DISCRETE SERIES MULTIPLICITIES FOR CLASSICAL GROUPS OVER $\mathbb{Z}$ AND LEVEL 1 ALGEBRAIC CUSP FORMS

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ABSTRACT. The aim of this paper is twofold. First, we introduce a new method for evaluating the multiplicity of a given discrete series in the space of level 1 automorphic forms of a split classical group G over $\mathbb{Z}$, and provide numerical applications in absolute rank ≤ 8. Second, we prove a classification result for the level one cuspidal algebraic automorphic representations of GL_n over $\mathbb{Q}$ (n arbitrary) whose motivic weight is ≤ 24.

In both cases, a key ingredient is a classical method based on the Weil explicit formula, which allows to disprove the existence of certain level one algebraic cusp forms on GL_n, and that we push further on in this paper. We use these vanishing results to obtain an arguably "effortless" computation of the elliptic part of the geometric side of the trace formula of G, for an appropriate test function.

Thoses results have consequences for the computation of the dimension of the spaces of (possibly vector-valued) Siegel modular cuspforms for $\mathrm{Sp}_{2g}(\mathbb{Z})$: we recover all the previously known cases without relying on any, and go further, by a unified and "effortless" method.

CONTENTS

1. Introduction

1.1. **Siegel modular forms for** $\mathrm{Sp}_{2g}(\mathbb{Z})$. We denote by $\mathrm{S}_k(\Gamma_g)$ and $\mathrm{S}_{\underline{k}}(\Gamma_g)$ respectively the space of cuspidal Siegel modular forms for the full Siegel modular group $\Gamma_g = \mathrm{Sp}_{2g}(\mathbb{Z})$, which are either scalar-valued of weight $k \in \mathbb{Z}$, or more generally vector-valued of weight $\underline{k} = (k_1, k_2, \ldots, k_g)$ in $\mathbb{Z}^g$ with $k_1 \geq k_2 \geq \cdots \geq k_g$ (we refer to [vdG08] for a general introduction to Siegel modular forms). Recall that $\mathrm{S}_{\underline{k}}(\Gamma_g)$ trivially vanishes when $\sum_i k_i$ is odd, and also for $k_g < g/2$ (Freitag, Reznikoff, Weissauer).

The question of determining the dimension of $\mathrm{S}_{\underline{k}}(\Gamma_g)$, very classical for $g = 1$, has a long and rich history for $g > 1$. It has first been attacked for $g = 2$ using geometric methods, in which case concrete formulas were obtained by Igusa (1962) in the scalar-valued case, and by Tsushima (1983) for the weights[1] $k_1 \geq k_2 \geq 3$. There is still no known formula for $k_2 = 2$, although we have $\mathrm{S}_{\underline{k}}(\Gamma_2) = 0$ for $k_2 = 1$ (Ibukiyama, Skoruppa): see [CvdG18] for a discussion of these singular cases. An analogue of Igusa's result for $g = 3$ was proved by Tsuyumine in 1986, but only quite recently a conjectural formula was proposed by Bergström, Faber and van der Geer, in the vector valued case $k_1 \geq k_2 \geq k_3 \geq 4$, based on counting genus three curves over finite fields (2011). Their formula, and more generally a formula[2] for $\dim \mathrm{S}_{\underline{k}}(\Gamma_g)$ for arbitrary $g \leq 7$ and $k_g > g$ was proved by the second author in [Taï17], using a method that we will recall in §1.4. Actually, the general formulas

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[1] More precisely, Tsushima could only prove that his formula works for $k_2 \geq 5$, and later Petersen (2013) and Taïbi (2016) independently showed that it holds as well for $k_2 \geq 3$ and $(k_1, k_2) \neq (3, 3)$, as conjectured by Ibukiyama.

[2] These are pretty huge formulas, which can't be printed here already for $g > 2$, but see Theorem A *loc. cit.* for their general shape.

given in [Taï17] apply to any genus g and any weights with $k_g > g$. However, they involve certain rational numbers, that we shall refer to later as *masses*, that are rather difficult to compute. Taïbi provided *loc. cit.* a number of algorithms to determine them (more precisely, certain local orbital integrals, see §1.4) that allowed him to numerically compute those masses for $g \leq 7$, using algorithms which were implemented and optimized case-by-case. It is fair to say that reproducing these computations from the generic algorithm explained in [Taï17] would require a considerable effort.

Our first main result in this paper is a completely different and comparatively much easier method to compute the aforementioned masses. This method allows us to recover, in a uniform and rather "effortless" way, all the computations of masses done in [Taï17] for $g \leq 7$, and even to go further:

Theorem 1. *There is an explicit and implemented formula computing* $\dim S_{\underline{k}}(\Gamma_g)$ *for any $g \leq 8$, and any $\underline{k}$ with $k_g > g$.*

See Theorems 6 & 7 for equivalent, better formulated, statements. Theorem 1 is about Siegel modular forms of arbitrary weights $\underline{k}$ such that $k_g > g$, but with genus $g \leq 8$. A second result concerns the Siegel modular forms of arbitrary genus, but of weights ≤ 13 (there are really finitely many relevant pairs $(\underline{k}, g)$ here). It is very much in the spirit of the determination of $\dim S_k(\Gamma_g)$ by Chenevier-Lannes in [CL19] in the cases $g \leq k \leq 12$.

Theorem 2. *The dimension of $S_{\underline{k}}(\Gamma_g)$ for $13 \geq k_1 \geq \cdots \geq k_g > g$, and $\underline{k}$ non scalar, is given by Table 5. The dimension of $S_k(\Gamma_g)$ for any $k \leq 13$ and any $g \geq 1$ is given by Table 6.*

The notations for these tables are explained in §5.3. Table 5 shows in particular that $S_{\underline{k}}(\Gamma_g)$ has dimension ≤ 1 for $\underline{k}$ non scalar and $13 \geq k_1 \geq \cdots \geq k_g > g$, and is nonzero for exactly 29 values of $\underline{k}$. Table 6 includes the fact that $S_k(\Gamma_g)$ vanishes whenever $k \leq 13$ and $g \geq k$, except in the three following situations:

$$\dim S_{12}(\Gamma_{12}) = \dim S_{13}(\Gamma_{16}) = \dim S_{13}(\Gamma_{24}) = 1$$

(a nonzero element in the first and last spaces has been constructed in [BFW98] and [Fre82]). We obtain for instance the following result.

Corollary 1. $S_{13}(\Gamma_g)$ *has dimension 1 for $g = 8, 12, 16, 24$, and 0 otherwise.*

An inspection of standard L-functions, and general results of Böcherer, Kudla-Rallis and Weissauer, show that these four spaces are spanned by certain Siegel theta series build on Niemeier lattices (see §5.4). In a companion paper [CT19a] we come back to these constructions and study them in a much more elementary way. Combined with [CL19, Sect. 9.5], *this provides an explicit construction of all the cuspidal Siegel modular eigenforms of weight $k \leq 13$ and level Γ_g for an arbitrary genus g.* In §5.4, we also prove that *Eichler's basis problem holds in*

weights $k = 8$ and 12 *for arbitrary genus g*, completing the results of [CL19] for $g \leq k$.

Last but not least, let us mention that in the past, several other authors have computed $\dim S_k(\Gamma_g)$ for a number of isolated and small pairs (g, k), sometimes with much efforts, e.g. Igusa, Witt, Erokhin, Duke-Immamoglu, [3] Nebe-Venkov, and Poor-Yuen among others. We would like to stress that *none of the results of this paper depend on a previous computation of the dimension of a space of Siegel modular forms*, not even of $S_k(\mathrm{SL}_2(\mathbb{Z}))$! see §1.4. Moreover, as far as we know, the dimensions given by Theorems 1 and 2 seem to recover all the previously known[4] dimensions of spaces of Siegel modular cuspforms for Γ_g.

Our proof of Theorems 1 and 2 will use automorphic methods, building on a strategy developed in the recent works [CR15, CL19, Taï17]: we will review this strategy in §1.4 and §5. An important ingredient is Arthur's *endoscopic classification* of the discrete automorphic spectrum of classical groups in terms of general linear groups [Art13], including the so-called *multiplicity formula*. A special feature of this approach is that even if we were only interested in $\dim S_{\underline{k}}(\Gamma_g)$, we would be forced to compute first the dimension of various spaces of automorphic forms for all the split classical groups over $\mathbb{Z}$ of smaller dimension. By a *split classical group over* $\mathbb{Z}$ we will mean here either the group scheme Sp_{2g} over $\mathbb{Z}$, or the special orthogonal group scheme SO_n of the integral quadratic form $\sum_{i=1}^{n/2} x_i x_{n+1-i}$ (case n even $\neq 2$) or $\sum_{i=1}^{(n-1)/2} x_i x_{n+1-i} + x_{(n+1)/2}^2$ (case n odd) over $\mathbb{Z}^n$. An important gain of this approach, however, is that in the end we do not only compute $\dim S_{\underline{k}}(\Gamma_g)$, but also the dimension of its subspace of cuspforms of any possible endoscopic type, a quantity which is arguably more interesting than the whole dimension itself: see Tables 5 & 6 for a sample of results.

1.2. **Level one algebraic cusp forms on** GL_m. Let $m \geq 1$ be an integer and π a cuspidal automorphic representation of PGL_m over $\mathbb{Q}$. We say that π has *level* 1 if π_p is unramified for each prime p. We say that π is *algebraic* if the infinitesimal character of π_∞, that we may view following Harish-Chandra and Langlands as a semi-simple conjugacy class in $\mathrm{M}_m(\mathbb{C})$, has its eigenvalues in $\frac{1}{2}\mathbb{Z}$, say $w_1 \geq w_2 \geq \cdots \geq w_m$, and with $w_i - w_j \in \mathbb{Z}$. Those eigenvalues w_i are called

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[3]We warn the reader that the proofs of Duke and Immamoglu in [DI98] are not valid in the case $g > k$ since they rely on the incorrect Corollary 3 p. 601 in [Miz91]: see [Miz19] for a recent erratum. Note also that a preliminary version of [CL19] did include a proof of the vanishing of $S_k(\Gamma_g)$ for $k \leq 12$, $g > k$ and $g \neq 24$, but this statement was deleted in the published version for the same reason. Our proofs here show that all these incriminated results for $g > k$ were nevertheless correct, and actually do not rely anymore on the results in [Miz91].

[4]More precisely, the only cases not covered by these two theorems seem to be the vanishing of $S_{\underline{k}}(\Gamma_2)$ for $k_2 = 1$, and for the pairs $\underline{k} = (k_1, 2)$ with $k_1 \leq 50$ [CvdG18, Thm. 1.3]. However, this vanishing for $k_2 = 1$ can also be proved by arguments in the spirit of the ones employed here, as explained by the first author in an appendix of [CvdG18], and we can actually prove it as well for all $\underline{k} = (k_1, 2)$ with $k_1 \leq 54$: see §2.4.7.

the weights of π, and the important integer $w(\pi) := 2w_1$ is called the *motivic weight* of π. The Jacquet-Shalika estimates imply $w_{m+1-i} = -w_i$ for all i, and in particular, $w(\pi) \geq 0$ (see §2.1).

The algebraic cuspidal π are especially interesting to number theorists, as for such a π standard conjectures (by Clozel, Langlands) predict the existence of a compatible system of pure and irreducible ℓ-adic Galois representations ρ with same L-function as $\pi|.|^{w_1}$, the Hodge-Tate weights of ρ being the $w_i + w_1$, and its Deligne weight being $w(\pi)$. The level 1 assumption in this work has to be thought as a simplifying, but still interesting, one (see [CL19] for several motivations).

An important problem is thus the following. For an integer $m \geq 1$, we denote by W_m the set of $\underline{w} = (w_i)$ in $\frac{1}{2}\mathbb{Z}^m$ with $w_1 \geq w_2 \geq \cdots \geq w_m$, $w_i - w_j \in \mathbb{Z}$ and $w_i = -w_{m+1-i}$ for all $1 \leq i, j \leq m$.

Problem 1. *For $\underline{w} \in W_m$, determine the (finite) number $N(\underline{w})$ of level 1 cuspidal algebraic automorphic representations of PGL_m whose weights are the w_i, and the number $N^\perp(\underline{w})$ of those π satisfying furthermore $\pi^\vee \simeq \pi$ (self-duality).*

Let us say that an element (w_i) in W_m is *regular* if for all $i \neq j$ we have either $w_i \neq w_j$, or $m \equiv 0 \bmod 4$, $i = j - 1 = m/2$ and $w_i = w_j = 0$ (hence $w_1 \in \mathbb{Z}$). Despite appearences, the question of determining the $N^\perp(\underline{w})$ for regular $\underline{w}$ is very close to that discussed in §1.1. Indeed, as was observed and used in [CR15, CL19, Taï17], the level 1 self-dual π of regular weights are the exact building blocks for Arthur's endoscopic classification of the discrete automorphic representations of split classical groups over $\mathbb{Z}$ which are unramified at all finite places and discrete series at the Archimedean place (with a very concrete form of Arthur's multiplicity formula, relying on [AMR18]). As an illustration of this slogan, the following fact was observed in [CR15, Chap 9].

Key fact 1. *Fix $g \geq 1$ and $\underline{k} = (k_1, \ldots, k_g) \in \mathbb{Z}^g$ with $k_1 \geq k_2 \geq \cdots \geq k_g > g$. Then the dimension of $S_{\underline{k}}(\Gamma_g)$ is an explicit function of the (finitely many) quantities $N^\perp(\underline{w})$ with $\underline{w} = (w_i) \in W_m$ regular, $m \leq 2g + 1$ and $w_1 \leq k_1 - 1$.*

See [CR15, Prop. 1.11] for explicit formulas for $g \leq 3$, [Taï17, Chap. 5] for $g = 4$, and [CL19, Thm. 5.2] for the general recipe. This general recipe will actually be recalled in §5, in which we will also apply the recent results of [MR] to give an analoguous statement for $S_k(\Gamma_g)$ (scalar-valued case) in the case $k \leq g$. This last case is quite more sophisticated, in particular it also involves certain slightly irregular weights. We will come back later on the relations between the Problem above and Theorems 1 and 2.

1.3. **Classification and inexistence results.** Let us denote by Π_{alg} the set of cuspidal automorphic representations of PGL_m, with $m \geq 1$ arbitrary, which are algebraic and of level 1. The second main result of this paper is a partial classification of elements π in Π_{alg} having motivic weight ≤ 24. The first statement of

this type, proved in [CL19, Thm. F], asserts that there are exactly 11 elements π in Π_{alg} of motivic weight ≤ 22: the trivial representation of PGL_1, the 5 representations Δ_{k-1} of PGL_2 generated by the 1-dimensional spaces $\mathrm{S}_k(\mathrm{SL}_2(\mathbb{Z}))$ for $k = 12, 16, 18, 20, 22$ (whose weights are $\pm\frac{k-1}{2}$), the Gelbart-Jacquet symmetric square of Δ_{11} (with weights $-11, 0, 11$), and four other 4-dimensional self-dual π with respective weights

$$\{\pm\, 19/2,\, \pm\, 7/2\},\ \ \{\pm\, 21/2,\, \pm\, 5/2\},\ \ \{\pm\, 21/2,\, \pm\, 9/2\}\ \text{ and }\ \{\pm\, 21/2,\, \pm\, 13/2\}.$$

In this paper we significantly simplify the proof of [CL19, Thm. F]: see §2.4.6. More importantly, these simplifications allow us to prove the following theorems for motivic weights 23 and 24.

Theorem 3. *There are exactly* 13 *level* 1 *cuspidal algebraic automorphic representations of* GL_m *over* $\mathbb{Q}$*, with* m *varying, with motivic weight* 23*, and having the weight* 23/2 *with multiplicity* 1*:*

 (i) 2 *representations of* PGL_2 *generated by the eigenforms in* $\mathrm{S}_{24}(\mathrm{SL}_2(\mathbb{Z}))$*,*

 (ii) 3 *representations of* PGL_4 *of weights* $\pm 23/2,\, \pm v/2$ *with* $v = 7, 9$ *or* 13*,*

 (iii) 7 *representations of* PGL_6 *of weights* $\pm 23/2,\, \pm v/2,\, \pm u/2$ *with*

$$(v, u) \,=\, (13, 5),\, (15, 3),\, (15, 7),\, (17, 5),\, (17, 9),\, (19, 3)\ \text{ and }\ (19, 11),$$

 (iv) 1 *representation of* PGL_{10} *of weights* $\pm 23/2,\, \pm 21/2,\, \pm 17/2,\, \pm 11/2,\, \pm 3/2$*.*

There are all self-dual (symplectic) and uniquely determined by their weights.

The representations in (ii) and (iii) above were first discovered in [CR15], using some conditional argument that was later made unconditional in [Taï19]. Their existence was also confirmed by the different computation of the second author in [Taï17], who also discovered the 10-dimensional form in (iv).

Despite our efforts, we have not been able to classify the π of motivic weight 23 such that multiplicity of the weight 23/2 is > 1. We could only prove that there is an explicit list $\mathcal{L}$ of 182 weights $\underline{w} = (w_i)$ with $w_1 = w_2 = 23$ such that (a) the weight of any such π belongs to $\mathcal{L}$, (b) for any $\underline{w}$ in $\mathcal{L}$ there is at most one π with this weight (necessarily self-dual symplectic), except for the single weight $\underline{w} = (v_i/2)$ in $\mathrm{W}_{14} \cap \mathcal{L}$ with $(v_1, v_2, \ldots, v_7) = (23, 23, 21, 17, 13, 7, 1)$, for which there might also be two such π which are the dual of each other: see Proposition 4.1. Nevertheless, we conjecture that all of those putative 183 representations do not exist, except perhaps one. Indeed, we prove in §4.3 the following result, assuming a suitable form of (GRH).

Theorem 4. *Assume* (GRH) *and that there exists* π *in* Π_{alg} *with motivic weight* 23 *and having the weight* 23/2 *with multiplicity* > 1*. Then we have* $m = 16$*, the weights of* π *are* $\pm 1/2$*,* $\pm 7/2$*,* $\pm 11/2$*,* $\pm 15/2$*,* $\pm 19/2$*,* $\pm 21/2$ *as well as* $\pm 23/2$ *with multiplicity* 2*, and* π *is the unique element of* Π_{alg} *having these* 16 *weights.*

We now state our (weaker) classification result in motivic weight 24.

Theorem 5. *There are exactly* 3 *level* 1 *algebraic cuspidal self-dual automorphic representations of* PGL_m *over* $\mathbb{Q}$, *with* m *varying, with motivic weight* 24 *and a regular weight. They have respective sets of weights*

$$\{\pm 12, \pm 8, \pm 4, 0\}, \{\pm 12, \pm 9, \pm 5, \pm 2\} \text{ and } \{\pm 12, \pm 10, \pm 7, \pm 1\}.$$

Again, those three forms were first discovered in [CR15, Cor. 1.10 & 1.12] and confirmed in [Taï17]. Interestingly, as explained in [CR15], we expect that their Sato-Tate groups are respectively the compact groups G_2, $\mathrm{Spin}(7)$ and $\mathrm{SO}(8)$. See [Che19, Thm. 6.12] for a proof that the first form, which is 7-dimensional, has G_2-valued ℓ-adic Galois representations.

Proofs. Our proofs of Theorems 3 and 5 are in the same spirit of the one of [CL19, Thm. F]. As already said, all the representations appearing in the theorem were already known to exist by the works [CR15, Taï19, Taï17] (and we will give another proof of their existence in §3), so the main problem is to show that there are no others. The basic idea that we will use for doing so is to consider an hypothetical π, consider an associate L-function of π, and show that this function cannot exist by applying to it the so-called explicit formula for suitable test functions. This is a classical method by now, inspired by the pioneering works of Stark, Odlyzko and Serre on discriminant bounds of number fields. It was developed by Mestre in [Mes86] and applied *loc. cit.* to the standard L-function of π (see also [Fer96]), then by Miller [Mil02] to the Rankin-Selberg L-function of π, and developed further more recently in [CL19, §9.3.4].

Two important novelties were discovered in [CL19] in order to obtain the aforementioned classification result in motivic weight ≤ 22. The first one, developed further in [Chea], is a finiteness result which implies that there are only finitely many level 1 cuspidal algebraic automorphic representations π of PGL_m, with m varying, of motivic weight ≤ 23. This finiteness is also valid in motivic weight ≤ 24 assuming a suitable form of GRH. This result is effective and produces a finite but large list of possible weights for those π (for instance, it leads to 12295 possible weights in motivic weight 23). The hardest part is then to eliminate most of those remaining weights. The second novelty found in [CL19] was the observation that we obtain efficient constraints by applying as well the explicit formula to all the L-functions $\mathrm{L}(s, \pi \times \pi_i)$, where the π_i are the known representations. See [CL19, Scholia 9.3.26 & 9.3.32] for the two useful criteria obtained there using this idea.

In this paper, we discovered a criterion that may be seen as a natural generalisation of [CL19, Scholia 9.3.26], and that happened to be (in practice, and quite surprisingly) much more efficient than the aforementioned ones. Moreover, contrary to [CL19, Scholia 9.3.32], we do not need to know any Satake parameter for the known elements of Π_{alg} (which allow us to use test functions with aribtrary

supports). Our basic idea here is to apply the explicit formula to the Rankin-Selberg L-function of all linear combinations $t_1\pi_1 \oplus \cdots \oplus t_s\pi_s$ where π_1 is unknown of given weights, the π_i with $i > 1$ are known (in the sense that they exist and we know their weights), and the t_i are arbitrary nonnegative *real* numbers. More precisely, for any test function F we associate a certain symmetric bilinear form C^F on the free vector space $\mathbb{R}\,\Pi_{\mathrm{alg}}$ over Π_{alg}, which represents the *computable* part of the explicit formula for the test function F. Assuming a certain positivity assumption on F, the quadratic form $x \mapsto C^F(x,x)$ is then ≥ 0 on the cone of $\mathbb{R}\,\Pi_{\mathrm{alg}}$ generated by Π_{alg}: see Proposition 2.2. In order to reach a contradiction we have thus to show that at least one quadratic form C^F takes a negative value on the cone generated by $\pi_1, \ldots, \pi_s$. See §2.4 for the description of the minimisation algorithm that we have used for this purpose, as well as the homepage [CT19b] for related sources. One charm of this method is that although it requires some computational work to find a concrete element x of that cone and a test function F leading to a contradiction (and all is fair for that!), once we have found it it is quick and easy to rigorously check that we have $C^F(x,x) < 0$: see §2.4.3.

