imply the same conjectures for cz. This implies the same invariance property for pure inner twists. Presumably a similar invariance property should be valid in the isocrystal case.

6.4. Reduction to the isocrystal case. Let G^* be a quasi-split connected reductive group over a p-adic field F. As explained above all inner forms of G^* can be reached using the rigid theory, and one might be tempted to simply forget the pure and isocrystal versions. They are simpler however, and the relative complexity of the rigid version is exacerbated in the global setting. Another reason to favor the isocrystal version is that it seems more naturally related to geometric incarnations of the correspondence, as in [**FS**]. It is thus useful to relate the isocrystal and rigid versions (the relation between the pure and isocrystal versions being rather obvious).

As explained in [**Kal18**, §4], for $z^{\mathrm{iso}} \in Z^1_{\mathrm{bas}}(\mathcal{E}^{\mathrm{iso}}, G^*)$ and $z^{\mathrm{rig}} \in Z^1_{\mathrm{bas}}(\mathcal{E}^{\mathrm{rig}}, G^*)$ its pullback via $\mathcal{E}^{\mathrm{rig}} \to \mathcal{E}^{\mathrm{iso}}$, the relevant representations of centralizers (relevant=given restriction to center) are the same and the endoscopic character relations are also the same. In §5 loc. cit. Kaletha construct an embedding $G^* \to \widetilde{G}^*$

with normal image and abelian cokernel such that the center of $\widetilde{G}^*$ is connected and such that Conjectures 6.12 and 6.13 for G^* and $\widetilde{G}^*$ are equivalent. Since these conjectures for $\widetilde{G}^*$ can be reduced to the isocrystal case, it would be enough to prove Conjectures 6.12 and 6.13 for all quasi-split groups in the isocrystal setting to deduce them for all quasi-split groups in the rigid setting, yielding "the" refined Langlands correspondence for all connected reductive groups.

6.5. Relation with the crude version. By [**Kal16**, Lemma 5.7] Conjecture 6.12 recovers the relevance condition on parameters discussed in 6.1.2.

One can formulate a more precise version of property (4) in Conjecture 6.1. Let $f : G_1^* \to G_2^*$ be a central isogeny between quasi-split connected reductive groups over F, inducing a dual map $\hat{f} : {}^L G_2 \to {}^L G_1$. Let $\phi_2 : \mathrm{WD}_F \to {}^L G_2$ be a tempered Langlands parameter and denote $\phi_1 = \hat{f} \circ \phi_2$. Let $? \in \{\mathrm{pur}, \mathrm{rig}, \mathrm{iso}\}$. We use the same notation as in Conjecture 6.12, choosing finite central subgroups Z_i^{rig} in the rigid case. Up to enlarging these groups we may assume that Z_1^{rig} contains the kernel of f and that its image is Z_2^{rig}. Let $z_1 \in Z^1(u \to \mathcal{E}^?, Z^? \to G_1^*)$ and let z_2 be its image in $Z^1(u \to \mathcal{E}^?, Z^? \to G_2^*)$. Denote $G_1 = G_{1,z_1}^*$ and $G_2 = G_{2,z_2}^*$. In all three cases $\hat{f}$ induces a morphism $S_{\phi_2}^? \to S_{\phi_1}^?$. Let $\mathfrak{w}$ be a Whittaker datum for G_1^* and G_2^*.

Conjecture 6.14. *For any $\pi_2 \in \Pi_{\phi_2}(G_2)$ we should have*

$$\pi_2|_{G_1(F)} \simeq \bigoplus_{\pi_1 \in \Pi_{\phi_1}(G_1)} m(\pi_1, \pi_2)\pi_1$$

where $m(\pi_1, \pi_2)$ is the multiplicity of $\iota_{\mathfrak{w}}(z_2, \pi_2)$ in the restriction of $\iota_{\mathfrak{w}}(z_1, \pi_1)$ to $S_{\phi_2}^?$.

6.6. A non-exhaustive list of known cases. In the case of real groups Conjectures 6.12 and 6.13 were proved by Shelstad in many papers, see [**She08a**], [**She10**], [**She08b**] and [**Kal16**, §5.6].

Hiraga and Saito [**HS12**] proved Conjectures 6.12 and 6.13[9] for inner forms of SL_n over non-Archimedean local fields of characteristic zero.