1.4. **The effortless computation of masses.** Fix G a rank n split classical group over $\mathbb{Z}$ in the sense of §1.1. In other words, G belongs to one of the families

$$(\mathrm{SO}_{2n+1})_{n\geq 1}, \quad (\mathrm{Sp}_{2n})_{n\geq 1} \quad \text{and} \quad (\mathrm{SO}_{2n})_{n\geq 2}$$

Assume that $G(\mathbb{R})$ has discrete series, *i.e.* that G is not isomorphic to SO_{2n} with n odd, and fix K a maximal compact subgroup of $G(\mathbb{R})$. There is an analogue of Key fact 1 with $\mathrm{S}_{\underline{k}}(\Gamma_g)$ replaced by the multiplicity of any discrete series of $G(\mathbb{R})$ in the space $\mathcal{A}^2(G)$ of K-finite square-integrable automorphic forms over $G(\mathbb{Z})\backslash G(\mathbb{R})$. In this paper, as in [Taï17], we will use a variant of this statement involving rather certain Euler-Poincaré characteristics. Our first aim is to state this variant (Key fact 2 below). For any dominant weight λ of $G(\mathbb{C})$, we denote by V_λ an irreducible representation of $G(\mathbb{C})$ with highest weight λ and consider

$$\mathrm{EP}(G;\lambda) = \sum_{i\geq 0}(-1)^i \dim \mathrm{H}^i(\mathfrak{g}, K; \mathcal{A}^2(G) \otimes V_\lambda^\vee) \ \in \ \mathbb{Z},$$

where $\mathrm{H}^*(\mathfrak{g}, K; -)$ denotes $(\mathfrak{g}, K)$-cohomology. Attached to G is a certain integer denoted $n_{\widehat{G}}$, defined as the dimension of the standard representation of $\widehat{G}$, the Langlands dual group of G: concretely, we have $n_{\widehat{G}} = 2n + 1$ for $G = \mathrm{Sp}_{2n}$, and $n_{\widehat{G}} = 2n$ for $G = \mathrm{SO}_{2n+1}$ or SO_{2n}. The infinitesimal character of V_λ, namely "$\lambda+\rho$", defines a unique regular element $w(\lambda)$ in W_m with $m = n_{\widehat{G}}$. Concretely, using the standard[5] notation $\lambda = \sum_{i=1}^n \lambda_i e_i$ for dominant weights of classical groups (as in

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[5]The conditions on the λ_i are the following: $\lambda_i \in \mathbb{Z}$ for each i, $\lambda_1 \geq \cdots \geq \lambda_n$, and either $\lambda_n \geq 0$ (cases $G = \mathrm{Sp}_{2n}$ or SO_{2n+1}) or $|\lambda_n| \leq \lambda_{n-1}$ (case $G = \mathrm{SO}_{2n}$ with $n \geq 2$).

[Taï17, §2]), $w(\lambda)$ is explicitely given by the following formulas:

$$w(\lambda)_i = \begin{cases} \lambda_i + n + 1/2 - i & \text{for } 1 \le i \le n \text{ if } G = \mathrm{SO}_{2n+1}, \\ \lambda_i + n + 1 - i & \text{for } 1 \le i \le n \text{ if } G = \mathrm{Sp}_{2n}, \\ |\lambda_i| + n - i & \text{for } 1 \le i \le n \text{ if } G = \mathrm{SO}_{2n} \text{ and } n \equiv 0 \bmod 2. \end{cases}$$

The promised second key fact, explained in §§4.1 and 4.2 of [Taï17] is:

Key fact 2. *Fix G and λ as above, and set $\underline{w} = w(\lambda) = (w_i)$. Assume we know $\mathrm{N}^\perp(\underline{v})$ for all regular $\underline{v} = (v_i) \in \mathrm{W}_m$ with $m < \mathrm{n}_{\widehat{G}}$ and $v_1 \le w_1$, then it is equivalent to know $\mathrm{EP}(G; \lambda)$ and $\mathrm{N}^\perp(\underline{w})$.*

It follows from the formula above for $w(\lambda)$ that any regular $\underline{w}$ in W_m, with $m \ge 1$ arbitrary, is of the form $w(\lambda)$ for a unique split classical group G over $\mathbb{Z}$ and some dominant weight λ of G. As a consequence, Key fact 2 paves the way for a computation of $\mathrm{N}^\perp(\underline{w})$ for all regular w, by induction on $\mathrm{n}_{\widehat{G}}$.

Contrary to the case of Key fact 1, we will not reproduce here the precise claimed relation between $\mathrm{EP}(G; \lambda)$ and the quantities $\mathrm{N}^\perp(-)$ stated in Key fact 2, and simply refer to [Taï17, §4]. As for Key fact 1, it crucially depends on Arthur's endoscopic classification [Art13] and on the explicit description of Archimedean Arthur packets in the regular algebraic case given by [AJ87] and [AMR18] (they are the so-called Adams-Johnson packets, see [Taï17, §4.2.2]). The full combinatorics, when written down explicitly and case-by-case, are rather complicated but computable, and implemented since [Taï17].

The second ingredient is Arthur's L^2-*Lefschetz trace formula* developed in [Art89] applied to the relevant test function. A detailed analysis of this formula has been made in [Taï17] that we will follow below. We use this version of the trace formula as it is the one with the simplest geometric terms. Its spectral side is exactly $\mathrm{EP}(G; \lambda)$, a quantity in principle harder to interpret (the price to pay for a simple geometric side), but this can be done precisely by Key fact 2 above.

We fix a Haar measure $dg = \prod_v' dg_v$ on $G(\mathbb{A})$, with $\mathbb{A}$ the adèle ring of $\mathbb{Q}$, such that the Haar measure dg_p on $G(\mathbb{Q}_p)$ gives $G(\mathbb{Z}_p)$ the volume 1. Fix a dominant weight λ of G. We apply Arthur's formula [Art88] to a test function φ of the form $\varphi_\infty \otimes_p' 1_{G(\mathbb{Z}_p)}$, where $1_{G(\mathbb{Z}_p)}$ is the characteristic function of $G(\mathbb{Z}_p)$, and where $\varphi_\infty(g)dg_\infty$ is the sum, over all the discrete series δ of $G(\mathbb{R})$ with same infinitesimal character as V_λ, of a pseudocoefficient of δ. According to [Art89] the resulting identity, which only depends on G and λ, is

$$(1.4.1) \qquad \mathrm{EP}(G; \lambda) = \mathrm{T}_{\mathrm{geom}}(G; \lambda).$$

The geometric side $\mathrm{T}_{\mathrm{geom}}(G; \lambda)$ is a finite sum of terms indexed by Levi subgroups of G of the form

$$(1.4.2) \qquad \mathrm{GL}_1^a \times \mathrm{GL}_2^b \times G'$$

with G' a split classical groups over $\mathbb{Z}$ and $a, b \geq 0$. The main term, corresponding to G itself and called the *elliptic term*, is

$$(1.4.3) \qquad \mathrm{T}_{\mathrm{ell}}(G; \lambda) = \sum_{\gamma} \mathrm{vol}(G_\gamma(\mathbb{Q}) \backslash G_\gamma(\mathbb{A})) \cdot \mathrm{O}_\gamma(1_{G(\widehat{\mathbb{Z}})}) \cdot \mathrm{tr}(\gamma \,|\, \mathrm{V}_\lambda),$$

where γ runs over representatives of the (finitely many) $G(\mathbb{Q})$-conjugacy classes of finite order elements in $G(\mathbb{Q})$ whose $G(\mathbb{Q}_p)$-conjugacy class meets $G(\mathbb{Z}_p)$ for each prime p (see §3.1). For each such γ, we have denoted by G_γ its centralizer in G (defined over $\mathbb{Q}$), choosed on $G_\gamma(\mathbb{A})$ a signed Haar measure $dh = \prod'_v dh_v$ with dh_∞ an Euler-Poincaré measure on $G_\gamma(\mathbb{R})$ (in the sense of [Ser71]), and we have denoted by $\mathrm{O}_\gamma(1_{G(\widehat{\mathbb{Z}})})$ the product over all primes p of the classical orbital integrals

$$(1.4.4) \qquad \int_{G(\mathbb{Q}_p)/G_\gamma(\mathbb{Q}_p)} 1_{G(\mathbb{Z}_p)} \, (g_p \gamma g_p^{-1}) \, \frac{dg_p}{dh_p}.$$

Taïbi developed in [Taï17] a number of algorithms to enumerate the γ, compute their local orbital integrals (with dh_p Gross's canonical measure), and the associated global volumes. Here we shall simply write

$$(1.4.5) \qquad \mathrm{T}_{\mathrm{ell}}(G; \lambda) = \sum_{c \in \mathcal{C}(G)} \mathrm{m}_c \, \mathrm{tr}(c \,|\, \mathrm{V}_\lambda),$$

where $\mathcal{C}(G)$ denotes the set of $G(\overline{\mathbb{Q}})$-conjugacy classes of finite order elements in $G(\mathbb{Q})$ (essentially, a characteristic polynomial: see §3.1) and m_c is a certain number depending only on c and called *the mass* of c (note that $\mathrm{tr}(c \,|\, \mathrm{V}_\lambda)$ is well-defined). By definition, m_c is a concrete sum of volumes times global integral orbitals; it essentially follows from [Gro97, Theorem 9.9] and Siegel's theorem on the rationality of the values of Artin L-functions at non-negative integers [Sie69] that we have $\mathrm{m}_c \in \mathbb{Q}$.

The character of V_λ may be either evaluated using the (degenerate) Weyl character formula as in [CR15], or much more efficiently for small λ using Koike-Terrada's formulas [KT87] as was observed in [Cheb]. Last but not least, the term in the sum defining $\mathrm{T}_{\mathrm{geom}}(G; \lambda)$ corresponding to a proper Levi subgroup of the form (1.4.2) is expressed in terms of $\mathrm{T}_{\mathrm{ell}}(G'; \lambda')$, as well as $\mathrm{T}_{\mathrm{ell}}(\mathrm{SO}_3; \lambda'')$ if $b \neq 0$, for suitable auxilliary λ', λ'': see [Taï17, §3.3.4] for the concrete formulas, that will not be repeated here. As a consequence, the key problem is to be able to compute the masses m_c for $c \in \mathcal{C}(G)$.

The strategy. We are finally able to explain our strategy. Fix $m \geq 1$ an integer. We may assume, by induction, that we have computed the masses m_c for all split classical groups H over $\mathbb{Z}$ such that $H(\mathbb{R})$ has discrete series and $\mathrm{n}_{\widehat{H}} < m$, and all $c \in \mathcal{C}(H)$. By the Key fact 2 and the trace formula (1.4.1), note that we have an explicit and computable formula for $\mathrm{N}^\perp(\underline{w})$ for all regular $w \in \mathrm{W}_{m'}$ with $m' < m$. Fix a split classical group G over $\mathbb{Z}$ such that $G(\mathbb{R})$ has discrete series

and $n_{\widehat{G}} = m$. Assume we have found a finite set Λ of dominant weights of $G(\mathbb{C})$ with the following two properties:

(P1) For all $\lambda \in \Lambda$ we have $N^\perp(w(\lambda)) = 0$.

(P2) the $\Lambda \times \mathcal{C}(G)$ matrix $(\mathrm{tr}(c\,|\,V_\lambda))_{(\lambda,c)}$ has rank $|\mathcal{C}(G)|$.

Then from (P1) and the Key fact 2 we know $EP(G; \lambda)$ for all $\lambda \in \Lambda$. It follows that for all $\lambda \in \Lambda$ we know $T_{\mathrm{geom}}(G; \lambda)$ as well, by the trace formula (1.4.1), hence also $T_{\mathrm{ell}}(G; \lambda)$ since the non-elliptic geometric terms are also known by induction. By (1.4.5) and (P2), we deduce the masses m_c for all $c \in \mathcal{C}(G)$ by solving a linear system. As a consequence, for an arbitrary dominant weight λ of G we may then compute $T_{\mathrm{geom}}(G; \lambda)$, hence $EP(G; \lambda)$ by (1.4.1), and $N^\perp(w(\lambda))$ by Key fact 2.

Amusingly, the conclusion is that we end by proving the existence of self-dual cusp forms for PGL_m mostly by showing that many others do not exist ! (namely the ones with weights of the form $w(\lambda)$ with λ in Λ.)

A simple example. Let us illustrate this method in the (admittedly too) simple case $m = 2$ and $G = \mathrm{SO}_3$. In this case $\mathcal{C}(G)$ has 5 classes, of respective order $1, 2, 3, 4$ and 6: say c_i has order i. The dominant weights of G are of the form ke_1 for a unique integer $k \geq 0$, and will be simply denoted by k. For $k \geq 0$, we have $w(k) = (k + 1/2, -k - 1/2)$, $N(w(k)) = \dim S_{2k+2}(\mathrm{SL}_2(\mathbb{Z}))$, the analysis of the spectral side gives

$$EP(G; k) = -N(w(k)) + \delta_{k,0},$$

with $\delta_{i,j}$ the Kronecker symbol, and the geometric side is

$$T_{\mathrm{geom}}(G; k) = T_{\mathrm{ell}}(G; k) + \frac{1}{2}.$$

Assume we know that there is no cuspidal modular form for $\mathrm{SL}_2(\mathbb{Z})$ of usual weight $2, 4, 6, 8, 10$. This may for instance be shown by applying the explicit formula to the Hecke L-function of a putative eigenform of such a weight, as observed in [Mes86, Rem. III.1]. This also follows very easily from the methods of §1.3. Using $\dim V_k = 2k+1$ and the identity $\mathrm{tr}(c_i | V_k) = \sin(\frac{(2k+1)\pi}{i}) / \sin \frac{2\pi}{i}$ for $i > 1$ we obtain with $\Lambda = \{0, 1, 2, 3, 4\}$ the linear system

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
3 & -1 & 0 & 1 & 2 \\
5 & 1 & -1 & -1 & 1 \\
7 & -1 & 1 & -1 & -1 \\
9 & 1 & 0 & 1 & -2
\end{bmatrix}
\cdot
\begin{bmatrix}
m_{c_1} \\
m_{c_2} \\
m_{c_3} \\
m_{c_4} \\
m_{c_6}
\end{bmatrix}
=
\begin{bmatrix}
1/2 \\
-1/2 \\
-1/2 \\
-1/2 \\
-1/2
\end{bmatrix}.
$$

Luckily, the matrix on the left-hand side is invertible: we find $m_{c_1} = -\frac{1}{12}$, $m_{c_2} = \frac{1}{4}$, $m_{c_3} = \frac{1}{3}$ and $m_{c_4} = m_{c_6} = 0$. As a consequence, we recover the classical formula for $\dim S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$.

Remark 1.1. In certain classes c in $\mathcal{C}(G)$, there is no element γ whose $G(\mathbb{Q})$-conjugacy class meets $G(\mathbb{Z}_p)$ for each p and such that $G_\gamma(\mathbb{R})$ has discrete series: this forces $m_c = 0$ by (1.4.3). This actually explains $m_{c_4} = m_{c_6} = 0$ above.

This remark will lead us in §3.2, and following [Taï17, Remark 3.2.8], to replace $\mathcal{C}(G)$ by a smaller set $\mathcal{C}_1(G)$, and to rather apply our strategy with $\mathcal{C}(G)$ replaced by $\mathcal{C}_1(G)$ in (P2). See also §3.1 for other more elementary reductions, using the center of G or an outer automorphism of SO_{2n}.

The crucial last ingredient for this method to work is to be able to find sufficiently many $w \in \mathrm{W}_{n_{\widehat{G}}}$ such that $\mathrm{N}^{\perp}(w) = 0$. We will use of course for this the method explained in §1.3 (the explicit formula for Rankin-Selberg L-functions). Rather miraculously, it provides enough vanishing results up to rather a high rank: see §3.3 for a proof of the following final theorem, obtained by applying only this "effortless" strategy:

Theorem 6. *("Effortless" computation) Assume $G = \mathrm{SO}_n$ with $n \leq 17$, or $G = \mathrm{Sp}_{2n}$ with $2n \leq 14$. Then the masses m_c, for all $c \in \mathcal{C}(G)$, are given in* [CT19b].

As already said, these results are in accordance with all the orbital integral computations done in [Taï17]. This method is both conceptually simpler and faster: for comparison, computing all the orbital integrals for Sp_{14} takes several weeks, whereas finding Λ and solving the linear system to determine all the m_c, for $c \in \mathcal{C}(\mathrm{Sp}_{14})$, only takes minutes on the same computer. The cases SO_m with $m = 15, 16, 17$ are new. Last but not least, if we combine the methods of this paper with Taïbi's, we obtain the following new result in the symplectic case (see §3.3).

Theorem 7. *The masses m_c, for all $c \in \mathcal{C}(\mathrm{Sp}_{16})$, are given in* [CT19b].

Following Key fact 2, these two theorems allow to compute $\mathrm{EP}(G; \lambda)$ for all those G and an arbitrary weight λ, as well as the quantity $\mathrm{N}^{\perp}(\underline{w})$ for any $w \in \mathrm{W}_m$ for $m \leq 16$: see *loc. cit.* for tables.

1.5. **Limits of the method and possible generalizations.** At present, it seems very difficult to us to improve any of the classification Theorems 3, 4 and 5, or to extend significantly the number of vanishing results needed for the effortless computation of masses in §3.3, without a really new idea. Our numerical experiments suggest that those results are at the limit of what can be extracted from the explicit formula, or at least from Proposition 2.2, but whether there is a deeper reason for that remains a mystery to us. As an example, we still do not know, even conjecturally, if there should be finitely many cuspidal level 1 algebraic π of PGL_n, with $n \geq 1$ arbitrary, whose motivic weight is 25: see Example 6.7 in [Chea].

These limitations have consequences for the applications to the dimensions of spaces of Siegel cuspforms: Theorems 1, 2 and 6 seem to be the optimal results that can be obtained using our method. In particular, as already explained in §1.4, our computation of the masses of Sp_{16} in Theorem 7 is already *not* "effortless" anymore. Also, it seems unlikely to be able soon to extend the computation of $\dim \mathrm{S}_k(\Gamma_g)$ to a weight k much higher than 13: already in the case $k = 16$ and arbitrary g,

this question is closely related (via Siegel theta series) to that of determining the size of the set X_{32} of isometry classes of even unimodular lattices of rank 32, a classical problem usually considered as out of reach using any known computational method. In particular, we have $1 + \sum_{g \geq 1} \dim S_{16}(\Gamma_g) \geq |X_{32}|$, and the huge lower bound $|X_{32}| > 10^9$ due to King [Kin03] (compare with Table 6).

There are nevertheless several possible generalizations of the results of this paper that would deserve to be studied, the most obvious ones being to work over arbitrary base number fields, to include non trivial conductors, other groups, or to compute traces of Hecke operators of small degree rather than dimensions. Along these lines, we mention the forthcoming Ph.D. dissertation of Lachaussée [Lac] for some variants of the results of §1.3 for cuspidal algebraic representations of GL_n over $\mathbb{Q}$ whose conductor is a small prime number.

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2. WEIL'S EXPLICIT FORMULA FOR RANKIN-SELBERG L-FUNCTIONS: A REFINED POSITIVITY CRITERION

2.1. **Algebraic Harish-Chandra modules and automorphic representations.** Let π be a cuspidal automorphic representation of PGL_n over $\mathbb{Q}$. As recalled in §1.2, we say that π has level 1 if π_p is unramified for every prime p. We also explained *loc. cit.* what it means for π to be algebraic. It will be useful to have an alternative point of view on this last condition in terms of the *Langlands correspondence* (see e.g. [CL19, §8.2.12]). Recall that Π_{alg} denotes the set of level 1 algebraic cuspidal automorphic representations of PGL_n (with $n \geq 1$ varying).

We denote by $W_{\mathbb{R}}$ the Weil group of $\mathbb{R}$: we have $W_{\mathbb{R}} = \mathbb{C}^\times \coprod j\mathbb{C}^\times$, where j^2 is the element -1 of $\mathbb{C}^\times$ and with $jzj^{-1} = \overline{z}$ for all $z \in \mathbb{C}^\times$. The Langlands correspondence for $GL_n(\mathbb{R})$ is a natural bijection $V \mapsto L(V)$ between the set of isomorphism classes of irreducible admissible Harish-Chandra modules for $GL_n(\mathbb{R})$, and the set of isomorphism classes of n-dimensional (complex, continuous and semisimple) representations of $W_{\mathbb{R}}$ [Kna94]. We say that the Harish-Chandra module V is *algebraic* if every element in the center $\mathbb{R}^\times$ of $W_{\mathbb{R}}$ acts as a homothety with factor ± 1 in $L(V)$. In particular, $L(V)$ factors through the (compact) quotient of $W_{\mathbb{R}}$ by $\mathbb{R}_{>0}$, which is an extension of $\mathbb{Z}/2$ by the unit circle. We denote by 1 the trivial representation of $W_{\mathbb{R}}$, by $\varepsilon_{\mathbb{C}/\mathbb{R}}$ its unique order 2 character, and for $w \in \mathbb{Z}$ we set $I_w = \mathrm{Ind}_{\mathbb{C}^\times}^{W_{\mathbb{R}}} \eta^w$ where $\eta(z) = z/|z|$. Up to isomorphism, the irreducible representations of $W_{\mathbb{R}}$ trivial on $\mathbb{R}_{>0}$ are

$$1, \quad \varepsilon_{\mathbb{C}/\mathbb{R}}, \quad \text{and} \quad I_w \text{ for } w > 0.$$

We also have $I_0 \simeq 1 \oplus \varepsilon_{\mathbb{C}/\mathbb{R}}$, and $I_w \simeq I_{w'}$ if and only if $w = \pm w'$.