Arthur [**Art13**] proved Conjectures 6.4 and 6.8 for quasi-split special orthogonal[10] and symplectic groups over non-Archimedean fields of characteristic zero using, among other tools, the stabilization of the twisted trace formula [**?**] [**?**]. In this case the stable characters $S\Theta$ are characterized by *twisted* endoscopy for the group GL_N with its automorphism $\theta : g \mapsto {}^t g^{-1}$ and the correspondence for general linear groups. Note that endoscopic groups of special orthogonal or symplectic groups are product of similar groups and general linear groups. Mok [**Mok15**] followed the same strategy to prove the conjectures for quasi-split unitary groups over non-Archimedean local fields of characteristic zero. For completeness we recall that to our knowledge the main results of [**Art13**] and [**Mok15**] still depend on unpublished results. These cases were then extended to certain inner forms:

- using the stabilization of the trace formula: to non-quasi-split unitary groups [**?**], to non-quasi-split special orthogonal and unitary groups [**?**], to non-quasi-split odd special orthogonal groups [**?**], and
- using theta correspondences: to non-quasi-split unitary groups [**?**].

[9]This was before [**Kal16**] so one should compare the normalizations of transfer factors.

[10]In the even orthogonal case Arthur proved these conjectures "up to outer automorphism".

Gan-Takeda [**GT11**] and Chan-Gan [**CG15**] proved Conjectures 6.12 and 6.13 for the groups GSp_4 over non-Archimedean local fields of characteristic zero, using theta correspondences and the stabilization of the trace formula.

The method of close fields of Deligne and Kazhdan allowed several authors to extend the existence of a map LL for certain types of groups over non-Archimedean fields from the characteristic zero case to the positive characteristic case:

- [**?**] for GSp_4 (assuming that the characteristic is not 2),
- [**?**] for split symplectic and special orthogonal groups (with a restriction on the characteristic),
- [**?**] for inner forms of SL_n.

This method gives internal structure of L-packets but does not seem to yield endoscopic character relations.

In the non-Archimedean cases mentioned above the characterizations of the local Langlands correspondence are rather indirect (using functoriality, global methods etc). Of course it is desirable to have a more direct construction, like in the case of real groups [**Lan89**]. The existence of (many) supercuspidal representations implies that such a direct construction has to be much more complicated than in the real case. Thanks to the work of many mathematicians (Moy and Prasad, Morris, Adler, Yu, Kim, Fintzen, Hakim and Murnaghan) we now have a "direct" classification of supercuspidal representations, i.e. one using representation theory rather than Langlands parameters, at least for tamely ramified connected reductive groups such that the residual characteristic p does not divide the order of the absolute Weyl group. We refer the reader to [**?**, §1.2] for more details and references. Using this classification and building on work of Adler, DeBacker, Reeder and Spice, Kaletha constructed [**?**] [**?**], under the above assumption on (G, p) and for supercuspidal (i.e. essentially discrete and trivial on SL_2) Langlands parameters ϕ, L-packets Π_ϕ and natural parametrizations as in the rigid case of Conjecture 6.12. Under additional assumptions (p large enough and F of characteristic zero) Fintzen, Kaletha and Spice [**?**, Theorem 4.4.4], improving on previous work of Adler, DeBacker, Kaletha and Spice, proved stability and the endoscopic character relations of the rigid case of Conjecture 6.13 for s in a certain subgroup of S_ϕ^+. For the so-called *regular* supercuspidal this subgroup is S_ϕ^+, i.e. Conjecture 6.13 holds.

There is much work to be done in this direction to handle, in order of increasing generality: all supercuspidal parameters, all essentially discrete parameters, all tempered parameters. Recently Aubert and Xu [**?**] constructed a map LL for the split group G_2 over a finite extension of $\mathbb{Q}_p$ for $p \notin \{2, 3\}$, together with a parametrization of L-packets (Conjecture 6.4). Their work uses Kaletha's parametrization for supercuspidal representations and Hecke algebra techniques for the non-supercuspidal ones. Gan and Savin [**?**] also constructed a map LL for the split group G_2 over a finite extension of $\mathbb{Q}_p$ using theta correspondences with "classical" groups, and gave a parametrization of L-packets for $p \neq 3$.