If π is a *cuspidal* automorphic representation of PGL_n over $\mathbb{Q}$, Clozel's purity lemma (or the Archimedean Jacquet-Shalika estimates) shows that the Harish-Chandra module π_∞ is algebraic in the sense above if, and only if, π is algebraic in the sense of §1.2 [CL19, Prop. 8.2.13]. Moreover, for all $v \in \mathbb{Z}$ the multiplicity of the weight $v/2$ of π is the same as the multiplicity of the character η^v in the restriction of $\mathrm{L}(\pi_\infty)$ to $\mathbb{C}^\times$. In other words, all the weights 0 (resp. $\pm v/2$ with $v > 0$) of π_∞ are explained by occurences of 1 or $\varepsilon_{\mathbb{C}/\mathbb{R}}$ (resp. of I_v) in $\mathrm{L}(\pi_\infty)$. It will be convenient to introduce:

- the Grothendieck ring K_∞ of complex, continuous, finite dimensional, representations of $\mathrm{W}_\mathbb{R}$ which are trivial on its central subgroup $\mathbb{R}_{>0}$,
- for w an integer, the sugroup $\mathrm{K}_\infty^{\leq w}$ of K_∞ generated by the I_v with $0 \leq v \leq w$ and $v \equiv w \bmod 2$, and also by 1 and $\varepsilon_{\mathbb{C}/\mathbb{R}}$ in the case w is even.

An element U of K_∞ will be called *effective* if it is the class of a representation of $\mathrm{W}_\mathbb{R}$. The *weights* of such a U are the $w/2$ in $\frac{1}{2}\mathbb{Z}$ such that η^w occurs in $U_{|\mathbb{C}^\times}$, counted with the multiplicity of η^w in $U_{|\mathbb{C}^\times}$ (a nonnegative number as U is effective).

It follows from this discussion that for π in Π_{alg}, we have $\mathrm{L}(\pi_\infty) \in \mathrm{K}_\infty^{\leq w}$ for some integer $w \geq 0$, and that the smallest such integer w coincides with the motivic weight $w(\pi)$ of π introduced in §1.2. Moreover, the weights of π are that of $\mathrm{L}(\pi_\infty)$.

For later use, we now recall Langlands' definition for the ε-factors and Γ-factors of algebraic Harish-Chandra modules. The Γ-factor of an element U in K_∞ is a meromorphic function $s \mapsto \Gamma(s, U)$ on the whole complex plane, characterized by the additivity property $\Gamma(s, U \oplus U') = \Gamma(s, U)\Gamma(s, U')$ and the following axioms [Tat79]:

$$\Gamma(s, 1) = \pi^{-\frac{s}{2}}\Gamma(\frac{s}{2}) \quad \text{and} \quad \Gamma(s, \mathrm{I}_w) = 2(2\pi)^{-s - \frac{|w|}{2}}\Gamma(s + \frac{|w|}{2}) \text{ for all } w \in \mathbb{Z},$$

in which $s \mapsto \Gamma(s)$ is the classical gamma function. Similarly,[6] the ε-factor of $U \in \mathrm{K}_\infty$ is the element $\varepsilon(U)$ of $\{\pm 1, \pm i\}$ characterized by the additivity property $\varepsilon(U \oplus U') = \varepsilon(U)\varepsilon(U')$ and the identities $\varepsilon(1) = 1$ and $\varepsilon(\mathrm{I}_w) = i^{w+1}$ for every integer $w \geq 0$.

2.2. **Regular and self-dual elements of Π_{alg}.** Let π be a level 1 algebraic cuspidal automorphic representation of PGL_n over $\mathbb{Q}$. We will say that π is *regular* if the representation $\mathrm{L}(\pi_\infty)$ of $\mathrm{W}_\mathbb{R}$ is multiplicity free. It is thus equivalent to say that for each weight w of π, either w has multiplicity 1 or we have $w = 0$ and $\mathrm{L}(\pi_\infty)$ contains both 1 and $\varepsilon_{\mathbb{C}/\mathbb{R}}$ with multiplicity 1. This latter case can only occur of course if both the motivic weight of π and n are even. Moreover, we observe that:

---

[6]What we denote by $\varepsilon(U)$ here is what Tate denotes $\varepsilon(U, \psi, dx)$ in [Tat79, (3.4)], with choice of additive character $\psi(x) = e^{2i\pi x}$, and with dx the standard Lebesgue measure on $\mathbb{R}$.

- if all the nonzero weights of π have multiplicity 1, and if the weight 0 has multiplicity 2, then π is regular if, and only if, we have $n \equiv 0 \bmod 4$.

- π is regular if, and only if, the vector $(w_i) \in \frac{1}{2}\mathbb{Z}^n$, where $w_1 \geq w_2 \geq \cdots \geq w_n$ are the weights of π, is regular in the sense of §1.2.

- if π is regular and $n = 2g + 1$ is odd, then $\mathrm{L}(\pi_\infty)$ contains $\varepsilon_{\mathbb{C}/\mathbb{R}}^g$.

Indeed, as π a trivial central character, we must have $\det \mathrm{L}(\pi_\infty) = 1$, and we conclude by the formula $\det \mathrm{I}_v = \varepsilon_{\mathbb{C}/\mathbb{R}}^{v+1}$ for $v \in \mathbb{Z}$.

Assume now that π is *self-dual*, that is, isomorphic to its contragredient (or *dual*) $\pi^\vee$. Then π is either *symplectic* or *orthogonal* in the sense of Arthur [Art13, Thm. 1.4.1]. Moreover, if π is symplectic (resp. orthogonal) then $\mathrm{L}(\pi_\infty)$ preserves a nondegenerate alternating (resp. symmetric) pairing [Art13, Thm. 1.4.2]. In particular, if $\pi \in \Pi_{\mathrm{alg}}$ is self-dual, and if some weight of π has multiplicity 1 (e.g. if π is regular), then π is symplectic if and only if $w(\pi)$ is odd [CL19, Prop. 8.3.3].

2.3. **The explicit formula for L-functions of pairs of elements in Π_{alg}.** Let π and π' be level 1 cuspidal automorphic representations of PGL_n and PGL_m respectively. For p prime we denote by $\mathrm{c}_p(\pi)$ the semi-simple conjugacy class in $\mathrm{SL}_n(\mathbb{C})$ associated with the unramified representation π_p, following Langlands, under the Satake isomorphism. The Rankin-Selberg L-function of π and π' is the Euler product

$$\mathrm{L}(s, \pi \times \pi') = \prod_p \det(1 - p^{-s}\mathrm{c}_p(\pi) \otimes \mathrm{c}_p(\pi'))^{-1}.$$

By fundamental works of Jacquet, Piatetski-Shapiro, and Shalika [JS81, JPSS83], this Euler product is absolutely convergent for $\mathrm{Re}\, s > 1$, and the *completed* L-function

(2.3.1) $$\Lambda(s, \pi \times \pi') = \Gamma(s, \mathrm{L}(\pi_\infty) \otimes \mathrm{L}(\pi'_\infty))\, \mathrm{L}(s, \pi \times \pi'),$$

has a meromorphic continuation to $\mathbb{C}$ and a functional equation of the form

(2.3.2) $$\Lambda(s, \pi \times \pi') = \epsilon(\pi \times \pi')\, \Lambda(1 - s, \pi^\vee \times (\pi')^\vee)$$

where $\epsilon(\pi \times \pi')$ is a certain nonzero complex number (it does not depend on s as π has level 1). We set $\epsilon(\pi) = \epsilon(\pi \times 1)$.

Assuming π and π' are algebraic, the only case of interest here, the $\Gamma(s, -)$ factor in (2.3.1) is given by the recipe recalled in §2.1, and we simply have

(2.3.3) $$\epsilon(\pi \times \pi') = \varepsilon(\mathrm{L}(\pi_\infty) \otimes \mathrm{L}(\pi'_\infty)).$$

Note that the ring structure of K_∞ is determined by the relations $\mathrm{I}_w \cdot \mathrm{I}_{w'} = \mathrm{I}_{|w+w'|} + \mathrm{I}_{|w-w'|}$ and $\varepsilon_{\mathbb{C}/\mathbb{R}} \cdot \mathrm{I}_w = \mathrm{I}_w$.

By Moeglin and Waldspurger [MW89, App.], $\Lambda(s, \pi \times \pi')$ is entire unless we have $\pi' \simeq \pi^\vee$, in which case the only poles are simple and at $s = 0, 1$. Moreover, $\Lambda(s, \pi \times \pi')$ is bounded in vertical strips away from its poles by Gelbart and

Shahidi [GS01]. All those analytic properties are key to establishing the Weil explicit formula (for which we refer to Poitou [Poi77b, §1]) in this context. The general formalism of Mestre [Mes86, §I] applies verbatim: we refer to [CL19, Chap. 9, Sect. 3] for the details and only recall here what we need to prove our criterion.

We denote by $\mathbb{R}\,\Pi_{\mathrm{alg}}$ the $\mathbb{R}$-vector space with basis Π_{alg}. We fix a *test function* F, that is an even function $\mathbb{R} \to \mathbb{R}$ satisfying the axioms (i), (ii) and (iii) of [Mes86, §I.2] with the constant *c loc. cit.* equal to 0 (see also [Poi77b, §1]). The reader will not lose anything here assuming simply that F is compactly supported and of class $\mathcal{C}^2$. We denote by $\widehat{F}$ the Fourier transform of F, with the convention $\widehat{F}(\xi) = \int_{\mathbb{R}} F(x) e^{-2i\pi x\xi} dx$. Following [CL19, Chap. 9, Sect. 3], we first define five symmetric bilinear forms on $\mathbb{R}\,\Pi_{\mathrm{alg}}$, that we denote by B_f^F, B_∞^F, Z^F, $\mathrm{e}^\perp$ and δ. The first three of them depend on the choice of F. They are uniquely determined by their values on any $(\pi, \pi') \in \Pi_{\mathrm{alg}} \times \Pi_{\mathrm{alg}}$:

(a) $\mathrm{B}_f^F(\pi, \pi') = \Re \sum_{p,k} F(k\log p) \frac{\log p}{p^{k/2}} \overline{\mathrm{tr}\,(\mathrm{c}_p(\pi)^k)}\,\mathrm{tr}\,(\mathrm{c}_p(\pi')^k)$, the sum being over all primes p and integers $k \geq 1$.

(b) $\mathrm{B}_\infty^F(\pi, \pi') = \mathrm{J}_F(\mathrm{L}(\pi_\infty) \otimes \mathrm{L}(\pi'_\infty))$, where $\mathrm{J}_F : \mathrm{K}_\infty \to \mathbb{R}$ is the linear map defined by

$$(2.3.4) \qquad \mathrm{J}_F(U) = -\int_{\mathbb{R}} \frac{\Gamma'}{\Gamma}\left(\frac{1}{2} + 2\pi t\,i, U\right) \widehat{F}(t)\,\mathrm{d}t.$$

We will also denote abusively by B_∞^F the real-valued symmetric bilinear form on K_∞ defined by $\mathrm{B}_\infty^F(U, V) = \mathrm{J}_F(U \cdot V)$. With these abusive notations we have $\mathrm{B}_\infty^F(\pi, \pi') = \mathrm{B}_\infty^F(\mathrm{L}(\pi_\infty), \mathrm{L}(\pi'_\infty))$.

(c) $\mathrm{Z}^F(\pi, \pi')$ is the limit of $\sum_\rho (\mathrm{ord}_{s=\rho} \Lambda(s, \pi^\vee \times \pi'))\,\Re\widehat{F}(\frac{1-2\rho}{4i\pi})$, the sum being over the zeros ρ of $\Lambda(s, \pi^\vee \times \pi')$ with $0 \leq |\Im\rho| \leq T$ and $0 \leq \Re\rho \leq 1$, when the real number T goes to $+\infty$.

(d) $\delta(\pi, \pi') = 1$ if $\pi \simeq \pi'$, and $\delta(\pi, \pi') = 0$ otherwise (Kronecker symbol).

(e) $\mathrm{e}^\perp(\pi, \pi') = 1$ if π and π' are self-dual with $\epsilon(\pi \times \pi') = -1$, and $\mathrm{e}^\perp(\pi, \pi') = 0$ otherwise.

The main result is that for any test function F we have the equality of bilinear forms

$$(2.3.5) \qquad \mathrm{B}_f^F + \mathrm{B}_\infty^F + \frac{1}{2}\,\mathrm{Z}^F = \widehat{F}\left(\frac{i}{4\pi}\right)\delta \qquad (\textit{the ``explicit formula''})$$

on the space $\mathbb{R}\,\Pi_{\mathrm{alg}}$: see [Mes86, §I.2] and [CL19, Prop. 9.3.9]. We finally define a last bilinear form on $\mathbb{R}\,\Pi_{\mathrm{alg}}$ by the formula

$$(2.3.6) \qquad \mathrm{C}^F := \widehat{F}\left(\frac{i}{4\pi}\right)\delta - \mathrm{B}_\infty^F - \frac{1}{2}\widehat{F}(0)\,\mathrm{e}^\perp.$$

In our applications, it will represent the "computable" part of the explicit formula. Note that for any test function F, both $\widehat{F}(0)$ and $\widehat{F}(i/4\pi)$ are real numbers, and if F is non-negative then they are both non-negative.

Definition 2.1. *Let F be a test function. We will say that F satisfies* (POS) *if we have $F(x) \geq 0$ for all $x \in \mathbb{R}$, and $\Re\, \widehat{F}(\xi) \geq 0$ for all $\xi \in \mathbb{C}$ with $|\mathrm{Im}\, \xi| \leq \frac{1}{4\pi}$.*

Proposition 2.2. *Let F be a test function satisfying* (POS). *Then for any integer $r \geq 1$, any $\pi_1, \ldots, \pi_r$ in Π_{alg} and any nonnegative real numbers $t_1, \ldots, t_r$, we have*

$$(2.3.7) \qquad \mathrm{C}^F\Big(\sum_i t_i \pi_i, \sum_i t_i \pi_i\Big) \geq 0.$$

Proof. By density of the rationals in $\mathbb{R}$, we may assume that the t_i are rational numbers, and even that they are integers by homogeneity of the quadratic form $x \mapsto \mathrm{C}^F(x, x)$. But in this case, the statement is [CL19, Cor. 9.3.12]. As the proof is very simple, we give a direct argument. By (2.3.5) we have $\mathrm{C}^F = \mathrm{B}_f^F + \frac{1}{2}(\mathrm{Z}^F - \widehat{F}(0)\, \mathrm{e}^\perp)$. By definition (a) and the assumption $F \geq 0$, the symmetric bilinear form B_f^F is positive (semi-definite) on $\mathbb{R}\Pi_{\mathrm{alg}}$. It is thus enough to show that $\frac{1}{2}(\mathrm{Z}^F - \widehat{F}(0)\, \mathrm{e}^\perp)$ has nonnegative coefficients in the natural basis Π_{alg} of $\mathbb{R}\,\Pi_{\mathrm{alg}}$, *i.e.* that we have $\mathrm{Z}^F(\pi, \pi') \geq \widehat{F}(0)\, \mathrm{e}^\perp(\pi, \pi')$ for all $\pi, \pi' \in \Pi_{\mathrm{alg}}$. But this follows from the definition of $\mathrm{Z}^F(\pi, \pi')$, the assumption on $\Re\, \widehat{F}$, and the fact that if we have $\mathrm{e}^\perp(\pi, \pi') = 1$ then $\Lambda(s, \pi^\vee \times \pi')$ has a zero at $s = 1/2$ by the functional equation (2.3.2). $\square$

2.4. **Applications.** In what follows we will apply Proposition 2.2 to disprove the existence of representations π in Π_{alg} such that π_∞ is a given algebraic representation, using the knowledge that there are representations in Π_{alg} with known Archimedean components.

2.4.1. *The basic inequalities.* Before doing so, we first recall the following basic but important consequence of the explicit formula, that we derive here as a very special case of Proposition 2.2 (see also [CL19, Cor. 9.3.12 & 9.3.14]).

Corollary 2.3. *Let F be a test function satisfying* (POS) *and fix U in K_∞. If there is an element π in Π_{alg} with $\mathrm{L}(\pi_\infty) = U$ then we have the inequality*

$$(2.4.1) \qquad \mathrm{B}_\infty^F(U, U) \leq \widehat{F}(i/4\pi).$$

More generally, if there are distinct elements $\pi_1, \ldots, \pi_m$ in Π_{alg} with $\mathrm{L}((\pi_j)_\infty) = U$ for all j, then we have

$$(2.4.2) \qquad \mathrm{B}_\infty^F(U, U) \leq \frac{1}{m}\widehat{F}(i/4\pi).$$

Proof. Consider the element $x = \sum_{i=1}^{m} \pi_i$ of $\mathbb{R}\Pi_{\text{alg}}$. We have $C^F(x,x) \geq 0$ by Proposition 2.2. We clearly have[7] $\widehat{F}(0)e^{\perp}(x,x) \geq 0$, by the inequality $\widehat{F}(0) \geq 0$. We conclude by the equalities $\delta(x,x) = m$ and $B_{\infty}^F(x,x) = m^2 B_{\infty}^F(U,U)$. □

Establishing inequality (2.4.1) is the original application of the explicit formula for Rankin-Selberg L-function to prove the nonexistence of certain π in Π_{alg} with given π_{∞}. It was used by [8] Miller in [Mil02] to show that for $n \leq 12$ there is no π in $\Pi_{\text{alg}} \smallsetminus \{1\}$ such that $L(\pi_{\infty})$ is either $I_1 + I_3 + \cdots + I_{2n+1}$ or $\varepsilon_{\mathbb{C}/\mathbb{R}}^n + I_2 + I_4 + \cdots + I_{2n}$. As explained in [CL19, Sect. 9.3] and [Chea], the simple inequality (2.4.1) is very constraining generally in motivic weight ≤ 23: for suitable F the bilinear form B_{∞}^F is positive definite on $K_{\infty}^{\leq w}$ for $w \leq 23$, and there is an explicit finite list $\mathcal{L}$ of elements of K_{∞} such that whenever U is in $K_{\infty}^{\leq w} - \mathcal{L}$ with $w \leq 23$, there is no π in Π_{alg} with $L(\pi_{\infty}) = U$. It has however some limitations: as we shall see, the list $\mathcal{L}$ is quite large, and far from optimal. For instance, it does not seem possible to exclude this way the possibility[9] $L(\pi_{\infty}) \simeq I_{13}$. Nevertheless, Inequality (2.4.1) will be extremely helpful to us in §3.3 and Sect. 4. Inequality (2.4.2) was first observed by Taïbi. In the case $B_{\infty}^F(U,U) > 0$, it may be seen as an effective form of Harish-Chandra's finiteness theorem. We will often use it to show that there is at most one π in Π_{alg} with given $L(\pi_{\infty}) = U$; note that such a π has to be self-dual if it exists, as we have $L((\pi^{\vee})_{\infty}) = L(\pi_{\infty})^{\vee} = L(\pi_{\infty})$.

2.4.2. *A general method.* For π in Π_{alg}, set $\text{sd}(\pi) = 1$ if π is self-dual, and $\text{sd}(\pi) = 0$ otherwise. In this section, we will develop a method trying to answer *in the negative* the following question.

Question 2.4. *Fix an integer $r \geq 1$, and for each $1 \leq i \leq r$ elements U_i in K_{∞} and δ_i in $\{0,1\}$. Does there exist distinct representations π_1, ..., π_r in Π_{alg} with $L((\pi_i)_{\infty}) = U_i$ and $\text{sd}(\pi_i) = \delta_i$ for each $1 \leq i \leq r$?*

To do so, assume we are given an integer $r \geq 1$ and for each $1 \leq i \leq r$, elements U_i in K_{∞}, δ_i in $\{0,1\}$, and an integer $m_i \geq 1$. In other words, we fix a quadruple

$$(2.4.3) \qquad\qquad \mathcal{Q} = (r, \underline{U}, \underline{\delta}, \underline{m})$$

---

[7] We actually have $e^{\perp}(x,x) = 0$. Indeed, if π, π' are in Π_{alg} with $L(\pi_{\infty}) = L(\pi'_{\infty})$, then $e^{\perp}(\pi,\pi') = 0$. To see this, we may assume π and π' are self-dual, either both symplectic or both orthogonal (they have the same motivic weight by assumption), and the assertion follows then from the general property $\epsilon(\pi \times \pi') = 1$ proved in [Art13, Thm. 1.5.3 (b)]. Alternatively, we can easily check $\varepsilon(U \cdot U) = 1$ for $U = L(\pi_{\infty})$.

[8] In the context of Artin L-functions, the advantages of considering Rankin-Selberg L-functions had already been noticed by Serre, see [Poi77a] p. 150.