7. Gerbs, Tannakian formalism and isocrystals

We briefly mention the more conceptual point of view on gerbs and Tannakian categories, and explain how it relates the above definition of $H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{iso}}, G)$ with the set $B(G)$ of isocrystals with G-structure [**Kot85**] [**Kot97**]. The latter point of view was motivated by the study of Shimura varieties over finite fields and historically came first.

7.1. Gerbs and Tannakian formalism. We first recall the equivalence between certain gerbs and Tannakian categories [**SR72**, Théorème 3] as corrected by [**Del90**]. We consider fpqc[11] stacks over F. Recall that a gerb is a stack in groupoids admitting local sections and such that any two objects are locally isomorphic. A gerb $\mathcal{C}$ is said to have affine band if for any scheme S over F and any two objects x, y of $\mathcal{C}_S$ the sheaf $\underline{\mathrm{Isom}}_S(x, y)$ is representable by an affine scheme over S. If this holds for one non-empty S and one pair (x, y) then $\mathcal{C}$ has affine band [**Del90**, p. 131].

A representation R of a gerb $\mathcal{C}$ is a morphism from $\mathcal{C}$ to the stack of quasi-coherent sheaves (over varying schemes over F). A representation may also be intuitively understood as a quasi-coherent sheaf on $\mathcal{C}$. For a scheme S over F and an object x of $\mathcal{C}_S$ the quasi-coherent sheaf $R(x)$ over S is automatically flat, and if it has finite rank n for some pair (S, x) then $R(y)$ has the same rank for any object y of $\mathcal{C}$ [**Del90**, §3.5]. In that case we may see R as a morphism from $\mathcal{C}$ to the stack of vector bundles of rank n (equivalently, GL_n-torsors). Finite-dimensional representations of $\mathcal{C}$ form a category $\mathrm{Rep}(\mathcal{C})$, that can be endowed with a tensor product (taking tensor products of vector bundles ...). In fact $\mathrm{Rep}(\mathcal{C})$ is a tensor category over F (in the sense of [**Del90**, §2.1]). Because $\mathcal{C}$ has local sections the tensor category $\mathrm{Rep}(\mathcal{C})$ is even Tannakian, i.e. it admits a fiber functor [**Del90**, §1.9] over some non-empty scheme over F.

To any tensor category $\mathcal{T}$ over F we can associate the fibered category (over schemes over F) of fiber functors of $\mathcal{T}$, denoted by $\mathrm{Fib}(\mathcal{T})$. If $\mathcal{T}$ is Tannakian then $\mathrm{Fib}(\mathcal{T})$ is a gerb having affine band and the natural tensor functor $\mathcal{T} \to \mathrm{Rep}(\mathrm{Fib}(\mathcal{T}))$ is an equivalence. Conversely for a gerb $\mathcal{C}$ we also have a natural morphism of stacks $\mathcal{C} \to \mathrm{Fib}(\mathrm{Rep}(\mathcal{C}))$ which is an equivalence if and only if $\mathcal{C}$ has affine band.

For a gerb $\mathcal{C}$ having affine band and a linear algebraic group G over F we can consider morphisms of stacks from $\mathcal{C}$ to the gerb BG of G-torsors, generalizing the notion of representation of $\mathcal{C}$. Such a morphism may also be interpreted as a G-torsor on $\mathcal{C}$ (see [**Dil**, §2.4]). By the correspondence recalled above such a morphism amounts to a morphism of tensor categories $\mathrm{Rep}(G) \to \mathrm{Rep}(\mathcal{C})$. Here we have identified $\mathrm{Rep}(BG)$ with the category of finite-dimensional representations of G over F. The set[12] of isomorphism classes of morphisms $\mathcal{C} \to BG$ will be denoted by $H^1(\mathcal{C}, G)$.

We now assume that F has characteristic zero and specialize to the case of a gerb $\mathcal{C}$ whose band u is commutative, so that u is an fpqc sheaf of commutative groups over F, and is representable by an affine (commutative group) scheme. Any group scheme over F is isomorphic to a projective limit, over a directed poset I, of group schemes of finite type $(u_i)_{i \in I}$. We assume further that I may be chosen to be countable. We make this assumption because it implies that any projective limit over I of non-empty sets with surjective transition maps is itself not empty. We may identify $\mathcal{C}$ with a projective limit of gerbs $\mathcal{C}_i$ bound by u_i (equivalently, we may identify the Tannakian category $\mathrm{Rep}(\mathcal{C})$ with a union of tensor subcategories admitting a tensor generator). Recall from [**SR72**, Chapitre III Théorème 3.1.3.3] or [**Del90**, Corollaire 6.20] that $\mathcal{C}_i$ admits a section over a finite extension of F. It follows that $\mathcal{C}$ admits a section over $\overline{F}$. By our assumption on I the fiber $\mathcal{C}_{\overline{F}}$

[11] Fix a universe ...

[12] We ignore set-theoretic issues here ...

has only one isomorphism class (i.e. every $u_{\overline{F}}$-torsor is trivial). This implies that the group $\mathrm{Aut}_{\mathcal{C}}(x)$ is an extension $\mathcal{E}$ of Γ by $\mathrm{Aut}_{\mathcal{C}_{\overline{F}}}(x) = u(\overline{F})$. Moreover for any $x \in \mathcal{C}_{\overline{F}}$ the two pullbacks $p_1^* x$ and $p_2^* x$ in $\mathcal{C}_{\overline{F} \otimes_F \overline{F}}$ are isomorphic [**Dil**, Lemma 2.61]. Fix such an isomorphism $\varphi : p_1^* x \simeq p_2^* x$. Pulling back φ via the morphisms

$$\begin{aligned} \overline{F} \otimes_F \overline{F} &\longrightarrow \overline{F} \\ x \otimes y &\longmapsto x\sigma(y) \end{aligned}$$

for $\sigma \in \Gamma$ gives us a (set-theoretic) splitting $\Gamma \to \mathrm{Aut}_{\mathcal{C}}(x)$. Taking the "coboundary" $d\varphi = (p_{13}^* \varphi)^{-1} \circ (p_{23}^* \varphi) \circ (p_{12}^* \varphi)$ yields an automorphism of the pullback of x via the first projection $\overline{F} \to \overline{F}^{\otimes 3}$, i.e. an element of $u(\overline{F}^{\otimes 3})$, and one can check that it is a Čech 2-cocycle [**Dil**, Fact 2.31]. Conversely any such 2-cocycle induces a gerb bound by u [**Dil**, Proposition 2.36], and two gerbs bound by u are isomorphic if and only if their associated class in $\check{H}^2(\overline{F}/F, u)$ are equal. Yet another projective limit argument shows that we have a natural isomorphism $\check{H}^2(\overline{F}/F, u) \simeq H^2_{\mathrm{cont}}(\Gamma, u(\overline{F}))$. For a linear algebraic group G over F and a morphism of stacks $R : \mathcal{C} \to BG$, R factors through $\mathcal{C}_i$ for some $i \in I$ (equivalently, $\mathrm{Rep}(G)$ has a tensor generator [**DM82**, Proposition 2.20 (b)] and so the tensor functor $\mathrm{Rep}(G) \to \mathrm{Rep}(\mathcal{C})$ factors through a sub-tensor category of $\mathrm{Rep}(\mathcal{C})$ generated by a single object). Choosing a trivialization of the $G_{\overline{F}}$-torsor $R(x)$ we obtain a morphism $u_{\overline{F}} \to G_{\overline{F}}$. (Some other choice of trivialization would give the same morphism conjugated by some element of $G(\overline{F})$.) One can check that restricting R to $\mathcal{E} = \mathrm{Aut}_{\mathcal{C}}(x)$ gives a continuous 1-cocycle $\mathcal{E} \to G(\overline{F})$ whose restriction to $u(\overline{F})$ is (induced by) the above morphism of group schemes over $\overline{F}$. We obtain a map $H^1(\mathcal{C}, G) \to H^1_{\mathrm{alg}}(\mathcal{E}, G(\overline{F}))$ and one can check that it is bijective[13].