[9] An intuitive reason for that is that there actually exists a π' in Π_{alg} with very close weights, namely $\pi' = \Delta_{11}$, with $L(\pi'_{\infty}) \simeq I_{11}$. See the discussion in [CL19, Sect. 9.3.19] for many other examples (and how to deal with this case differently), which allow to develop some intuition.

with $\underline{U} = (U_i)_{1 \leq i \leq r}$ in K_∞^r, $\underline{\delta} = (\delta_i)_{1 \leq i \leq r}$ in $\{0,1\}^r$ and $\underline{m} = (m_i)_{1 \leq i \leq r}$ in $\mathbb{Z}_{\geq 1}^r$. View $\mathbb{R}^r$ as an Euclidean space for the standard scalar product $(x_i) \cdot (y_i) = \sum_i x_i y_i$. To the choice of $\mathcal{Q}$ and of a test function F, we associate the symmetric bilinear form $\beta_{\mathcal{Q}}^F$ on $\mathbb{R}^r$ defined by the formula

$$(2.4.4) \qquad \beta_{\mathcal{Q}}^F(e_i, e_j) = \frac{1}{m_i} \widehat{F}(i/4\pi) \; e_i \cdot e_j - \mathrm{J}_F(U_i \cdot U_j) - \widehat{F}(0) \delta_i \delta_j \frac{1 - \varepsilon(U_i \cdot U_j)}{4},$$

where $e_1, \ldots, e_r$ is the canonical (orthonormal) basis of $\mathbb{R}^r$. We will discuss the practical numerical evaluation of $\beta_{\mathcal{Q}}^F$ (i.e. of J_F, $\widehat{F}(0)$ and $\widehat{F}(\frac{i}{4\pi})$) in §2.4.3. Set $\mathbb{S}_+^{r-1} = \{(t_i) \in \mathbb{R}^r \mid \sum_{i=1}^r t_i^2 = 1 \text{ and } \forall i, t_i \geq 0\}$.

Problem 2. *Fix a test function F and a quadruple $\mathcal{Q} = (r, \underline{U}, \underline{\delta}, \underline{m})$ as in (2.4.3). Determine wether the map $x \mapsto \beta_{\mathcal{Q}}^F(x, x)$ takes a negative value on $\mathbb{S}_+^{r-1}$.*

The relationship between Question 2.4 and this problem (which does not involve automorphic representations) is the following. Suppose $m_i = 1$ for each $1 \leq i \leq r$ (the general m_i will play a role only later). Assume there are distinct $\pi_1, \ldots, \pi_r$ in Π_{alg} with $\mathrm{L}((\pi_i)_\infty) = U_i$ and $\mathrm{sd}(\pi_i) = \delta_i$ for each $1 \leq i \leq r$. Denote $V = \bigoplus_{i=1}^r \mathbb{R}\pi_i \subset \mathbb{R}\Pi_{\mathrm{alg}}$ viewed as an Euclidean space with orthonormal basis $(\pi_1, \ldots, \pi_r)$. As the π_i are distinct we actually have $x \cdot x = \delta(x, x)$ for all $x \in V$. We also have

$$(2.4.5) \quad \mathrm{B}_\infty^F(\pi_i, \pi_j) = \mathrm{B}_\infty^F(U_i, U_i) = \mathrm{J}_F(U_i \cdot U_j), \;\; \mathrm{e}^\perp(\pi_i, \pi_j) = \delta_i \, \delta_j \, \frac{1 - \varepsilon(U_i \cdot U_j)}{2}.$$

In other words, the linear map $\iota : \mathbb{R}^r \to V$ defined by $e_i \mapsto \pi_i$ is an isometry satisfying $\mathrm{C}^F(\iota(x), \iota(y)) = \beta_{\mathcal{Q}}^F(x, y)$ for all $x, y \in \mathbb{R}^r$. If we are able to find an element $\underline{t} = (t_i) \in \mathbb{S}_+^{r-1}$ with $\beta_{\mathcal{Q}}^F(\underline{t}, \underline{t}) < 0$, then the element $\iota(\underline{t}) = \sum_{i=1}^r t_i \pi_i$ contadicts Proposition 2.2: we have answered Question 2.4 in the negative.

From now on we thus focus on Problem 2. We fix an arbitrary quadruple $\mathcal{Q} = (r, \underline{U}, \underline{\delta}, \underline{m})$ as in (2.4.3) and a test function F. To simplify the notations we also set $E = \mathbb{R}^r$ and $D = \mathbb{S}_+^{r-1}$. Let us introduce, for each non-empty $I \subset \{1, \ldots, r\}$:

- the subspace $E_I := \bigoplus_{i \in I} \mathbb{R}e_i$ of E, the intersection $D_I = D \cap E_I$ and its interior $\mathring{D}_I := \{\sum_{i \in I} t_i e_i \in D \mid \forall i \in I, t_i > 0\}$. We have $D = \bigsqcup_I \mathring{D}_I$.

- the minimal eigenvalue λ_I of the Gram matrix $(\beta_{\mathcal{Q}}^F(e_i, e_j))_{i,j \in I}$ of the restriction of $\beta_{\mathcal{Q}}^F$ to $E_I \times E_I$, and the corresponding eigenspace E_{I,λ_I}.

We also denote by $\mu_{\mathcal{Q}}^F$ the minimum of $x \mapsto \beta_{\mathcal{Q}}^F(x, x)$ on D.

Proposition 2.5. *Fix a test function F and a quadruple $\mathcal{Q} = (r, \underline{U}, \underline{\delta}, \underline{m})$ as in (2.4.3). Let $\mathcal{I}$ be the set of non-empty $I \subset \{1, \ldots, r\}$ such that E_{I,λ_I} intersects $\mathring{D}_I$. Then $\mathcal{I}$ is non-empty and we have $\mu_{\mathcal{Q}}^F = \min_{I \in \mathcal{I}} \lambda_I$.*

Proof. The minimum $\mu_{\mathcal{Q}}^F$ of $x \mapsto \beta_{\mathcal{Q}}^F(x,x)$ on the compact $D = \bigsqcup_{I \in \mathcal{I}} \mathring{D}_I$ is reached in $\mathring{D}_J$ for some J. By Lemma 2.6 below applied to the Euclidean space E_J and to the restriction b of $\beta_{\mathcal{Q}}^F$ to $E_J \times E_J$, any local minimum of $x \mapsto \beta_{\mathcal{Q}}^F(x,x)$ on $\mathring{D}_J$ is an eigenvector for λ_J and we have $\mu_{\mathcal{Q}}^F = \lambda_J$. We have $J \in \mathcal{I}$, and the other inequality $\mu_{\mathcal{Q}}^F \leq \lambda_I$ for any $I \in \mathcal{I}$ is obvious. $\qquad\square$

Lemma 2.6. *Let E be an Euclidean space with scalar product $x \cdot y$, S its unit sphere, b a symmetric bilinear form on E and u the (symmetric) endomorphism of E satisfying $b(x,y) = x \cdot u(y)$ for all x, y in E. Assume that the map $S \to \mathbb{R}, x \mapsto b(x,x)$ has a local minimum at the element v in S. Then v is an eigenvector of u whose eigenvalue $b(v,v)$ is the minimal eigenvalue of u.*

Proof. Set $q(x) = b(x,x)$. We have $q(\frac{v+w}{|v+w|}) = q(v) + 2b(w,v) + \mathrm{O}(w^2)$ when w goes to 0 in $v^\perp$. As v is a local minimum of q, this shows $b(w,v) = w \cdot u(v) = 0$ for all w in $v^\perp$. So v is an eigenvector of u. Denote by λ be the corresponding eigenvalue. Assume u has an eigenvalue $\lambda' < \lambda$, and choose v' in S with $u(v') = \lambda'v'$. We have $b(v,v') = 0$ and $q((1-\epsilon^2)^{1/2}v + \epsilon v') = \lambda + \epsilon^2(\lambda' - \lambda) < \lambda$ for all $0 < \epsilon < 1$, a contradiction. $\qquad\square$

Example 2.7. Assume $r = 2$ and set $(\beta_{\mathcal{Q}}^F(e_i, e_i))_{1 \leq i,j \leq 2} = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$. We have $\lambda_{\{1\}} = a$ and $\lambda_{\{2\}} = c$. We may assume a and c are ≥ 0, otherwise Problem 2 is solved. For $I = \{1,2\}$, the eigenvalue λ_I is < 0 if and only if the determinant $ac - b^2$ is < 0. In this case, we have $b \neq 0$ and the eigenspace E_{I,λ_I} is a line: we easily check that this line meets $\mathring{D}_I$ if and only if $b < 0$. Proposition 2.5 implies that assuming $ac < b^2$ and $b < 0$, or equivalently $b + \sqrt{ac} < 0$, we have $\mu_{\mathcal{Q}}^F < 0$.

Lemma 2.8. *Fix a test function F and a quadruple $\mathcal{Q} = (r, \underline{U}, \underline{\delta}, \underline{m})$ as in (2.4.3). Assume $\mu_{\mathcal{Q}}^F < 0$, $\widehat{F}(i/4\pi) > 0$, as well as $(U_i, \delta_i, m_i) = (U_j, \delta_j, m_j)$ for some indices $i \neq j$. Then any element $\underline{t}$ in D with $\beta_{\mathcal{Q}}^F(\underline{t}, \underline{t}) = \mu_{\mathcal{Q}}^F$ satisfies $t_i = t_j$.*

Proof. Set $q(x) = \beta_{\mathcal{Q}}^F(x,x)$. Consider the set $B = \bigcup_{0 \leq \lambda < 1} \lambda D$; then $B \cup D$ is convex and we have $q(x) > \mu_{\mathcal{Q}}^F$ for $x \in B$. Fix $\underline{t} \in D$ with $\beta_{\mathcal{Q}}^F(\underline{t}, \underline{t}) = \mu_{\mathcal{Q}}^F$. An inspection of Formula (2.4.4) shows that for any real numbers s_i, s_j we have

$$(2.4.6) \quad q\left(s_i e_i + s_j e_j + \sum_{l \neq i,j} t_l e_l\right) = -\frac{2}{m_i} s_i s_j \widehat{F}(i/4\pi) + (\text{function of } s_i + s_j).$$

The set

$$(2.4.7) \qquad \{(s_i, s_j) \,|\, s_i, s_j \geq 0,\, s_i + s_j = t_i + t_j,\, s_i^2 + s_j^2 + \sum_{l \neq i,j} t_l^2 \leq 1\}$$

is a compact interval in $\mathbb{R}^2$ with end points (t_i, t_j) and (t_j, t_i). By assumption we have $\widehat{F}(i/4\pi) > 0$, and so the minimum of (2.4.6) on (2.4.7) is reached for

$s_i = s_j = (t_i + t_j)/2$. If we assume $t_i \neq t_j$ then $s_i e_i + s_j e_j + \sum_{l \neq i,j} t_l e_l$ belongs to B, a contradiction. $\qquad\square$

This lemma leads to the following considerations. Start with a quadruple $\mathcal{Q} = (r, \underline{U}, \underline{\delta}, \underline{m})$ with the property $m_i = 1$ for $i = 1, \dots, r$. Assume we have a partition

$$\{1, \dots, r\} = \coprod_{l=1}^{r'} P_l$$

with the property that for each $1 \leq l \leq r'$, and each $i, j \in P_l$, we have $(U_i, \delta_i) = (U_j, \delta_j)$. Consider the new quadruple $\mathcal{Q}' = (r', \underline{U'}, \underline{\delta'}, \underline{m'})$ where for each $1 \leq l \leq r'$ we define U_l' (resp. δ_l') as the element U_i (resp. δ_i) with $i \in P_l$ (this does not depend on the choice of such an i), and set $m_l = |P_l|$. We have a natural inclusion

$$i : \mathbb{R}^{r'} \longrightarrow \mathbb{R}^r$$

sending e_l to $\frac{1}{\sqrt{m_l}} \sum_{i \in P_l} e_i$ for each $1 \leq l \leq r'$. This embedding is an isometry for the standard Euclidean structures on both sides, and it follows from Formula (2.4.4) that we have $\beta_{\mathcal{Q}}^F(i(x), i(y)) = \beta_{\mathcal{Q}'}^F(x, y)$ for all $x, y \in \mathbb{R}^{r'}$ and all test functions F. Lemma 2.8 shows then (the inequality $\mu_{\mathcal{Q}}^F \leq \mu_{\mathcal{Q}'}^F$ being obvious):

Corollary 2.9. *Let $\mathcal{Q}$ and $\mathcal{Q}'$ be as above, and fix a test function F with $\widehat{F}(i/4\pi) > 0$. We have $\mu_{\mathcal{Q}}^F < 0$ if and only if $\mu_{\mathcal{Q}'}^F < 0$, and if these inequalities hold we have*

$$\mu_{\mathcal{Q}}^F = \mu_{\mathcal{Q}'}^F.$$

Remark 2.10. Assume we have two quadruples of the form $\mathcal{Q} = (r, \underline{U}, \underline{\delta}, \underline{m})$ and $\mathcal{Q}' = (r, \underline{U}, \underline{\delta'}, \underline{m})$ with $\delta_i' \geq \delta_i$ for each $1 \leq i \leq r$. Choose a test function F with $\widehat{F}(0) \geq 0$. Then we have $\beta_{\mathcal{Q}'}^F(x, y) \leq \beta_{\mathcal{Q}}^F(x, y)$ for all x, y in $\mathbb{R}_{\geq 0}^r$ by Formula (2.4.4). This shows $\mu_{\mathcal{Q}'}^F \leq \mu_{\mathcal{Q}}^F$. In particular, $\mu_{\mathcal{Q}}^F < 0$ implies $\mu_{\mathcal{Q}'}^F < 0$.

2.4.3. *A digression on numerical evaluation.* Before discussing the natural algorithm that follows from Propositions 2.5 and Corollary 2.9, let us discuss the numerical evaluation of the bilinear form C^F. Given a test function F, we will have to be able to compute with enough and certified precision the quantities

(2.4.8) $\qquad \widehat{F}(0), \quad \widehat{F}(i/4\pi) \quad \text{and} \quad \mathrm{J}_F(U) \text{ for } U = 1 \text{ and } U = \mathrm{I}_w \ (w \in \mathbb{Z}).$

It amounts to computing certain indefinite integrals. Numerical integration routines of computer packages such as `PARI` allow to compute approximations of such integrals, with increasing and in principle arbitrarily large accuracy. Although these routines have been very useful in our preliminary computations, and experimentally return highly accurate values when properly used, it would be delicate to rigorously bound the differences between these computed values and the exact ones. This is why we proceed differently.

In this paper, we only use Odlyzko's function $F = F_\ell$ with parameter $\ell > 0$. This is the function defined by $F_\ell(x) = g(x/\ell)/\cosh(x/2)$, where $g : \mathbb{R} \to \mathbb{R}$ is twice the convolution square of the function $x \mapsto \cos(\pi x)1_{|x| \le 1/2}$: see [Poi77b, Sect. 3] and [CL19, Sect. 9.3.17]. These functions satisfy (POS), $\widehat{F}_\ell(i/4\pi) = \frac{8}{\pi^2}\ell$, and Proposition 9.3.18 of [CL19] provides alternative closed formulas for all the other quantities in (2.4.8) (see Proposition 4.4 for similar expressions). Each is a sum of a linear combination of a few special values of the classical digamma function $\psi = \Gamma'/\Gamma$ and of its derivative $\psi'(z) = \sum_{n \ge 0} 1/(n+z)^2$, and of a simple rapidly converging series with given tail estimates [CL19, (3) p.127]. Using these formulas and estimates, we implemented functions in Python using Sage [S+14] to compute certified intervals containing the real numbers (2.4.8) for $F = F_\ell$. See [CT19b] for the source code. For interval arithmetic, Sage relies on the Arb library http://arblib.org/. Our computations only use the four operations, the exponential and logarithm functions, the constant π, the function ψ (`acb_digamma` in this library), and its derivative (a special case of `acb_polygamma`).

Remark 2.11. Fix an integer $0 \le w \le 23$. For suitable $\ell > 0$, the restriction of $B_\infty^{F_\ell}$ to $K_\infty^{\le w}$ is positive definite (see e.g. Lemma 9.3.37 and Proposition 9.3.40 in [CL19], as well as [Chea]). By Corollary 2.3, it is important to be able to enumerate, for $c > 0$, all the (finitely many) effective elements U in $K_\infty^{\le w}$ satisfying $B_\infty^{F_\ell}(U, U) \le c$. We use for this the Fincke-Pohst algorithm enumerating the short vectors in a lattice. Using interval arithmetic as explained above we can obtain rational lower bounds for the coefficients of the Gram matrix of $B_\infty^{F_\ell}$, and since we are only interested in effective elements of $K_\infty^{\le w}$ we can work with this rational Gram matrix instead. Unfortunately PARI's `qfminim` does not (yet?) include an *exact* variant of the Fincke and Pohst algorithm for Gram matrices with integral entries. For this reason we reimplemented the first (simple) algorithm of Fincke-Pohst [FP85] using only exact computations, adding the condition of effectivity in the recursion to avoid unnecessary computations. Of course in practice this algorithm always leads to the same conclusions as PARI's `qfminim` algorithm, if the latter is properly used. See [CT19b] for our source code.

2.4.4. *The algorithm.* The following algorithm tries to solve Problem 2 using the method discussed in §2.4.2.

Input: A quadruple $\mathcal{Q} = (r, \underline{U}, \underline{\delta}, \underline{m})$ as in 2.4.3.

Output: (if the algorithm terminates) A triple $(\ell, I, \underline{t})$ with $\ell > 0$, a non empty $I \subset \{1, \ldots, r\}$, and $\underline{t} \in \mathbb{R}^I$ with $\beta_{\mathcal{Q}}^{F_\ell}(\underline{t}, \underline{t}) < 0$.

Step 1. Choose a real number $\ell > 0$ and compute an approximation $(G_{i,j})_{1 \le i,j \le r}$ of the Gram matrix $(\beta_{\mathcal{Q}}^{F_\ell}(e_i, e_j))_{1 \le i,j \le r}$. We do this using the formulas (2.4.4) of §2.4.2 and the expressions of [CL19] for the quantities (2.4.8) with $F = F_\ell$ discussed in §2.4.3.

Step 2. Choose a nonempty subset I of $\{1, \ldots, r\}$ and compute an approximation λ_I of the minimal eigenvalue of the Gram matrix $(G_{i,j})_{i,j \in I}$, as well as an approximate corresponding eigenvector $(t_i)_{i \in I}$. For doing so, we apply `PARI`'s `qfjacobi` function to $(G_{i,j})_{i,j \in I}$ (an implementation of Jacobi's method).

Step 3. If we have $\lambda_I < 0$ and $t_i > 0$ for all $i \in I$, return ℓ, I and $\underline{t} = (t_i)_{i \in I}$ and go to Step 4. Otherwise, go back to Step 2 and change the subset I. If all the I have been tried, go back to Step 1 and change the parameter ℓ.

Step 4. Check rigorously, using interval arithmetic as discussed in §2.4.3, that we have indeed $\beta_{\mathcal{Q}}^{\mathrm{F}_\ell}(\underline{t}, \underline{t}) < 0$. If it fails go back to the second part of Step 3.

Let us comment this algorithm and discuss the unexplained choices involved:

- The choice of ℓ in Step 1 is based on some preliminary experiments, and it seems quite hard to guess a priori a range for the best ones. In our applications, we will choose ℓ in $[\frac{1}{2}, 15] \cap 10^{-2}\mathbb{Z}$.

- The loop consisting of Steps 2 and 3, for a given ℓ, can be very long if r is large, as there are $2^r - 1$ possibilities for I. In practice, we order the subsets I by increasing cardinality, often restrict to I of small cardinality. In practice again, the eigenspace E_{I,λ_I} is just a line.

- Whenever we reached Step 4, the rigorous check with interval arithmetic of the inequality $\beta_{\mathcal{Q}}^{\mathrm{F}_\ell}(\underline{t}, \underline{t}) < 0$ never fails in practice. *This single check is enough to justify that the algorithm does show that $x \mapsto \beta_{\mathcal{Q}}^{\mathrm{F}_\ell}(x, x)$ takes a negative value on $\mathbb{S}_+^{r-1}$.* This is the most important remark regarding this algorithm. In particular, we do not have to justify any of the computations done in Steps 1, 2 and 3 before: all is fair in order to find a candidate $(\ell, I, \underline{t})$. Of course, the experimental fact that the last check in Step 4 never fails just reflects that the computations made with `PARI` are highly accurate.

In the end, a charm of this algorithm is that even if the loop of Steps 1, 2 and 3 can be very long, once we get the candidate $(\ell, I, \underline{t})$ we just have to store it, and then inequality $\beta_{\mathcal{Q}}^{\mathrm{F}_\ell}(\underline{t}, \underline{t}) < 0$ can be rechecked instantly.

2.4.5. *Final algorithm.* For our applications in §2.4.6, §4 and §3.3, it will be convenient to apply Algorithm 2.4.4 in the following slightly more restrictive context.

Set up. We fix U in K_∞, δ in $\{0, 1\}$, and an integer $m \geq 1$. We fix as well a *known* set $\mathbf{S}$ of elements of Π_{alg} and our aim is to show that there does not exist distinct elements $\pi_1, \ldots, \pi_m$ in $\Pi_{\mathrm{alg}} \setminus \mathbf{S}$ with $\mathrm{L}((\pi_i)_\infty) = U$ and $\mathrm{sd}(\pi_i) \geq \delta$ for each $1 \leq i \leq m$. By known we mean that we assume given $\mathrm{L}(\varpi_\infty)$ and [10] $\mathrm{sd}(\varpi)$ for all

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[10]Let us mention that, at present, the authors are not aware of the existence of any non self-dual element in Π_{alg}, so in practice will always actually have $\mathrm{sd}(\varpi) = 1$ for ϖ in $\mathbf{S}$.

$\varpi \in S$. We denote by S the set of triples (U', δ', m') in $\mathrm{K}_\infty \times \{0, 1\} \times \mathbb{Z}_{\geq 1}$ such that there are exactly m' elements ϖ in $\mathbf{S}$ with $(\mathrm{L}(\varpi_\infty), \mathrm{sd}(\varpi)) = (U', \delta')$.

Algorithm. Set $r = 1 + |S|$. Assuming $|S| \geq 1$ it is convenient choose a bijection

$$(2.4.9) \qquad\qquad S \xrightarrow{\sim} \{2, \dots, r\}$$

and write $S = \{(U_i, \delta_i, m_i) \mid 2 \leq i \leq r\}$. Set also $(U_1, \delta_1, m_1) = (U, \delta, m)$. This defines a quadruple $\mathcal{Q} = (r, \underline{U}, \underline{\delta}, \underline{m})$. We now apply Algorithm 2.4.4 to $\mathcal{Q}$. In Step 2 we obviously may, and do, restrict to subsets I containing 1, *i.e.* of the form $I = \{1\} \coprod S'$ with $S' \subset S$, via the identification (2.4.9).