7.2. Isocrystals. For F a non-Archimedean local field of characteristic zero the gerb corresponding to $\mathcal{E}^{\mathrm{iso}}$ was historically first introduced via its corresponding Tannakian category, the category of isocrystals. We briefly recall this notion. Let L be the completion of the maximal unramified extension of F. Denote by σ the Frobenius automorphism of L. An isocrystal is a finite-dimensional vector space V over L endowed with a σ-linear bijection $\Phi : V \to V$. They form a tensor category Isoc_F for the obvious notion of tensor product. (Among other axioms, we indeed have $\mathrm{End}_{\mathrm{Isoc}_F}(1) = L^\sigma = F$.) We have an obvious fiber functor for Isoc_F over L, namely $(V, \Phi) \mapsto V$, and so Isoc_F is Tannakian. By the Dieudonné-Manin classification theorem the tensor category Isoc_F has a simple structure: it is semi-simple and its simple objects are parametrized by $\mathbb{Q}$. We briefly recall this classification and refer the reader to [**SR72**, Chapitre VI §3.3] for more details and references. Fix a uniformizer ϖ of F. For $r/s \in \mathbb{Q}$ for coprime $r, s \in \mathbb{Z}$ with $s > 0$ we may construct the corresponding simple object of Isoc_F as follows. Let

[13]The least obvious part is perhaps the fact that a morphism $R_c : \mathcal{C} \to BG$ can be constructed from a cocycle $c \in Z^1_{\mathrm{alg}}(\mathcal{E}, G)$. As usual one reduces to constructing a morphism $\mathcal{C}_i \to \mathrm{Rep}(G)$ for some $i \in I$. Thanks to the fact that u_i is of finite type we may reconstruct from c the image under R_c of the fpqc sheaf $\underline{\mathrm{Iso}}(p_1^* x, p_2^* x)$ on $\mathrm{Spec}(\overline{F}^{\otimes 2})$. For an object y of $\mathcal{C}_S$ for some scheme S over F, letting $\pi_1 : S \times_F \overline{F} \to S$ be the first projection we can first define $R_c(\pi_1^* y)$. There exists a cover $S' \to S \times_F \overline{F}$ and an isomorphism between the pullbacks of y and x to S'. Via this isomorphism we can see the descent datum for y along $S' \to S$ as a section of $\underline{\mathrm{Iso}}(p_1^* x, p_2^* x)$ on the natural map $S' \times_S S' \to \mathrm{Spec}(\overline{F}^{\otimes 2})$. Taking the image of this section by R_c allows us to define $R_c(y)$ from $R_c(\pi_1^* y)$ by descent.

$S(r/s)$ be L^s and define a σ-linear automorphism of $S(r/s)$ as σ (on coordinates) post-composed with the linear automorphism of L^s with matrix

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & & \\ & & & \ddots & \\ 0 & 0 & \dots & 0 & 1 \\ \varpi^r & 0 & \dots & 0 & 0 \end{pmatrix}.$$

This defines a simple object $S(r/s)$ in Isoc_F. The isomorphism class of $S(r/s)$ does not depend on the choice of uniformizer ϖ, and any simple object is isomorphic to $S(q)$ for a uniquely determined $q \in \mathbb{Q}$. Denote by F_s the unramified extension of degree s of F in L. The F-algebra $\mathrm{End}_{\mathrm{Isoc}_F}(S(r/s))$ embeds in the matrix algebra $M_s(L)$, in fact it embeds in $M_s(F_s)$ and it is a central simple algebra over F which is a division ring and is split by F_s. Its invariant in $H^2(F, \mathbb{G}_m) \simeq \mathbb{Q}/\mathbb{Z}$ is simply the image of r/s. Any isocrystal (V, Φ) decomposes canonically as $\bigoplus_{r/s \in \mathbb{Q}} V_{r/s}$ where

$$V_{r/s} = L \otimes_{F_s} V^{\varpi^{-r}\Phi^s}$$

is the isotypic component isomorphic to a finite sum of copies of $S(r/s)$. The rational numbers q for which $V_q \neq 0$ are called the slopes of (V, f) and the above decomposition is called the slope decomposition. An isocrystal (V, Φ) is said to be pure of slope $q \in \mathbb{Q}$ if $V_{q'} = 0$ for all $q' \neq q$. The tensor product of two isocrystals which are pure of slopes q_1 and q_2 is also pure, of slope $q_1 + q_2$. The tensor category Isoc_F is the union of its tensor subcategories $\mathrm{Isoc}_{F,s}$ consisting of all isocrystals (V, f) whose slopes q all satisfy $qs \in \mathbb{Z}$. The Tannakian category $\mathrm{Isoc}_{F,s}$ admits a fiber functor over F_s, namely

$$\omega_s : (V, \Phi) \longmapsto \bigoplus_{r \in \mathbb{Z}} V^{\varpi^{-r}\Phi^s}.$$