Output. When this algorithm terminates, it produces $(\ell, I, \underline{t})$ such that $\beta_{\mathcal{Q}}^{\mathrm{F}_\ell}(\underline{t}, \underline{t}) < 0$. For $j = 2, \dots, m$, set $x_j = \frac{1}{\sqrt{m_j}} \sum \varpi$, the sum being over the $\varpi \in \mathbf{S}$ with $(\mathrm{L}(\varpi_\infty), \delta(\varpi)) = (U_j, \delta_j)$. Assume there are distinct elements $\pi_1, \dots, \pi_m$ in $\Pi_{\mathrm{alg}} \setminus \mathbf{S}$ with $\mathrm{L}((\pi_i)_\infty) = U$ and $\mathrm{sd}(\pi_i) \geq \delta$ for each $1 \leq i \leq m$. Then for the element $x = t_1 \frac{1}{\sqrt{m}}(\pi_1 + \cdots + \pi_m) + \sum_{i=2}^r t_i x_i$ of $\mathbb{R}\,\Pi_{\mathrm{alg}}$ we have $\mathrm{C}^{\mathrm{F}_\ell}(x, x) \leq \beta_{\mathcal{Q}}^{\mathrm{F}_\ell}(\underline{t}, \underline{t}) < 0$ (see Remark 2.10 for the first inequality), contradicting Proposition 2.2.

Remark 2.12. In the case $S = \emptyset$, this method just amounts to applying Corollary 2.3. In the case $|S| = 1$, it amounts to applying Scholium 9.3.26 of [CL19], by the discussion of Example 2.7. The case of arbitrary $|S|$ can thus be viewed as a generalisation of these criteria *loc. cit.* See [CT19b] for our source code in `PARI` of the algorithm above.

2.4.6. *An illustration.* Algorithm 2.4.5 can be used to give another proof of the Chenevier-Lannes classification theorem [CL19, Thm. 9.3.3] mentionned in §1.3 of the introduction, which is both very fast (a few seconds of computations) and systematic. Although this alternative proof shares many steps with the one *loc. cit.*, it bypasses the geometric criterion involving Satake parameters explained in §9.3.29 therein (and does not rely at all on any computation of Satake parameters of known elements in Π_{alg}). Such an improvement, although not decisive here, will be crucial in the proof of Theorem 3, because at present we only know rather few Satake parameters for the known elements of Π_{alg} of dimension > 3 (see however [BFvdG17] and [Még18]).

For the convenience of the reader, and in order to illustrate our new method, let us now explain the aforementioned proof of [CL19, Thm. 9.3.3] in the most complicated case of motivic weight 22. So we want to prove that there is a unique π in Π_{alg} of motivic weight 22, namely $\pi = \mathrm{Sym}^2\Delta_{11}$ (for which we have $\mathrm{L}(\pi_\infty) = \mathrm{I}_{22} + \varepsilon_{\mathbb{C}/\mathbb{R}}$). We refer to the working sheet in [CT19b] for the numerical verifications used below.

Step 1. We first observe that $\mathrm{B}_\infty^{\mathrm{F}_\ell}$ is positive definite on the lattice $\mathrm{K}_\infty^{\leq 22}$ for $\ell = 4.38$ (Lemma [CL19, 9.3.37]). Using the PARI `qfminim` command, or better

Remark 2.11, we may and do list all the effective elements U in $K_\infty^{\leq 22}$ satisfying

$$B_\infty^{F_\ell}(U, U) \leq \widehat{F}_\ell(i/4\pi)$$

for $\ell = 4.38$. We retain furthermore only those satisfying $\det U = 1$ and containing I_{22}. The resulting list $\mathcal{U}$ has 158 elements. If π in Π_{alg} has motivic weight 22, then $L(\pi_\infty)$ is in $\mathcal{U}$ by Corollary 2.3. We will study each of these 158 possibilities for $L(\pi_\infty)$ mostly case by case.

Step 2. Denote by $N(U)$ be the number of elements π in Π_{alg} with $L(\pi_\infty) = U$. We want to bound $N(U)$ for each U in $\mathcal{U}$ by applying Inequality (2.4.2) of Corollary 2.3. For this we check that for all U in $\mathcal{U}$ we have $B_\infty^{F_\ell}(U, U) > \frac{1}{2}\widehat{F}_\ell(i/4\pi)$, unless U belongs to the subset $\mathcal{U}' = \{I_{22} + I_{12}, I_{22} + I_{10}, I_{22} + I_8\}$, in which case we only have $B_\infty^{F_\ell}(U, U) > \frac{1}{3}\widehat{F}_\ell(i/4\pi)$ (here ℓ is still 4.38). This shows $N(U) \leq 1$ for U in $\mathcal{U} \setminus \mathcal{U}'$, and $N(U) \leq 2$ for U in $\mathcal{U}'$.

Step 3. Fix U in $\mathcal{U}'$. We want to show $N(U) \leq 1$. Assume $N(U) = 2$, *i.e.* that there exist distinct π_1, π_2 in Π_{alg} with $L((\pi_1)_\infty) = L((\pi_2)_\infty) = U$. We apply Algorithm 2.4.5 to U, $\delta = 0$ (see Remark 2.10), $m = 2$ and to the known set $\mathbf{S} = \{1, \Delta_{11}, \Delta_{15}, \Delta_{17}, \Delta_{19}, \Delta_{21}, \mathrm{Sym}^2\Delta_{11}\}$. For $U = I_{22} + I_{12}$ and $\ell = 3.5$ it returns for instance an element close to

$$x = 0.924 \, \frac{1}{\sqrt{2}}(\pi_1 + \pi_2) + 0.383 \, \Delta_{11}.$$

We verify (using interval arithmetic, see §2.4.3) that we have $C^{F_\ell}(x, x) \simeq -0.173$ up to 10^{-3}: this contradicts Proposition 2.2. The algorithm produces a similar element x in the case $U = I_{22} + I_{10}$, with $(0.924, 0.383)$ replaced by $(0.900, 0.436)$, and we have then $C^{F_\ell}(x, x) \simeq -0.198$ up to 10^{-3}. Nevertheless, it does not seem to produce any contradiction in the remaining case $U = I_{22} + I_8$, even if we let ℓ vary. To deal with this last U we add to $\mathbf{S}$ the known element $\Delta_{21,9}$ of Π_{alg}, whose Archimedean L-parameter is $I_{21} + I_9$ (which is "close" to U). The algorithm returns for $\ell = 3.5$ the element $x := 0.942 \, \frac{1}{\sqrt{2}}(\pi_1 + \pi_2) + 0.335 \, \Delta_{21,9}$ and we verify that we have $C^{F_\ell}(x, x) \simeq -0.147$ up to 10^{-3}, which is indeed < 0.

Step 4. We have proved so far $N(U) \leq 1$ for all $U \in \mathcal{U}$. In particular, any π in Π_{alg} with $L(\pi) \in \mathcal{U}$ is self-dual. Fix U in $\mathcal{U}$. We now apply Algorithm 2.4.5 to U, $\delta = 1$, $m = 1$ and to the same set S as above (with $|S| \leq 7$). Using the nine ℓ in $[3, 5] \cap \frac{1}{4}\mathbb{Z}$, it yields a contradiction in each case! Actually, if we restrict to subsets $S' \subset S$ with $|S'| = 1$ in Step 2 of the algorithm (in other words, if we only apply the Scholium of [CL19] mentioned in Remark 2.12) we already get a contradiction for all but the 7 elements U mentioned in Table 1 below. These remaining cases were exactly the ones dealt with using the geometric criterion involving Satake parameters explained in [CL19, §9.3.29]. In these 7 cases, our algorithm produces contradictions for subsets S' of size 2, such as the ones gathered in Table 1. This concludes the proof. $\square$

U	x	$\mathrm{C}^{\mathrm{F}_4}(x,x)$ up to 10^{-3}
$\mathrm{I}_{22} + \mathrm{I}_{16} + 1$	$0.625\,\pi_1\ +\ 0.611\,\Delta_{19}\ +\ 0.485\,\mathrm{Sym}^2\Delta_{11}$	-0.427
$\mathrm{I}_{22} + \mathrm{I}_{12}$	$0.640\,\pi_1\ +\ 0.582\,\Delta_{15}\ +\ 0.502\,\Delta_{17}$	-0.511
$\mathrm{I}_{22} + \mathrm{I}_{12} + 1$	$0.709\,\pi_1\ +\ 0.432\,\Delta_{11}\ +\ 0.558\,\Delta_{15}$	-0.204
$\mathrm{I}_{22} + \mathrm{I}_{20} + \mathrm{I}_{10} + \epsilon_{\mathbb{C}/\mathbb{R}}$	$0.636\,\pi_1\ +\ 0.393\,\Delta_{19}\ +\ 0.664\,\mathrm{Sym}^2\Delta_{11}$	-0.037
$\mathrm{I}_{22} + \mathrm{I}_{16} + \mathrm{I}_{10} + \epsilon_{\mathbb{C}/\mathbb{R}}$	$0.701\,\pi_1\ +\ 0.531\,\Delta_{19}\ +\ 0.476\,\mathrm{Sym}^2\Delta_{11}$	-0.246
$\mathrm{I}_{22} + \mathrm{I}_4$	$0.630\,\pi_1\ +\ 0.608\,\Delta_{11}\ +\ 0.483\,\mathrm{Sym}^2\Delta_{11}$	-0.204
$\mathrm{I}_{22} + \mathrm{I}_{20} + \mathrm{I}_{14} + \mathrm{I}_4$	$0.696\,\pi_1\ +\ 0.297\,1\ +\ 0.654\,\Delta_{21}$	-0.047

TABLE 1. Some elements x with $\mathrm{L}((\pi_1)_\infty) = U$ and $\mathrm{C}^{\mathrm{F}_\ell}(x,x) < 0$ for $\ell = 4$.

2.4.7. *Another illustration: a strengthening of a vanishing result in* [CvdG18]. As another example, let us show that for all odd $1 \leq w \leq 53$, there is no cuspidal selfdual algebraic level 1 automorphic representation π of PGL_4 with $\mathrm{L}(\pi_\infty) = \mathrm{I}_w + \mathrm{I}_w$. We apply for this Algorithm 2.4.5 to $U = 2\,\mathrm{I}_w$, $\delta = m = 1$ and to the set $\mathbf{S}$ of $\dim \mathrm{S}_{w+1}(\mathrm{SL}_2(\mathbb{Z}))$ cuspidal automorphic representations generated by level 1 cuspidal eigenforms for $\mathrm{SL}_2(\mathbb{Z})$. Note that we have $|S| = 0$ for $w = 13$ and $w < 11$, and $|S| = 1$ otherwise. We obtain a contradiction in each case using $S' = S$ and $\ell = 5$. This shows $\mathrm{S}_{(k_1,2)}(\Gamma_2) = 0$ for all $k_1 \leq 54$ by [CvdG18, Lemma A.2].

3. EFFORTLESS COMPUTATION OF MASSES IN THE TRACE FORMULA

Let G be a split classical group over $\mathbb{Z}$ such that $G(\mathbb{R})$ admits discrete series. In other words, G belongs to one of the three families

$$(\mathrm{SO}_{2n+1})_{n \geq 1}, \quad (\mathrm{Sp}_{2n})_{n \geq 1} \quad \text{and} \quad (\mathrm{SO}_{4n})_{n \geq 1}$$

In this section, we explain how to implement the strategy explained in §1.4 in order to determine the masses m_c for $c \in \mathcal{C}(G)$. In §3.1, we first make elementary observations that will allow us to replace $\mathcal{C}(G)$ by a concrete set $\mathcal{P}(G)/\sim$ of equivalence classes of polynomials, and to rewrite the elliptic terms accordingly. In §3.2, we restrict further $\mathcal{C}(G)$ by an explicit subset containing all conjugacy classes c such that $\mathrm{m}_c \neq 0$. It turns out that finding a subset significantly smaller than $\mathcal{C}(G)$ is possible in the case of special orthogonal groups, using spinor norms considerations. In the last §3.3, we finally prove Theorems 6 and 7, by discussing how to produce sets Λ of dominant weights satisfying a variant of conditions (P1) and (P2) alluded in §1.4.

3.1. **Conjugacy classes and characteristic polynomials: elementary observations.** Let G be one of SO_{2n+1}, Sp_{2n} or SO_{4n}. We shall denote by n_G the dimension of the standard (or tautological) representation of G, so n_G is respectively $2n+1, 2n$ or $4n$. (Do not confuse n_G with the $\mathrm{n}_{\widehat{G}}$ introduced in §1.4).

The indexing set for the sum defining the elliptic part $T_{ell}(G; \lambda)$ of the geometric side in Arthur's L^2-Lefschetz trace formula [Art89] recalled in (1.4.3), is the set of conjugacy classes of semi-simple elements $\gamma \in G(\mathbb{Q})$ which are $\mathbb{R}$-*elliptic* (i.e. γ belongs to an anisotropic maximal torus of $G_{\mathbb{R}}$, in particular the eigenvalues of γ have absolute value 1) and such that the conjugacy class of γ in $G(\mathbb{A}_f)$ meets the compact support of the smooth function we put in the trace formula, in our case the characteristic function of $G(\widehat{\mathbb{Z}})$. In particular, the characteristic polynomial P_γ of such a γ, a monic polynomial of degree n_G in $\mathbb{Q}[X]$, belongs to $\mathbb{Z}[X]$ and has all its complex roots of absolute value 1. Using a celebrated theorem of Kronecker, these conditions imply that the roots of P_γ are roots of unity, hence that the semi-simple element γ has finite order. This explains the discussion of Formula (1.4.3) in §1.4, and the indexing set $\mathcal{C}(G)$ of finite order elements of $G(\mathbb{Q})$ taken up to conjugacy by $G(\overline{\mathbb{Q}})$ in the sum (1.4.5).

Definition 3.1. *Let $\mathcal{P}(G)$ be the set of polynomials P in $\mathbb{Q}[X]$ having degree n_G, which are products of cyclotomic polynomials and in which $X + 1$ has even multiplicity (or equivalently, with $P(0) = (-1)^{n_G}$).*

If c is a class in $\mathcal{C}(G)$, then all the elements $\gamma \in c$ have the same P_γ, and we will denote by P_c this polynomial. It is an element of $\mathcal{P}(G)$ by the above discussion. We have thus defined a map

(3.1.1) $$\text{char} : \mathcal{C}(G) \to \mathcal{P}(G), \ c \mapsto P_c.$$

It is well-known that if two semi-simple conjugacy classes c_1, c_2 in the classical group $G(\overline{\mathbb{Q}})$ have the same characteristic polynomial P, then they are equal, except in the case $G = SO_{4n}$, $P(-1)P(1) \neq 0$ and c_1 and c_2 are conjugate under $O_{4n}(\overline{\mathbb{Q}})/SO_{4n}(\overline{\mathbb{Q}}) \simeq \mathbb{Z}/2\mathbb{Z}$. In particular, for $P \in \mathcal{P}(G)$ the fiber[11] $\text{char}^{-1}(P)$ has at most 1 element if $G \neq SO_{4n}$ or $P(-1)P(1) = 0$, and 0 or 2 elements otherwise. The following elementary lemma (see [Taï17, Remark 3.2.11]) shows that this latter case does not create complications:

Lemma 3.2. *For $G = SO_{4n}$ and $c, c' \in \mathcal{C}(G)$ with $P_c = P_{c'}$, we have $m_c = m_{c'}$.*

Thus we may write

$$T_{ell}(G; \lambda) = \sum_{P \in \mathcal{P}(G)} m_P \ \text{tr}(P; \lambda)$$

with

$$m_P := \begin{cases} m_c & \text{if there is } c \in \mathcal{C}(G) \text{ with } \text{char}(c) = P \\ 0 & \text{if } P \text{ does not belong to } \text{char}(\mathcal{C}(G)) \end{cases}$$

---

[11]Beware that the map char is not surjective in general. For instance, Corollary 3.6 shows that for any prime $p \equiv 1 \mod 4$, there is no order p element in $SO_{p-1}(\mathbb{Q})$, as $\Phi_p(1)\Phi_p(-1) = p$ is not a square.

and

$$\mathrm{tr}(P; \lambda) := \begin{cases} \mathrm{tr}(c \,|\, V_\lambda) & \text{if } G \neq \mathrm{SO}_{4n} \text{ or } P(1)P(-1) = 0 \\ \mathrm{tr}(c \,|\, V_\lambda) + \mathrm{tr}(c' \,|\, V_\lambda) & \text{otherwise} \end{cases}$$

with $\mathrm{char}^{-1}(P) = \{c\}$ in the first case and $\mathrm{char}^{-1}(P) = \{c, c'\}$ in the second case. This also implies $\mathrm{T}_{\mathrm{ell}}(\mathrm{SO}_{4n}; \theta(\lambda)) = \mathrm{T}_{\mathrm{ell}}(\mathrm{SO}_{4n}; \lambda)$ where θ is the non-trivial outer automorphism of SO_{4n} induced by $\mathrm{O}_{4n}(\mathbb{Z})/\mathrm{SO}_{4n}(\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. This invariance is fortunate also because Koike and Terada's simple (and most importantly very effective for small weights) formulas [KT87] for traces in algebraic representations apply to irreducible representations of symplectic and orthogonal (rather than *special* orthogonal) groups. Equivalently, their formula gives $\mathrm{tr}(P; \lambda)$ in terms of P, but not $\mathrm{tr}(c \,|\, V_\lambda)$ in the second case above if $\theta(\lambda) \neq \lambda$.

There is another obvious invariance property of masses. For $G = \mathrm{SO}_{4n}$ or Sp_{2n}, the element -1 of $G(\mathbb{Z})$ is in the center of G, and $c \mapsto -c$ preserves $\mathcal{C}(G)$. Formula (1.4.4) thus shows:

Lemma 3.3. *For $G = \mathrm{SO}_{4n}$ or Sp_{2n}, and $c \in \mathcal{C}(G)$, we have $\mathrm{m}_{-c} = \mathrm{m}_c$.*

For G and c as above, we have $P_{-c}(X) = (-1)^{\deg P_c} P_c(-X)$, $\mathrm{tr}(P_{-c}; \lambda) = \lambda(-1)\mathrm{tr}(P_c; \lambda)$, as well as $\mathrm{m}_{P_c} = \mathrm{m}_{P_{-c}}$ by the lemma. A consequence is that $\mathrm{T}_{\mathrm{ell}}(G; \lambda) = 0$ if the restriction of λ to the center $\mathrm{Z}(G)$ of G is non-trivial.[12] Define the following equivalence relation on $\mathcal{P}(G)$: $P_1 \sim P_2$ if $P_1 = P_2$ or $P_2(X) = (-1)^{\mathrm{n}_G} P_1(-X)$. Assuming $\lambda_{|\mathrm{Z}(G)} = 1$ we may thus finally write

$$(3.1.2) \qquad \mathrm{T}_{\mathrm{ell}}(G; \lambda) = \sum_{P \in \mathcal{P}(G)/\sim} \mathrm{e}_P \, \mathrm{m}_P \, \mathrm{tr}(P; \lambda)$$

where $\mathrm{e}_P \in \{1, 2\}$ denotes the size of the equivalence class of P.

To sum up, for the purpose of implementing our strategy introduced in §1.4 we can replace the indexing set $\mathcal{C}(G)$ by $\mathcal{P}(G)/\sim$, which is computable, and we may as well restrict to dominant weights λ such that $\lambda|_{\mathrm{Z}(G)} = 1$, and even to a set of representatives for the orbits under $\{1, \theta\}$ in the even orthogonal case.

TABLE 2.

G	Sp_2	Sp_4	Sp_6	Sp_8	Sp_{10}	Sp_{12}	Sp_{14}	Sp_{16}
$|\mathcal{P}(G)/\sim|$	3	12	32	92	219	530	1157	2521

---

[12] From the perspective of the strategy discussed in §1.4, this vanishing is in agreement with the vanishing of $\mathrm{N}^\perp(w(\lambda))$ for all dominant weights λ of Sp_{2n} or SO_{4n} such that $\lambda(-1) = -1$, which is a consequence of the property $\epsilon(\pi) = \varepsilon(\mathrm{L}(\pi_\infty)) = 1$ for orthogonal π in Π_{alg}: see [Art13, Thm. 1.5.3] and [CR15, Prop. 1.8].

3.2. **Conjugacy classes and characteristic polynomials in the orthogonal case: spinor norms.** As announced in Remark 1.1, it turns out that in the orthogonal cases we can further reduce the set parametrizing conjugacy classes. Let $\mathcal{C}_0(G) \subset \mathcal{C}(G)$ be the subset of equivalence classes containing a finite order element in $G(\mathbb{Q})$ whose $G(\mathbb{A}_f)$-conjugacy class meets $G(\widehat{\mathbb{Z}})$. In particular $\mathcal{C}_0(G)$ contains the set of $c \in \mathcal{C}(G)$ such that $\mathrm{m}_c \neq 0$. A priori it may happen that $\mathcal{C}_0(G)$ is smaller than $\mathcal{C}(G)$. Using the analysis in [Taï17, §3.2.2] and Jacobson's hermitian analogue of the Hasse-Minkowski theorem [Jac40], one can argue that $\mathcal{C}_0(\mathrm{Sp}_{2n}) = \mathcal{C}(\mathrm{Sp}_{2n})$ for any $n \geq 1$. Since this fact is rather unfortunate for our strategy, we leave the details to the interested reader.

We now focus on special orthogonal groups. Proposition 3.7 below gives an explicit subset $\mathcal{P}_1(G)$ of $\mathcal{P}(G)$ such that its preimage $\mathcal{C}_1(G) \subset \mathcal{C}(G)$ under the map char (3.1.1) contains $\mathcal{C}_0(G)$. In contrast with the symplectic case we will see that $\mathcal{C}_1(G) \subsetneq \mathcal{C}(G)$ in general, owing to the fact that special orthogonal groups are not simply connected.

Remark 3.4. The second author had already observed that there was such a restriction on classes c satisfying $\mathrm{m}_c \neq 0$ in [Taï17, Remark 3.2.8], unfortunately without giving details or proofs ... He was also unaware of related previous work of Gross and McMullen: [GM02, Theorem 6.1] is similar to Proposition 3.7. Unfortunately we could not deduce Proposition 3.7 from the results of [GM02], so we give a slightly different proof below, relying on the Zassenhaus formula for spinor norms.

Let R be a commutative ring, V a projective R-module of finite constant rank n and $q : V \to R$ a quadratic form. We say that q is *non-degenerate* if the associated R-bilinear form $\beta_q(x, y) = q(x + y) - q(x) - q(y)$ is a perfect pairing on V. We say that q is *regular* if either q is non-degenerate, or n is odd and Zariski-locally on R the half-discriminant of q is invertible: see [Knu91, Ch. IV §3] (who rather uses the terminology *semi-regular* in this case). When V is a free R-module and q is non-degenerate, we denote by $\mathrm{disc}(q) \in R^\times/R^{\times,2}$ the class of the determinant of a Gram matrix of β_q, where $R^{\times,2}$ denotes the subgroups of squares in $R^\times$.

First we recall a few definitions from [Con14, Appendix C] or [Knu91, Ch. IV]. Assume $n \geq 3$ and q regular. Associated to (V, q) are (reductive) groups schemes $\mathrm{Spin}(V, q) \subset \mathrm{GSpin}(V, q)$ and $\mathrm{SO}(V, q)$ over R. The group $\mathrm{GSpin}(V, q)$ is the group of even degree invertible elements in the Clifford algebra $\mathrm{C}(V, q)$ which stabilize the submodule $V \subset \mathrm{C}(V, q)$ under conjugation. This conjugation action gives a morphism $\pi : \mathrm{GSpin}(V, q) \to \mathrm{SO}(V, q)$, with kernel the central GL_1 (invertible scalars in the Clifford algebra). See *e.g.* Propositions C.2.8 and C.4.6 of [Con14] for these properties and the fact that π factors through the special orthogonal group. The *Clifford norm* morphism $\nu : \mathrm{GSpin}(V, q) \to \mathrm{GL}_1$ is defined in (C.4.2) and (C.4.4) *loc. cit.* The restriction of ν to the central GL_1 is $t \mapsto t^2$. The group $\mathrm{Spin}(V, q)$ can be defined as the kernel of the Clifford norm: see the proof of Lemma

C.4.1 *loc. cit.* for the case n even and the proof of Proposition C.4.10 loc. cit. and the paragraph following it for the case n odd. We have [Knu91, (6.2.3) p.231] an exact sequence of sheaves in groups on the Zariski site of R

(3.2.1) $$1 \to \mathrm{GL}_1 \to \mathrm{GSpin}(V, q) \to \mathrm{SO}(V, q) \to 1,$$

and thus an exact sequence of sheaves in groups on the fppf site of R

$$1 \to \mu_2 \to \mathrm{Spin}(V, q) \to \mathrm{SO}(V, q) \to 1.$$

The (not so) long exact sequence in cohomology associated to the second short exact sequence above gives the spinor norm $\mathrm{sn} : \mathrm{SO}(V, q) \to H^1_{\mathrm{fppf}}(R, \mu_2)$. If $\mathrm{Pic}(R) = 1$, which will always be the case in this paper, the fppf exact sequence $1 \to \mu_2 \to \mathrm{GL}_1 \to \mathrm{GL}_1 \to 1$ gives the isomorphism $H^1_{\mathrm{fppf}}(R, \mu_2) \simeq R^\times/R^{\times,2}$, and we will implicitly consider the spinor norm in this last group. The spinor norm of $\gamma \in \mathrm{SO}(V, q)(R)$ is then represented by $\nu(\tilde{\gamma})$ where $\tilde{\gamma} \in \mathrm{GSpin}(V, q)(R)$ is any lift of γ; such a lift exists by (3.2.1) and $\mathrm{Pic}(R) = 1$. The spinor norm is additive: if $(V, q) = (V_1, q_1) \perp (V_2, q_2)$, $\gamma_i \in \mathrm{SO}(V_i, q_i)(R)$ for $i = 1, 2$, and if we set $\gamma = \gamma_1 \times \gamma_2 \in \mathrm{SO}(V, q)(R)$, then we have $\mathrm{sn}\,\gamma = \mathrm{sn}\,\gamma_1 \times \mathrm{sn}\,\gamma_2$.

Theorem 3.5 (Zassenhaus). *Let k be a field of characteristic different from 2. Let V be a finite-dimensional vector space over k, endowed with a non-degenerate quadratic form q. Let $\gamma \in \mathrm{SO}(V, q)(k)$ and write the characteristic polynomial of γ as $(X - 1)^a (X + 1)^{2b} Q(X)$ with $Q(1)Q(-1) \neq 0$. Then the spinor norm $\mathrm{sn}\,\gamma$ of γ is represented by $\mathrm{disc}(q \,|\, \ker(\gamma + 1)^{2b}) \, Q(-1)$ in $k^\times/k^{\times,2}$.*

Recall that our convention for $\mathrm{disc}(q)$ was given after Remark 3.4.

Proof. This is just a reformulation of the main theorem of [Zas62]. Here is a short argument for the convenience of the reader. Using the orthogonal decomposition $V = \ker(\gamma - 1)^a \perp \ker(\gamma + 1)^{2b} \perp \ker Q(\gamma)$, and the additivity of spinor norms, we may assume $a = 0$, so $\dim V \equiv 0 \bmod 2$, and either $Q = 1$ or $b = 0$. In the case $Q = 1$, we have $\mathrm{sn}\,\gamma = \mathrm{sn}(-\mathrm{id}_V)$ as unipotent elements are squares, and we conclude by the classical formula $\mathrm{sn}(-\mathrm{id}_V) = 2^{\dim V} \mathrm{disc}(q)$ (that could be proved using an orthogonal basis of V). The arguments so far have used that the characteristic of k is $\neq 2$, but the following ones will not.

Assume $b = 0$, write $Q(X) = \prod_{i=1}^n (X - t_i)(X - t_i^{-1})$ in $\overline{k}[X]$, and choose $\tilde{\gamma} \in \mathrm{GSpin}(V, q)(k)$ a lift of γ. Write $\tilde{\gamma} = du = ud$ its Jordan decomposition in $\mathrm{GSpin}(V, q)(\overline{k})$, with d semi-simple and u unipotent. There is a decomposition $V \otimes_k \overline{k} = P_1 \perp \cdots \perp P_n$, where each P_i is a d-stable hyperbolic plane on which the two eigenvalues of d are $t_i^{\pm 1}$. Using the natural isomorphism between $C(V, q) \otimes_k \overline{k}$ and the graded tensor product of the Clifford algebras of the P_i (see e.g. [Knu91, IV. Prop. 1.3.1]) we easily sees that there is a pair (s, λ) in $\overline{k}^\times \times \overline{k}^\times$ such that: $s^2 = t_1 \ldots t_n$, the Clifford norm of d (or equivalently, of $\tilde{\gamma}$) is λ^2, and the trace of d (or equivalently, of $\tilde{\gamma}$) in the spin representation of $\mathrm{GSpin}(V, q)(k)$ is $\lambda s \prod_{i=1}^n (1 + t_i^{-1})$. The spinor norm of γ is thus represented by $\lambda^2 \in k^\times$. Note that although the spin

representation may not be defined over k, its trace is. Indeed, this representation is defined as the tautological morphism $\mathrm{GSpin}(V, q)(k) \subset \mathrm{C}(V, q)^\times$ and $\mathrm{C}(V, q)$ is a central simple algebra over k, whose reduced trace is k-valued. Since we have $1 + t_i^{-1} \neq 0$ for each i as $Q(-1) \neq 0$, the spinor norm of γ is represented by $s^2 \prod_{i=1}^n (1 + t_i^{-1})^2 = \prod_{i=1}^n t_i (1 + t_i^{-1})^2 = \prod_{i=1}^n (1 + t_i)(1 + t_i^{-1}) = (-1)^{2n} Q(-1)$. $\square$

We deduce the following discriminant formula, which can also be proved directly (see [GM02, Proposition A.3]).

Corollary 3.6. *Under the same assumptions, assume moreover that $a = b = 0$. Then $\mathrm{disc}(q) \in k^\times / k^{\times, 2}$ is represented by $Q(1)Q(-1)$.*

Proof. The assumption $a = 0$ implies $\dim V \equiv 0 \bmod 2$, hence $-\mathrm{id}_V \in \mathrm{SO}(V, q)(k)$. The discriminant of q is thus the spinor norm of $-\mathrm{Id}_V$, or equivalently of $\gamma(-\gamma)$ since $\mathrm{sn}(\gamma)^2 = 1$. We conclude by applying the theorem to γ and $-\gamma$. $\square$

Proposition 3.7. *Let V be a free $\mathbb{Z}$-module of rank n endowed with a regular quadratic form q, $G = \mathrm{SO}(V, q)$, and $P(X) = (X - 1)^a (X + 1)^{2b} Q(X)$ a monic polynomial of degree n in $\mathbb{Q}[X]$ with $Q(1)Q(-1) \neq 0$. Assume that for every prime p there exists $\gamma_p \in G(\mathbb{Z}_p)$ having characteristic polynomial P. If $b = 0$ then the integer $|Q(-1)|$ is square, and if $a = 0$ then the integer $|Q(1)|$ is a square.*

Note that the existence of γ_p for all primes p implies that Q has integer coefficients.

Proof. Fix a prime p. Since $\gamma_p \in G(\mathbb{Z}_p)$ and $\mathrm{Pic}\, \mathbb{Z}_p = 1$ the element γ_p can be lifted to an element of $\mathrm{GSpin}(V, q)(\mathbb{Z}_p)$ by (3.2.1), so the spinor norm of γ_p lies in the image of $\mathbb{Z}_p^\times$ in $\mathbb{Q}_p^\times / \mathbb{Q}_p^{\times, 2}$. Together with Theorem 3.5, this implies that

$$\mathrm{disc}(q \,|\, \ker(\gamma_p + 1)^{2b}) \times Q(-1) \ \in \mathbb{Q}_p^\times / \mathbb{Q}_p^{\times, 2}$$

lies as well in the image of $\mathbb{Z}_p^\times$. Assuming $b = 0$, it follows that the integer $Q(-1)$ has an even valuation at each prime p, so $|Q(-1)|$ is a square. Assume now $a = 0$. In particular, n is even and we have $\mathrm{disc}(q) \in \mathbb{Z}^\times$. By Corollary 3.6 applied to the orthogonal of $\ker(\gamma_p + 1)^{2b}$ in $V \otimes \mathbb{Q}_p$, we have $\mathrm{disc}(q) \times \mathrm{disc}(q \,|\, \ker(\gamma_p + 1)^{2b}) \equiv Q(-1)Q(1)$ in $\mathbb{Q}_p^\times / \mathbb{Q}_p^{\times, 2}$, or equivalently:

$$\mathrm{disc}(q) \times \mathrm{disc}(q \,|\, \ker(\gamma_p + 1)^{2b}) \times Q(-1) \equiv Q(1) \mod \mathbb{Q}_p^{\times, 2}$$

But have seen that the left-hand side is in the image of $\mathbb{Z}_p^\times$. So the integer $Q(1)$ has an even valuation at each prime p, and $|Q(1)|$ is a square. $\square$

The astute reader will notice that the second statement could also be proved by considering $(-1)^{\deg P} P(-X)$ and $(-\gamma_p)_p$. In odd dimension this would require the introduction of GPin and Pin groups to define the spinor norm (see complications in [Con14, Appendix C]).

Definition 3.8. *For $G = \mathrm{SO}_{2n+1}$ or SO_{4n} let $\mathcal{P}_1(G)$ be the subset of $\mathcal{P}(G)$ consisting of all polynomials of the form $(X-1)^a(X+1)^{2b}Q(X)$, where $Q(X)$ is a product of cyclotomic polynomials Φ_m with $m \geq 3$, which satisfy*

- *$b > 0$ if the positive integer $Q(-1)$ is not a square, and*
- *$a > 0$ if the positive integer $Q(1)$ is not a square.*

For $G = \mathrm{Sp}_{2n}$ denote $\mathcal{P}_1(G) = \mathcal{P}(G)$ (Definition 3.1).

The positive integers $\Phi_m(\pm 1)$ for $m \geq 3$ may be computed inductively in terms of the prime decomposition of m: see [GM02, Theorem 7.1]. In terms of the notation m_P defined in §3.1, Proposition 3.7 asserts:

Corollary 3.9. *If $P \in \mathcal{P}(G)$ satisfies $\mathrm{m}_P \neq 0$, then we have $P \in \mathcal{P}_1(G)$.*

This constraint is very useful, particularly in the even case, as the following table shows. In practice, we will see that it is almost sharp: see Remark 3.10. Observe that $\mathcal{P}_1(G)$ is stable under the equivalence relation $\sim$ introduced in the end of §3.1.

<div align="center">TABLE 3.</div>

G	SO_3	SO_4	SO_5	SO_7	SO_8	SO_9	SO_{11}	SO_{12}	SO_{13}	SO_{15}	SO_{16}	SO_{17}
$\lvert \mathcal{P}(G)/\sim \rvert$	5	12	19	59	92	165	419	530	1001	2257	2521	4877
$\lvert \mathcal{P}_1(G)/\sim \rvert$	3	6	12	34	40	99	244	211	598	1339	992	2948

3.3. **Non-existence of level one regular algebraic automorphic cuspidal representations.** In this paragraph, we prove Theorems 6 and 7 of the introduction. To implement the strategy explained in §1.4, taking into account the reduced Formula (3.1.2) and Corollary 3.9, it remains to actually produce, for as many "small rank" groups G as possible in the families $(\mathrm{SO}_{2n+1})_{n \geq 1}$, $(\mathrm{Sp}_{2n})_{n \geq 1}$ and $(\mathrm{SO}_{4n})_{n \geq 1}$, sets Λ of dominant weights for G satisfying the following properties:

(P1') For all $\lambda \in \Lambda$ we have $\lambda_{|Z(G)} = 1$ and $\mathrm{N}^{\perp}(w(\lambda)) = 0$.

(P2') For $\mathcal{P} = \mathcal{P}_1(G)/\sim$, the $\Lambda \times \mathcal{P}$ matrix $(e_P \, \mathrm{tr}(P; \lambda))_{\lambda \in \Lambda, P \in \mathcal{P}}$ has rank $|\mathcal{P}|$.

Of course (P2') implies $|\Lambda| \geq |\mathcal{P}|$ so our aim is roughly to produce as many dominant weights satisfying (P1') as possible. See also Footnote 12 for an important remark regarding the condition on $\lambda_{|Z(G)}$ in (P1').

NOTATIONS: For $w \geq 0$ an integer we denote by $\Lambda_G(w)$ the (finite) set of all dominant weights λ of G such that: $2w(\lambda)_1 \leq w$, $\lambda_{|Z(G)} = 1$, as well as $\lambda_m \geq 0$ in the case $G = \mathrm{SO}_{2m}$. For a dominant weight λ of G, there is a unique effective element $U(\lambda) \in \mathrm{K}_\infty$ with $\det U(\lambda) = 1$ and such that the multi-set of weights of $U(\lambda)$ (as defined in §2.1) coincides with $w(\lambda)$ (viewed of course as the multi-set $\{w(\lambda)_i \mid 1 \leq i \leq \mathrm{n}_{\widehat{G}}\}$). The representation $U(\lambda)$ is multiplicity free, and for any $\pi \in \Pi_{\mathrm{alg}}$ having weights $w(\lambda)$ we have $\mathrm{L}(\pi_\infty) = U(\lambda)$.

In order to produce Λ we will first use the inequality (2.4.1) in Corollary 2.3. We choose w big enough, and for every $\lambda \in \Lambda_G(w)$, and for all parameters $\ell \in \frac{1}{4}\mathbb{Z} \cap [1/2, 20]$, we compute $\widehat{F}_\ell(i/4\pi) - B_\infty^{F_\ell}(U(\lambda), U(\lambda))$. Whenever we find a negative value (certified using interval arithmetic as explained in §2.4.3), we know that $N^\perp(w(\lambda)) = 0$ by Corollary 2.3 and thus we add λ to Λ. In other words, we choose

$$\Lambda_w^{\text{test}} := \{\lambda \in \Lambda_G(w) \mid \exists \ell \in \frac{1}{4}\mathbb{Z} \cap [1/2, 20], \ \widehat{F}_\ell(i/4\pi) < B_\infty^{F_\ell}(U(\lambda), U(\lambda))\},$$

for the set Λ. For our purpose this simple method is already very effective. Table 4 below displays all groups for which it works, *i.e.* for which, using the given w, the set Λ_w^{test} satisfies the rank condition (P2').

TABLE 4.

G	$|\mathcal{P}_1(G)/\sim|$	$|\Lambda_w^{\text{test}}|$	w
SO_3	3	3	5
Sp_2	3	4	14
SO_5	12	44	23
SO_4	6	30	26
Sp_4	12	28	28
SO_7	34	183	27
Sp_6	32	97	28
SO_9	99	498	27
SO_8	40	335	28
Sp_8	92	255	28
SO_{11}	244	923	27
Sp_{10}	219	446	28
SO_{13}	598	1294	27
SO_{12}	211	1061	28
Sp_{12}	530	597	28
SO_{15}	1339	1924	35

See [CT19b] for the Sage program which checks that each set Λ in the table satisfies (P1') (using Corollary 2.3 and interval arithmetic) and inductively computes, for each group G appearing in the table:

(1) all masses $(m_P)_{P \in \mathcal{P}_1(G)}$,
(2) $T_{\text{ell}}(G; \lambda)$, $EP(G; \lambda)$ and $N^\perp(w(\lambda))$ for all dominant weights λ in any desired range (only limited by computer memory).

Note that we obtain in particular, independently of the computation of masses in [CR15] and [Taï17], the existence of 27 self-dual elements of Π_{alg} having regular weight and motivic weight ≤ 24. In §4 we will prove that there is no other such element in Π_{alg}. The only case which was obtained in [Taï17] and that we cannot recover using this much simpler method is Sp_{14}. The case of SO_{15} is new. For $G = \mathrm{Sp}_{14}$, considering all dominant weights λ in $\Lambda_G(90)$, we only find a set $\Lambda_{90}^{\mathrm{test}}$ of cardinality 974, whereas we have $|\mathcal{P}_1(\mathrm{Sp}_{14})/\sim| = 1157$. Higher motivic weights do not seem to provide any new non-existence results. Similarly this method does not allow us to conclude either in the case of SO_{17}.

To go further we use the algorithm explained in §2.4.5 to find larger sets Λ satisfying (P1'). More precisely, for a large enough w and each dominant weight $\lambda \in \Lambda_G(w)$ we applied this algorithm with $U = U(\lambda)$, $\delta = 1$, $m = 1$ and taking for $\mathbf{S}$ [13] the set of 27 known elements of Π_{alg} having motivic weight ≤ 24 found above. As before we try various $\ell \in \frac{1}{4}\mathbb{Z} \cap [1/2, 20]$. Using this refined method we obtain the following three new cases:

- For $G = \mathrm{Sp}_{14}$ we have $|\mathcal{P}(G)/\sim| = 1157$ and we found a subset $\Lambda \subset \Lambda_G(36)$ of cardinality 1274 satisfying (P1') and (P2').
- The case $G = \mathrm{SO}_{16}$ is easier: we have $|\mathcal{P}_1(G)/\sim| = 992$ and we found a subset $\Lambda \subset \Lambda_G(28)$ of cardinality 1810 satisfying (P1') and (P2').
- For $G = \mathrm{SO}_{17}$ we have $|\mathcal{P}_1(G)| = 2948$ and we found a subset $\Lambda \subset \Lambda_G(63)$ of cardinality 3477 satisfying (P1') and (P2'). (Restricting to $\Lambda_G(61)$ is not enough, as it yields a set of dominant weights of cardinality 3461 which does not satisfy (P2')).

Again our program checking rigorously that these sets satisfy (P1') and the inductive computation of masses and of the numbers $\mathrm{N}^\perp(w(\lambda))$ can be found at [CT19b]. This concludes the proof of Theorem 6.

Remark 3.10. Let G be as in Theorem 6. An inspection of the masses found above shows $\mathrm{m}_P \neq 0$ for all $P \in \mathcal{P}_1(G)$, except for 6 polynomials P in the case $G = \mathrm{SO}_{13}$, and 6 others in the case $G = \mathrm{SO}_{17}$. This shows that the spinor norms constraints established in §3.2 are almost sharp.

This second method only gives us these three additional cases for which we can compute all masses by solving a linear system. For example the set $\mathcal{P}_1(\mathrm{Sp}_{16})/\sim$ has 2521 elements, but we are not even close to producing Λ's with enough elements: we were only able to produce a subset $\Lambda \subset \Lambda_{\mathrm{Sp}_{16}}(116)$ having 1427 elements satisfying (P1'). To overcome this scarcity of dominant weights satisfying (P1'), we computed a lot of masses for Sp_{16}, namely for all P in a certain subset $\mathcal{P}(\mathrm{Sp}_{16})_{\mathrm{easy}}$ of $\mathcal{P}(\mathrm{Sp}_{16})$, using the method of [Taï17], *i.e.* by computing orbital integrals directly, and then we computed the remaining ones by solving a linear system.

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[13]We actually have $|S| = 26$, explained by the equality $\dim \mathrm{S}_{23}(\mathrm{SL}_2(\mathbb{Z})) = 2$.