If s divides s' then we have an obvious identification between $F_{s'} \otimes_{F_s} \omega_s$ and $\omega_{s'}$. We obtain a fiber functor ω for Isoc_F over the maximal unramified extension of F. Thanks to the description of $\mathrm{End}_{\mathrm{Isoc}_F}(S(r/s))$ recalled above we can compute the band u_s of (the gerb of fiber functors of) $\mathrm{Isoc}_{F,s}$ as the (commutative!) multiplicative group $\mathbb{G}_m$ over F. For an F_s-algebra A and $x \in A^\times$, x acts on the slope r/s part $A \otimes_{F_s} V^{\varpi^{-r}\Phi^s}$ by multiplication by x^r. For s dividing s' the natural morphism $u_{s'} \to u_s$ can be checked to be $x \mapsto x^{s'/s}$, and so the band u of (the gerb of fiber functors of) Isoc_F is the split protorus with character group $\mathbb{Q}$. The class of the gerb in

$$H^2(F, u) \simeq H^2_{\mathrm{cont}}(\Gamma, u(\overline{F})) \simeq \varprojlim_s H^2(F, u_s) \simeq \varprojlim_s \mathbb{Q}/\mathbb{Z} \simeq \widehat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$$

(the second isomorphism because each $H^1(F, u_s)$ vanishes and so $\varprojlim^1_s H^1(F, u_s)$ also vanishes) can be computed from the above description of endomorphisms of simple isocrystals and is simply equal to 1.

For a connected linear algebraic group G over F we can identify the set of isomorphism classes of tensor functors $\mathrm{Rep}(G) \to \mathrm{Isoc}_F$ with $B(G) := G(L)/\sim$ where $g_1 \sim g_2$ if and only if there exists $x \in G(L)$ for which $g_2 = x g_1 \sigma(x)^{-1}$ (σ-conjugacy). This is because $H^1(L, G_L)$ is trivial [**Lan52**] [**Ste65**, Theorem 1.9] and so there is up to isomorphism only one fiber functor for $\mathrm{Rep}(G)$ over L, namely $\omega_{G,L} : (V, \rho) \mapsto L \otimes_F V$. It follows that any tensor functor $\mathrm{Rep}(G) \to \mathrm{Isoc}_F$ is isomorphic to one of the form $(V,\rho) \mapsto (L \otimes_F V, \Phi_{V,\rho})$. It is clear that setting $\Phi_{V,\rho} = \sigma \otimes \mathrm{id}_V$ gives a tensor functor, and any other tensor functor differs from this one by an automorphism of the fiber functor $\omega_{G,L}$, i.e. by an element of $G(L)$. A similar argument shows that two elements of $G(L)$ induce isomorphic tensor functors if and only if they are σ-conjugated. This point of view on "isocrystals with additional structure" is historically the first one [**Kot85**] and was motivated by the study of Shimura varieties and Rapoport-Zink spaces.

We now briefly discuss the set $B(G) \simeq H^1_{\mathrm{alg}}(\mathcal{E}^{\mathrm{iso}}, G)$ in the case where G is a connected reductive group over F. We refer the reader to [**Kot97**] for more details. The basic subset $B(G)_{\mathrm{bas}} \simeq H^1_{\mathrm{bas}}(\mathcal{E}^{\mathrm{iso}}, G)$ is completely described by the map κ_G of Theorem 6.10. Kottwitz constructed [**Kot90**, Lemma 6.1] maps

$$\kappa_G : B(G) \to X^*(Z(\widehat{G})^\Gamma)$$

which as the notation suggests extend the maps of Theorem 6.10. (In fact the definition in the general and basic case are not different: Kottwitz first defined isomorphisms κ_T for all tori T and then extended the map to arbitrary connected reductive groups by reducing to the case where the derived subgroup is simply connected using z-extensions.) We also have obvious maps of pointed sets

$$\nu_G : B(G) \to \left(\mathrm{Hom}(u_{\overline{F}}, G_{\overline{F}})/G(\overline{F}) - \mathrm{conj}\right)^\Gamma$$

The kernel of ν_G is $B(G)_{\mathrm{bas}}$ by definition. The pair (ν_G, κ_G) is injective on $B(G)$ [**Kot97**, Theorem 5.4].

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