To describe the set $\mathcal{P}(\mathrm{Sp}_{16})_{\mathrm{easy}}$ explicitly, for $P \in \mathcal{P}(\mathrm{Sp}_{16})$ and p a prime write

$$P = \prod_m \prod_{k \in S(p,m)} \Phi_{mp^k}^{d(p,m,k)}$$

where the first product is over all integers m coprime to p, $S(p,m) \subset \mathbb{Z}_{\geq 0}$ and $d(p,m,k) \geq 1$. Then $P \in \mathcal{P}(\mathrm{Sp}_{16})_{\mathrm{easy}}$ if and only if for any prime number p and any m coprime to p we have $|S(p,m)| \leq 2$ and

$$\begin{cases} 0 \in S(p,m) & \text{if } p > 2 \text{ and } |S(p,m)| = 2, \\ 0 \in S(p,m) \text{ or } 1 \in S(p,m) & \text{if } p = 2 \text{ and } |S(p,m)| = 2. \end{cases}$$

For such a polynomial P the computation using the method explained *loc. cit.* of the orbital integrals (1.4.4) occurring in the mass m_P is purely a combinatorial matter and does not require any bilinear algebra. To be more precise, in general computing an orbital integral using the method loc. cit. involves enumerating totally isotropic subspaces stable under a given unipotent automorphism γ in (possibly degenerate) symplectic or hermitian spaces $(V, \langle \cdot, \cdot \rangle)$ over a finite field, enumerating isomorphisms between such triples $(V, \langle \cdot, \cdot \rangle, \gamma)$, and/or computing the complete invariants attached by Wall [Wal63] to isomorphism classes of such triples with $\langle \cdot, \cdot \rangle$ non-degenerate; we restrict to cases where no such computation is necessary. Although these easier cases have the obvious benefit of being much easier to implement, the second advantage here is that these orbital integrals are computed (by a computer) in a matter of seconds. In contrast, there are relevant orbital integrals for Sp_{16} for which the implementation of [Taï17] does not terminate in any reasonable time.

Denoting $\mathcal{P}(\mathrm{Sp}_{16}) = \mathcal{P}(\mathrm{Sp}_{16})_{\mathrm{easy}} \sqcup \mathcal{P}(\mathrm{Sp}_{16})_{\mathrm{hard}}$, we have $|\mathcal{P}(\mathrm{Sp}_{16})_{\mathrm{hard}}/ \sim| = 766$. We found a subset of dominant weights $\Lambda \subset \Lambda_G(36)$ for $G = \mathrm{Sp}_{16}$ of cardinality 1086 satisfying (P1') and the analogue of (P2') for $\mathcal{P}(\mathrm{Sp}_{16})_{\mathrm{hard}}$. This concludes the proof of Theorem 7.

4. Classification results in motivic weights 23 and 24

This section is in the natural continuation of Sect. 2, of which we shall use freely the notations. We have decided to postpone it here because, in a few places, we will use below existence or inexistence results of certain self-dual regular elements of Π_{alg}, results which have been proved in Sect. 3.

4.1. Motivic weight 23. We now prove Theorem 3 along with the following supplementary result.

Proposition 4.1. *Let U be an effective element of $\mathrm{K}_\infty^{\leq 23}$ containing I_{23} with multiplicity ≥ 2. Let T be the subset of elements π in Π_{alg} with $\mathrm{L}(\pi_\infty) = U$.*

(1) *If $|T| \geq 2$ then we have $U = I_1 + I_7 + I_{13} + I_{17} + I_{21} + 2\,I_{23}$ and $T = \{\pi, \pi^\vee\}$ for some non-self-dual π.*

(2) *If $T = \{\pi\}$ then U belongs to an explicit set of* 181 *elements and π is of symplectic type.*

The set of 181 possible U mentioned above can be found in [CT19b]. They all satisfy $14 \leq \dim U \leq 42$.

Proof of Theorem 3 and Proposition 4.1. For $\ell = 9.74$ the restriction of the symmetric bilinear form $\widehat{F}_\ell(i/4\pi)^{-1} B_\infty^{F_\ell}$ to $K_\infty^{\leq 23}$ is positive definite. As explained in Remark 2.11, using interval arithmetic we obtain rational lower bounds (we take them in $10^{-6}\mathbb{Z}$) for the coefficients of its Gram matrix in the basis $I_1, \ldots, I_{23}$. Applying the Fincke-Pohst algorithm, we obtain the set $\mathcal{U}_2$ of all 265 effective elements U in $K_\infty^{\leq 23}$ containing I_{23} and satisfying $B_\infty^{F_\ell}(U, U) \leq \widehat{F}_\ell(i/4\pi)/2$. By Corollary 2.3, $\mathcal{U}_2$ contains all the elements U such that there exist two distinct elements π_1, π_2 in Π_{alg} of motivic weight 23 and with $L((\pi_1)_\infty) = L((\pi_2)_\infty) = U$.

For each U in $\mathcal{U}_2$, we systematically applied Algorithm 2.4.5 to U, $\delta = 0$, $m = 2$ and to the set $\mathbf{S}$ of 27 known elements of Π_{alg} having motivic weight ≤ 24, and various ℓ. For all but one U, namely the one of assertion (1), it led to a contradiction with Inequality (2.3.7). Let us be more precise about the choices of ℓ and of the subset $S' \subset S$ that we can make *a posteriori* in order to reach these contradictions more quickly (see also the source code [CT19b] for a working sheet). We first replace for the rest of the proof the $\mathbf{S}$ above by its subset whose elements have motivic weight ≤ 23. We now have $|\mathbf{S}| = 24$ and $|S| = 23$. If we apply Algorithm 2.4.5 with all ℓ in $[3, 12] \cap \mathbb{Z}$ and all subsets $S' \subset S$ with $1 \leq |S'| \leq 2$, we obtain in a few seconds (on a personal computer) a contradiction for all but 12 elements of $\mathcal{U}_2$. Using then all ℓ in $[7, 9.5] \cap \frac{1}{2}\mathbb{Z}$ and all $S' \subset S$ with $3 \leq |S'| \leq 4$ for those 12 elements, the algorithm finds again in a few seconds a contradiction in all but 6 cases. Two of these six are eliminated in a minute about using $\ell = 11$ and all the 33649 subsets S' with $|S'| = 5$. The remaining 4 elements have the form $U = U' + I_{21} + 2\,I_{23}$ with U' in the following list: $I_3 + I_7 + I_{13} + I_{17}$, $I_1 + I_5 + I_9 + I_{11} + I_{15} + I_{17} + I_{19}$, $I_1 + I_7 + I_{11} + I_{15} + I_{19}$, $I_1 + I_7 + I_{13} + I_{17}$. In the case $U' = I_1 + I_7 + I_{11} + I_{15} + I_{19}$ we use $\ell = 8.75$ and all the 100947 subsets S' with $|S'| = 6$. To give an example, the algorithm produces in about 2 minutes a linear combination close to

$$x = 0.860 \frac{1}{\sqrt{2}}(\pi_1 + \pi_2) + 0.0834\,1 + 0.150\,\Delta_{11} + 0.108\,\Delta_{15} + 0.335\,\Delta_{19,7} + 0.172\,\Delta_{23,7} + 0.280\,\Delta_{23,15,7}$$

with $C^{F_{8.75}}(x, x) = -0.0023$ up to 10^{-4}. In the case $U' = I_1 + I_5 + I_9 + I_{11} + I_{15} + I_{17} + I_{19}$, we use similarly $\ell = 11.75$ and $|S'| = 6$. The case $U' = I_3 + I_7 + I_{13} + I_{17}$ is quite harder to discard. After many tries, we found a contradiction using $\ell = 10.25$

and a certain 11 element subset S' of S: see the source code in [CT19b] for the details. So far, we have thus proved the following:

(a) For any $U \neq \mathrm{I}_1 + \mathrm{I}_7 + \mathrm{I}_{13} + \mathrm{I}_{17} + \mathrm{I}_{21} + 2\,\mathrm{I}_{23}$ there is at most one element π of Π_{alg} with motivic weight 23 and $\mathrm{L}(\pi_\infty) = U$. In particular any such π is self-dual.

Despite our efforts, we could not find a contradiction in the case of the last element $U = \mathrm{I}_1 + \mathrm{I}_7 + \mathrm{I}_{13} + \mathrm{I}_{17} + \mathrm{I}_{21} + 2\,\mathrm{I}_{23}$. We have however $\mathrm{B}_\infty^{F_\ell}(U,U) > \widehat{\mathrm{F}}_\ell(i/4\pi)/3$ for $\ell = 9.74$. By Corollary 2.3, this shows:

(b) For $U = \mathrm{I}_1 + \mathrm{I}_7 + \mathrm{I}_{13} + \mathrm{I}_{17} + \mathrm{I}_{21} + 2\,\mathrm{I}_{23}$, there are at most 2 elements π of Π_{alg} of motivic weight 23 and with $\mathrm{L}(\pi_\infty) = U$.

Note that we have proved assertion *(1)* except for the non self-duality assertion. To go further, we determine the set of effective elements U in $\mathrm{K}_\infty^{\leq 23}$ containing I_{23} and satisfying $\mathrm{B}_\infty^{\mathrm{F}_\ell}(U,U) \leq \widehat{\mathrm{F}}_\ell(i/\pi)$ for $\ell = 9.74$. For this we proceed as in the first paragraph of the proof and obtain an explicit set $\mathcal{U}_1$ with 12230 elements. By Corollary 2.3, $\mathcal{U}_1$ contains all the elements U such that there exists π in Π_{alg} of motivic weight 23 with $\mathrm{L}(\pi_\infty) = U$. For each U in $\mathcal{U}_1$ we applied Algorithm 2.4.5 to U, $\delta = 1$ (we restrict to self-dual elements), $m = 1$, to the set $\mathbf{S}$ of all 27 known elements of Π_{alg} having motivic weight ≤ 24, for various choices of ℓ and subsets $S' \subset S$. We obtained contradictions with Inequality (2.3.7) for all but 187 elements of $\mathcal{U}_2$. We refer to [CT19b] for an explicit list of $12293 - 187 = 12106$ triples $(\ell, S', \underline{t})$ leading to a contradiction in each case (checked using interval arithmetic). It would be tedious to explain here in details which ℓ and S' we did choose to find these triples: this is unnecessary anyway as all the necessary information for our proof is contained in the aforementioned list! We nevertheless refer to [CT19b] for the log file of our computations (which took several months).

Among the 187 aforementioned "resistant" elements of $\mathcal{U}_2$, six of them are multiplicity free: $\mathrm{I}_3 + \mathrm{I}_{11} + \mathrm{I}_{17} + \mathrm{I}_{21} + \mathrm{I}_{23}, \mathrm{I}_7 + \mathrm{I}_{15} + \mathrm{I}_{21} + \mathrm{I}_{23}, \mathrm{I}_3 + \mathrm{I}_9 + \mathrm{I}_{15} + \mathrm{I}_{21} + \mathrm{I}_{23}, \mathrm{I}_1 + \mathrm{I}_9 + \mathrm{I}_{17} + \mathrm{I}_{21} + \mathrm{I}_{23}, \mathrm{I}_5 + \mathrm{I}_{13} + \mathrm{I}_{17} + \mathrm{I}_{21} + \mathrm{I}_{23}$ and $\mathrm{I}_5 + \mathrm{I}_{13} + \mathrm{I}_{19} + \mathrm{I}_{23}$. These six regular weights have dimension ≤ 10, and we know from the results of Sect. 3 that there is no self-dual π with these Archimedean components. An inspection of the list $\mathcal{V}$ of remaining 181 elements reveals that for any U in $\mathcal{V}$:

   (i) U contains I_{23} with multiplicity ≥ 2,

   (ii) U contains I_w for some $w \in \{1, 3, 5\}$,

   (iii) for any $w \in \{1, 3, 5, 7, 9\}$, the multiplicity of I_w in U is at most one.

Assertion (i) concludes the proof of Theorem 3. Assertion (b) above and the fact that $\mathrm{I}_1 + \mathrm{I}_7 + \mathrm{I}_{13} + \mathrm{I}_{17} + \mathrm{I}_{21} + 2\,\mathrm{I}_{23}$ is not in $\mathcal{V}$ imply assertion *(1)* of Proposition 4.1. By (ii) and (iii) above, for any U in $\mathcal{V}$ there is some I_w which occurs in U with multiplicity 1. In particular, such a U has no $\mathrm{W}_\mathbb{R}$-equivariant nondegenerate symmetric pairing. This shows that any self-dual π with $\mathrm{L}(\pi_\infty) = U$ is of symplectic type by [Art13, Theorem 1.4.2], and proves assertion *(2)* of Proposition 4.1. $\qquad\square$

Remark 4.2. For a given (U, δ, m), it seems hard to us to guess *a priori* what will be the best choices of ℓ and $\mathbf{S}$ (or S') to plug into Algorithm 2.4.5 for the purpose of reaching a contradiction with Inequality (2.3.7). Athough the authors have developed their own intuition and artisanal methods to find good ℓ and $\mathbf{S}$, they are mostly based on numerical experiments. In the same vein, in the cases where we did not find any contradiction, it seems difficult to prove that there cannot be any, as it is always possible to let ℓ vary and increase the size of $\mathbf{S}$. However, based on the large number of experiments we made, we find it likely that it is not possible to discard any of the elements of the remaining list $\mathcal{V}$ by changing ℓ or $\mathbf{S}$.

4.2. **Motivic weight** 24. The following lemma is the first step in the proof of Theorem 5.

Lemma 4.3. *Let* $n \geq 13$. *Let* π *be a self-dual level* 1 *cuspidal algebraic regular automorphic representation of* PGL_n *over* $\mathbb{Q}$ *of motivic weight* 24. *Then* $\mathrm{L}(\pi_\infty)$ *belongs to the following list:*

- $1 + \mathrm{I}_6 + \mathrm{I}_8 + \mathrm{I}_{14} + \mathrm{I}_{20} + \mathrm{I}_{22} + \mathrm{I}_{24}$ *for* $n = 13$,
- $1 + \mathrm{I}_6 + \mathrm{I}_{10} + \mathrm{I}_{16} + \mathrm{I}_{20} + \mathrm{I}_{22} + \mathrm{I}_{24}$ *for* $n = 13$,
- $\mathrm{I}_2 + \mathrm{I}_4 + \mathrm{I}_{12} + \mathrm{I}_{14} + \mathrm{I}_{18} + \mathrm{I}_{20} + \mathrm{I}_{22} + \mathrm{I}_{24}$ *for* $n = 16$,
- $\mathrm{I}_2 + \mathrm{I}_8 + \mathrm{I}_{12} + \mathrm{I}_{14} + \mathrm{I}_{18} + \mathrm{I}_{20} + \mathrm{I}_{22} + \mathrm{I}_{24}$ *for* $n = 16$.

In particular we have $n \leq 16$.

Proof. Let π be as in the lemma and set $U = \mathrm{L}(\pi_\infty)$. This is a multiplicity free effective element of $\mathrm{K}_\infty^{\leq 24}$ containing I_{24} and with $\det U = 1$. There are only finitely many such elements, with $\dim U \leq 25$ in all cases. Moreover, π is orthogonal as $w(\pi)$ is even (see the last paragraph of §2.2). By [Art13, Theorem 1.5.3] loc. cit. we have thus $\epsilon(\pi) = +1$. Since $\epsilon(\pi) = \varepsilon(U)$ this gives an extra constraint on U.

A straightforward computer-aided enumeration gives us the list of the 1260 effective multiplicity free elements U in $\mathrm{K}_\infty^{\leq 24}$ containing I_{24}, and satisfying $\dim U > 12$, $\det U = 1$ and $\varepsilon(U) = 1$. We applied Algorithm 2.4.5 to each such U, $\delta = 1$, $m = 1$, to the set $\mathbf{S}$ of 15 elements of Π_{alg} having motivic weight ≤ 23 and dimension ≤ 4. Except in the four cases given in the statement, we obtained a contradiction with Inequality (2.3.7). It is actually enough to choose ℓ in $[3, 7] \cap \frac{1}{2}\mathbb{Z}$ and to restrict to the subsets $S' \subset S$ with $|S'| \leq 7$. We refer to [CT19b] for an explicit list of 1256 triples $(\ell, S', \underline{t})$ leading to a contradiction in each case (checked using interval arithmetic). $\square$

Proof. (Of Theorem 5) Using Theorem 6 we may compute, for any effective, multiplicity free, element $U \in \mathrm{K}_\infty^{\leq 24}$ containing I_{24}, and with $\dim U \leq 16$, the number of self-dual π in Π_{alg} with $\mathrm{L}(\pi_\infty) = U$ (this uses SO_n for $n \leq 16$ and Sp_{2n} for $2n \leq 8$). Remarkably, we find only three such π, namely the ones in the statement of Theorem 5. We conclude by Lemma 4.3. $\square$

4.3. **Classification results conditional to** (GRH). By (GRH) we shall mean here: for all $\pi, \pi' \in \Pi_{\mathrm{alg}}$, the zeros $s \in \mathbb{C}$ of $\Lambda(s, \pi \times \pi')$ satisfy $\Re s = \frac{1}{2}$. Assuming (GRH), Proposition 2.2 holds more generally for any test function F satisfying $F(x) \geq 0$ and $\widehat{F}(\xi) \geq 0$ for all x and ξ in $\mathbb{R}$ (a condition weaker than (POS)). This condition is trivially satisfied by the function $\mathrm{G}_\ell(x) = \mathrm{g}(x/\ell)$, where g is the function recalled in §2.4.3 and ℓ is a positive real number (these are the classical functions of Odlyzko "under (GRH)"). In order to apply Algorithms 2.4.4 and 2.4.5 with G_ℓ instead of F_ℓ, we need the following variant of [CL19, Prop. 9.3.18]. We set $\phi(z) = \frac{1}{2}\psi(\frac{z+1}{2}) - \frac{1}{2}\psi(\frac{z}{2})$ and $\mathrm{r}(z) = 2\pi^2 \frac{e^{-z}}{(z^2+\pi^2)^2}$.

Proposition 4.4. *Let $\ell > 0$ be a real number. For any integer $w \geq 0$ we have*

$$\mathrm{J}_{\mathrm{G}_\ell}(\mathrm{I}_w) = \log 2\pi - \Re \psi(b + \frac{i\pi}{\ell}) + \frac{1}{\pi} \Im \psi(b + \frac{i\pi}{\ell}) - \frac{1}{\ell}\Re \psi'(b + \frac{i\pi}{\ell}) + s_1(b,\ell),$$

with $b = \frac{1+w}{2}$ and $s_1(b,\ell) = \ell \sum_{n=0}^{\infty} \mathrm{r}(\ell(b+n))$. Moreover, we also have

$$\mathrm{J}_{\mathrm{G}_\ell}(1 - \varepsilon_{\mathbb{C}/\mathbb{R}}) = \Re \phi(\frac{1}{2} + \frac{i\pi}{\ell}) - \frac{1}{\pi} \Im \phi(\frac{1}{2} + \frac{i\pi}{\ell}) + \frac{1}{\ell} \Re \phi'(\frac{1}{2} + \frac{i\pi}{\ell}) + s_2(\ell)$$

with $s_2(\ell) = \ell \sum_{n=0}^{\infty}(-1)^n \mathrm{r}(\ell(n+1/2))$, as well as $\widehat{\mathrm{G}_\ell}(0) = 8\ell/\pi^2$ and

$$\widehat{\mathrm{G}_\ell}(i/4\pi) = 4\pi^2\ell \frac{1 + \cosh(\ell/2)}{(\ell^2/4 + \pi^2)^2}.$$

Proof. We follow the proof of [CL19, Prop. 9.3.18] and omit the straightforward details. For real numbers $b, \ell > 0$, set $S(b,\ell) = \int_0^\infty (\mathrm{g}(x/\ell)\frac{e^{-bx}}{1-e^{-x}} - \frac{e^{-x}}{x})\mathrm{d}x$. A computation almost identical to p. 276 *loc. cit.* shows that we have $S(b,\ell) = -\Re \psi(b + \frac{i\pi}{\ell}) + \frac{1}{\pi} \Im \psi(b + \frac{i\pi}{\ell}) - \frac{1}{\ell}\Re \psi'(b + \frac{i\pi}{\ell}) + s_1(b,\ell)$. On the other hand, by [CL19, Prop. 9.3.8] we have $\mathrm{J}_{\mathrm{G}_\ell}(\mathrm{I}_w) = \log 2\pi + S(\frac{1+w}{2}, \ell)$ for any integer $w \geq 0$ and $\mathrm{J}_{\mathrm{G}_\ell}(1 - \varepsilon_{\mathbb{C}/\mathbb{R}}) = \frac{1}{2}(S(\frac{1}{4}, 2\ell) - S(\frac{3}{4}, 2\ell))$. This shows the first two formulas. Set $h(\alpha) = \int_0^\infty \mathrm{g}(x)e^{-\alpha x}\mathrm{d}x$ for α in $\mathbb{C}$. By p. 275 *loc. cit.* we have $h(\alpha) = \frac{\alpha}{\alpha^2+\pi^2} + 2\pi^2 \frac{1+e^{-\alpha}}{(\alpha^2+\pi^2)^2}$. We conclude by the relations $\widehat{\mathrm{G}_\ell}(0) = 2\ell \, h(0)$ and $\widehat{\mathrm{G}_\ell}(i/4\pi) = \ell (h(\ell/2) + h(-\ell/2))$. $\square$

Upper bounds for the tails of the series s_1 and s_2 are given in [CL19, (3) p. 277].

Proof. (Of Theorem 4) In this proof, whenever we apply Algorithms 2.4.4 & 2.4.5 we do it using G_ℓ instead of F_ℓ. Applying Algorithm 2.4.5 to the element U of Proposition 4.1 (1), $\delta = m = 2$ and the set **S** of 27 known elements of Π_{alg} with motivic weight ≤ 24, we obtain a contradiction with Inequality (2.3.7) with $\ell = 5$ and $S' = \{\Delta_{23,7}, \Delta_{23,13,5}, \Delta_{23,15,7}\}$ (three elements in the list of Thm. 3). It thus only remains to show that for any of the 181 elements of the list $\mathcal{V}$ of Proposition 4.1 (2), there is no selfdual π in Π_{alg} with $\mathrm{L}(\pi_\infty) = U$. For each U in $\mathcal{V}$, we applied Algorithm 2.4.5 to U, $\delta = m = 1$ and the set **S** above, using various ℓ. We

found a contradiction in all but one cases. More precisely, we may reach all these contradictions but one using $\ell \in [4,7] \cap \frac{1}{2}\mathbb{Z}$, S' of size ≤ 7, and S' not containing any of the 3 elements of Π_{alg} with motivic weight 24 (see [CT19b] for a working sheet). The two remaining elements of $\mathcal{V}$ are then

$$A = \mathrm{I}_1 + \mathrm{I}_7 + \mathrm{I}_{11} + \mathrm{I}_{15} + \mathrm{I}_{19} + \mathrm{I}_{21} + 2\,\mathrm{I}_{23} \;\text{ and }\; B = \mathrm{I}_1 + \mathrm{I}_9 + \mathrm{I}_{15} + \mathrm{I}_{19} + 2\,\mathrm{I}_{23}.$$

For $U = B$ we eventually found a contradiction using $\ell = 6.36$ and a certain subset $S' \subset S$ with 13 elements ! (see *loc. cit.*) The remaining case $U = A$ is the one of the statement of Theorem 4. $\qquad\square$

5. Siegel modular cusp forms for $\mathrm{Sp}_{2g}(\mathbb{Z})$

In this section, we explain how to use our classification Theorems 3 & 5 to prove Theorem 2. Along the way, we will also reformulate much more precisely the **Key fact 1** stated in the introduction.

5.1. **Brief review of Arthur's results for** Sp_{2g}. Fix $g \geq 1$ be an integer. We denote by $\Pi_{\mathrm{disc}}(\mathrm{Sp}_{2g})$ the set of isomorphism classes of discrete automorphic representations of Sp_{2g} with $\pi_p^{\mathrm{Sp}_{2g}(\mathbb{Z}_p)} \neq 0$ for all primes p. Recall that the Langlands dual group of Sp_{2g} "is" $\mathrm{SO}_{2g+1}(\mathbb{C})$; it has a tautological (often called *standard*) representation St of dimension $2g + 1$. Let π be in $\Pi_{\mathrm{disc}}(\mathrm{Sp}_{2g})$. For each prime p, the Satake parameter $c(\pi_p)$ of π_p will be viewed following Langlands as a semi-simple conjugacy class in $\mathrm{SO}_{2g+1}(\mathbb{C})$. Similarly, the infinitesimal character $c(\pi_\infty)$ of π_∞ will be viewed as a semi-simple conjugacy class in the Lie algebra of $\mathrm{SO}_{2g+1}(\mathbb{C})$ (and most of the time, as the collection of its $2g + 1$ eigenvalues).

Let $\Psi(\mathrm{Sp}_{2g})$ denote the set of level 1 *global Arthur parameters* for Sp_{2g}. An element of $\Psi(\mathrm{Sp}_{2g})$ is by definition a finite collection ψ of distinct triples (π_i, n_i, d_i), for i in I, with $n_i, d_i \geq 1$ a collection of integers satisfying $2g + 1 = \sum_{i \in I} n_i d_i$, and with π_i a level 1 self-dual cuspidal automorphic representation of PGL_{n_i} over $\mathbb{Q}$ which is orthogonal if d_i is odd, symplectic otherwise. It suggestive to view ψ as the isobaric automorphic representation of GL_{2g+1} over $\mathbb{Q}$ defined as

$$(5.1.1) \qquad\qquad \psi \;=\; \underset{i \in I}{\boxplus} \; \underset{0 \leq r_i \leq d_i - 1}{\boxplus} \; \pi_i \left| . \right|^{\frac{d_i - 1}{2} - r_i}.$$

We often simply write for short [14]

$$\psi = \underset{i \in I}{\oplus} \pi_i[d_i].$$

To any ψ in $\Psi(\mathrm{Sp}_{2g})$, viewed as in (5.1.1) as an irreducible admissible representation of $\mathrm{GL}_{2g+1}(\mathbb{A})$, we may attach a collection of Satake parameters ψ_p (semi-simple conjugacy classes in $\mathrm{GL}_{2g+1}(\mathbb{C})$), as well as an infinitesimal character

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[14]For typographical reasons we also replace the symbol $\pi_i[d_i]$ with $[d_i]$ if we have $\pi_i = 1$, and by π_i if we have $d_i = 1$ and $\pi_i \neq 1$.

ψ_∞ (a semi-simple conjugacy class in $\mathrm{M}_{2g+1}(\mathbb{C})$). We shall say that ψ is *algebraic* when the $2g + 1$ eigenvalues of ψ_∞ are in $\mathbb{Z}$. In this case, the only one that we shall need to study here, all the π_i are algebraic (see §2.1).

Assume $\psi \in \Psi(\mathrm{Sp}_{2g})$ is algebraic. Using the local Langlands correspondence for the $\mathrm{GL}_{n_i}(\mathbb{R})$, we may attach to ψ a morphism $\psi_\mathbb{R} : \mathrm{W}_\mathbb{R} \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{SO}_{2g+1}(\mathbb{C})$, uniquely defined up to $\mathrm{SO}_{2g+1}(\mathbb{C})$-conjugacy, with the property

$$\mathrm{St} \circ \psi_\mathbb{R} \simeq \bigoplus_i \mathrm{L}((\pi_i)_\infty) \boxtimes \mathrm{Sym}^{d_i-1}\mathbb{C}^2.$$

(Recall the notation $\mathrm{L}(-)$ from §2.1) By §2.1, note that $\psi_\mathbb{R}$ is trivial on $\mathbb{R}_{>0} \times 1$, and in particular, $\psi_\mathbb{R}(\mathrm{W}_\mathbb{R})$ is bounded (it is thus an Archimedean *Arthur parameter*). If r is a representation of $\mathrm{W}_\mathbb{R}$, and $d \geq 1$ is an integer, it will be convenient to write $r[d]$ for the representation $r \boxtimes \mathrm{Sym}^{d-1}\mathbb{C}^2$ of $\mathrm{W}_\mathbb{R} \times \mathrm{SL}_2(\mathbb{C})$.

Arthur's first main result [Art13, Thm. 1.5.2] attaches to any π in $\Pi_{\mathrm{disc}}(\mathrm{Sp}_{2g})$ a unique $\psi(\pi)$ in $\Psi(\mathrm{Sp}_{2g})$ such that we have $\psi(\pi)_v = \mathrm{St} \circ c(\pi_v)$ for all place v of $\mathbb{Q}$ (see also [Taï17, Lemma 4.1.1]). Arthur's second main result is a converse statement, the so-called *multiplicity formula*, on which we shall focus from now on and until the end of this section.

Fix $\psi = \oplus_{i \in I} \pi_i[d_i]$ in $\Psi(\mathrm{Sp}_{2g})$. We assume that ψ is algebraic for our purposes. There are both a local and a global ingredient in the multiplicity formula.

We start with the global one. Write $I = I_{\mathrm{even}} \coprod I_{\mathrm{odd}}$ with $i \in I_{\mathrm{even}}$ if, and only if, $n_i d_i$ is even. Define C_ψ as the abelian group generated by the symbols s_i for $i \in I_{\mathrm{even}}$, and by the symbols s_{ij} for all $i, j \in I_{\mathrm{odd}}$, with relations $1 = s_i^2 = s_{ij}^2$ and $s_{ij}s_{jk} = s_{ik}$ (note $s_{ii} = 1$). This is an elementary abelian 2-group of order $2^{|I|-1}$. Arthur defines a global character ϵ_ψ of this group in [Art13, p. 48], that we now recall. For each $i \in I$ consider the sign

$$(5.1.2) \qquad \epsilon(i) = \prod_{j \neq i} \epsilon(\pi_i \times \pi_j)^{\mathrm{Min}(d_i,d_j)}.$$

The term $\epsilon(\pi_i \times \pi_j)$ here is the Rankin-Selberg root number already encountered in §2.3, a (purely Archimedean) sign that we already explained how to compute *loc. cit.* It is necessarily $+1$ by [Art13, Theorem 1.5.3] if π_i and π_j are both orthogonal or both symplectic. As the adjoint representation of $\mathrm{SO}_{2g+1}(\mathbb{C})$ is isomorphic to $\Lambda^2\mathrm{St}$, Arthur's definition reads (see e.g. [CL19, Sect. 8.3.5] for more details):

$$(5.1.3) \qquad \epsilon_\psi(s_i) = \epsilon(i) \quad \forall i \in I_{\mathrm{even}} \quad \text{and} \quad \epsilon_\psi(s_{ij}) = \epsilon(i)\epsilon(j) \quad \forall i, j \in I_{\mathrm{odd}}.$$

We now describe the local ingredient. Fix K a maximal compact subgroup of $\mathrm{Sp}_{2g}(\mathbb{R})$ and $\mathfrak{g}$ the Lie algebra of $\mathrm{Sp}_{2g}(\mathbb{R})$. Arthur associates to $\psi_\mathbb{R}$ a finite multi-set $\Pi(\psi_\mathbb{R})$, also called an *Arthur packet*, of unitary irreducible admissible $(\mathfrak{g}, K)$-modules. One important property he shows is that we have $\pi_\infty \in \Pi(\psi(\pi)_\mathbb{R})$ for all $\pi \in \Pi_{\mathrm{disc}}(\mathrm{Sp}_{2g})$. Moreover, $\Pi(\psi_\mathbb{R})$ is equipped with a map

$$\Pi(\Psi_\mathbb{R}) \to \mathrm{Hom}(\mathrm{C}_{\psi_\mathbb{R}}, \{\pm 1\}), \quad U \mapsto \chi_U,$$

where $C_{\psi_\mathbb{R}}$ denotes the centralizer of the image of $\psi_\mathbb{R}$ in $SO_{2g+1}(\mathbb{C})$.

Remark 5.1. The map $U \mapsto \chi_U$ depends on the choice of an equivalence class of Whittaker datum for $Sp_{2g}(\mathbb{R})$. From now on we fix a global Whitaker datum Wh for Sp_{2g} such that Wh_p is unramified with respect to $Sp_{2g}(\mathbb{Z}_p)$, for each prime p, in the sense of Casselman and Shalika. Up to conjugating Wh if necessary by the outer action of $GSp_{2g}(\mathbb{Z})$, its Archimedian component Wh_∞ can belong to any of the two classes of Whittaker data for $Sp_{2g}(\mathbb{R})$.

We can now state Arthur's multiplicity formula. Fix an algebraic ψ in $\Psi(G)$. There is a natural group embedding $\iota : C_\psi \hookrightarrow C_{\psi_\mathbb{R}}$ ("local-global" map). Choose U in $\Pi(\psi_\mathbb{R})$ and assume for simplicity that it has multiplicity one in this multiset (this assumption will be satisfied in the cases that we will consider below). Then there is a π in $\Pi_{\mathrm{disc}}(G)$ with $\psi(\pi) = \psi$ and $\pi_\infty \simeq U$ if, and only if, we have

$$(5.1.4) \quad \epsilon_\psi(s_i) = \chi_U(\iota(s_i)) \quad \forall i \in I_{\mathrm{even}} \quad \text{and} \quad \epsilon_\psi(s_{ij}) = \chi_U(\iota(s_{ij})) \quad \forall i, j \in I_{\mathrm{odd}}.$$

Moreover, if these equalities are satisfied then the multiplicity of π in the automorphic discrete spectrum of Sp_{2g} is equal to 1. There is a slightly more complicated statement when we do not assume U has multiplicity one in $\Pi(\psi_\mathbb{R})$. This multiplicity one property will always be the case in our applications (see §5.2). It is believed but not known that it holds in general, although Moeglin and Renard have a number of results in this direction.

Remark 5.2. It is important to remark that (5.1.4) trivially holds when we have $\psi = \varpi[d]$ for some cuspidal ϖ of $PGL_{(2g+1)/d}$, because the group C_ψ is trivial.

5.2. **Lowest-weight modules: results of Arancibia-Moeglin-Renard and of Moeglin-Renard.** For $\underline{k} = (k_1, k_2, \ldots, k_g) \in \mathbb{Z}^g$ with $k_1 \geq k_2 \geq \cdots \geq k_g \geq 0$ we denote by $\rho_{\underline{k}}$ the holomorphic, unitary, lowest weight $(\mathfrak{g}, K)$-module of (lowest) weight $\underline{k}$. [15] The precise meaning here for "lowest" or "holomorphic" is a convention that we may fix as in [MR, §3] to fix ideas, nevertheless this choice will play no role in the sequel as we shall see. We are interested in $\rho_{\underline{k}}$ for the following classical reason. Let us denote by $M_{\underline{k}}(\Gamma_g)$ the vector-space of vector-valued Siegel modular forms of weight $\underline{k}$ for Γ_g, and by $L^2_{\underline{k}}(\Gamma_g)$ its subspace of square-integrable forms. We have

$$S_{\underline{k}}(\Gamma_g) \subset L^2_{\underline{k}}(\Gamma_g) \subset M_{\underline{k}}(\Gamma_g).$$

Assume F is a Hecke eigenform in $L^2_{\underline{k}}(\Gamma_g)$. Then F generates an element $\pi(F)$ in $\Pi_{\mathrm{disc}}(Sp_{2g})$ with $\pi(F)_\infty \simeq \rho_{\underline{k}}$. Better, $\dim L^2_{\underline{k}}(\Gamma_g)$ (resp. $\dim S_{\underline{k}}(\Gamma_g)$) is exactly the number of π in $\Pi_{\mathrm{disc}}(Sp_{2g})$ with $\pi_\infty \simeq \rho_{\underline{k}}$ counted with their global discrete (resp. cuspidal) multiplicity.

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[15]These modules have been classified by Enright, Howe and Wallach: they exist if, and only if, we have $k_g \geq g - (u + v/2)$, with $u = |\{i, k_i = k_g\}|$ and $v = |\{i, k_i = k_g + 1\}|$, and they are unique up to isomorphism if they exist.

An important property to have in mind is that the $2g + 1$ eigenvalues of the infinitesimal character of $\rho_{\underline{k}}$ are 0 and the $2g$ elements $\pm(k_i - i)$ for $i = 1, \ldots, g$. Note that these $2g + 1$ integers are distinct if, and only if, we have $k_g > g$. This is also exactly the condition under which $\rho_{\underline{k}}$ is a (holomorphic) discrete series. If F is a Hecke eigenform in $\mathrm{L}^2_{\underline{k}}(\Gamma_g)$ as above, the shape of the infinitesimal character of $\rho_{\underline{k}}$ implies that $\psi(\pi(F))$ is always algebraic. Moreover, $\rho_{\underline{k}}$ is an element of the Arthur packet $\Pi(\psi(\pi(F))_{\mathbb{R}})$.

Conversely, let us fix until the end of §5.2 a global Arthur parameter

$$\psi = \oplus_{i \in I} \pi_i[d_i]$$

in $\Psi(\mathrm{Sp}_{2g})$ such that the eigenvalues of ψ_∞ are 0 and the $\pm(k_i - i)$, with $i = 1, \ldots, g$ (in particular, ψ is algebraic). In order to apply the Arthur multiplicity formula, we want to know under which condition on ψ the module $\rho_{\underline{k}}$ belongs to $\Pi(\psi_{\mathbb{R}})$, whether it has multiplicity one in this multi-set, and if so, we want to know $\chi_{\rho_{\underline{k}}}$. We shall consider only the two following special, but important, cases.

5.2.1. *Vector-valued case, with $k_g > g$.* This situation is studied at length in [CR15, Chap. 9] and [CL19, Sect. 8.4.7]. In this case, $\rho_{\underline{k}}$ is a discrete series, the eigenvalues of ψ_∞ are distinct, and we have $\mathrm{S}_{\underline{k}}(\Gamma_g) = \mathrm{L}^2_{\underline{k}}(\Gamma_g)$ by a general result of Wallach. We have $I_{\mathrm{odd}} = \{i_0\}$ (a singleton) and π_i is regular for all i in I, so we have $n_i d_i \equiv 0 \bmod 4$ for $i \neq i_0$ by §2.2. The parameter $\psi_{\mathbb{R}}$ is necessarily an Adams-Johnson parameter (see e.g. [CR15, §3.8, App. A],[CL19, Sect. 8.4.15],[Taï17, §4.2.2]), and the main result of [AMR18] shows that $\Pi(\psi_{\mathbb{R}})$ coincides with the packet that Adams and Johnson associate to $\psi_{\mathbb{R}}$ in [AJ87] (any element of this packet having multiplicity one). Arancibia, Moeglin and Renard also prove the expected form of the map $U \mapsto \chi_U$. As was observed in [CR15, §9] (see also [CL19, Sect 8.5.1]), this packet contains $\rho_{\underline{k}}$ if and only if we have $d_{i_0} = 1$, and in this case the corresponding character $\chi_{\rho_{\underline{k}}}$ is given by the formula, for all i in I_{even}:

$$(5.2.1) \qquad \chi_{\rho_{\underline{k}}}(\iota(s_i)) \;=\; \begin{cases} (-1)^{\frac{n_i d_i}{4}} & \text{if } d_i \equiv 0 \bmod 2, \\ (-1)^{\mathrm{e}_i} & \text{otherwise,} \end{cases}$$

where e_i is the number of *odd* integers $1 \leq j \leq g$ such that $k_j - j$ is a weight of π_i. Note that the quantity $\mathrm{e}_i \bmod 2$ does not change if we replace *odd* with *even* in the definition of e_i, as we have $n_i \equiv 0 \bmod 4$ for d_i odd. This property expresses the fact that the character above does not depend on the choice of the Whittaker datum Wh_∞ in Remark 5.1. All in all, we have explained fully, and much more precisely, the Key fact 1 of the introduction.

5.2.2. *Scalar-valued case, arbitrary genus.* In this case we have $\underline{k} = (k, k, \dots, k)$ in $\mathbb{Z}^g$ with $k \geq 1$, and we rather write $\rho_k(g)$ for $\rho_{\underline{k}}$. If we have $k > g$ we are in the case of §5.2.1, so from now on we assume $g \geq k$. The $2g + 1$ eigenvalues of the infinitesimal character of $\rho_{\underline{k}}$ are now 0 and the $2g$ elements $\pm(k - i)$ for $i = 1, \dots, g$: the eigenvalue 0 has thus the multiplicity 3, and for $g > k > 1$ the eigenvalues $\pm 1, \pm 2, \dots, \pm \min(k - 1, g - k)$ have multiplicity 2.

We will use as a key ingredient the recent local results of Moeglin and Renard [MR] [16], that we will specialize in what follows to this level 1 situation. The first main result of [MR] is that $\rho_k(g)$ belongs to $\Pi(\psi_\mathbb{R})$ if, and only if, we are in one of the two cases called (I) and (H) below. In both cases they show that $\rho_k(g)$ has multiplicity 1 in $\Pi(\psi_\mathbb{R})$ and they determine $\chi_{\rho_k(g)}$. We use the letter I for the case reminiscent of Ikeda lifts, and the letter H for those related to the Howe (or theta) correspondence. The formula for $\chi_{\rho_k(g)}$ given in [MR, Prop. 18.3] depends on the class of Wh_∞, which is represented there by a certain sign δ loc. cit. and that we represent the same way here (it may be either one of ± 1: see Remark 5.1). We will express below only the restriction of $\chi_{\rho_k(g)}$ to C_ψ, which is the information we need for applying the global multiplicity formula (5.1.4). In all cases we will see in particular that this restriction does not depend on Wh_∞, i.e. on δ, hence neither on the choices discussed in the beginning of §5.2 that we made (or rather didn't) to define $\rho_{\underline{k}}$: changing of choice amounts to replace δ with $-\delta$ by [MR].

Preliminary general notations and remarks

Recall we have already defined a partition $I = I_{\mathrm{even}} \coprod I_{\mathrm{odd}}$ according to the parity of $n_i d_i$ for i in I. We now define $I_0 \subset I$ as the subset of elements i in I such that 0 is a weight of π_i. We clearly have $I_{\mathrm{odd}} \subset I_0$, $I_{\mathrm{odd}} \neq \emptyset$ and $|I_0| \leq 3$. Set

$$d_{\max} = \max_{i \in I_0} d_i.$$

It will be convenient to introduce the following definition:

Definition 5.3. *Let k and n be integers ≥ 1, and let π be a cuspidal algebraic automorphic representation of PGL_n. We denote by $\mathrm{r}(\pi)$ the multiplicity of the weight 0 of π. We will say that π satisfies (R_k) if: (i) its weights are $\leq k - 1$, (ii) its nonzero weights have multiplicity 1, and (iii) $\mathrm{r}(\pi) \leq 3$ and each of 1 and $\varepsilon_{\mathbb{C}/\mathbb{R}}$ have multiplicity at most 2 in $\mathrm{L}(\pi_\infty)$.*

We shall see below that all the π_i for $i \in I$ do satisfy (R_k), and that at most one of them is not regular. Our last remark is a simple identity of signs that we have found useful when deciphering the formulas of [MR, Prop. 18.3]. Denote by $\lfloor x \rfloor \in \mathbb{Z}$ the floor of the real number x; for $\epsilon = \pm 1$, $a \in \mathbb{Z}_{\geq 1}$ and $b \in \mathbb{Z}$, we have

$$(5.2.2) \qquad (-1)^{\lfloor \epsilon a/2 \rfloor} = \prod_{i=1}^{a} (-1)^{i-1} \epsilon \quad \text{and} \quad (-1)^{\lfloor (-1)^b \epsilon a/2 \rfloor} = \prod_{i=b+1}^{b+a} (-1)^{i-1} \epsilon.$$

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[16] Note that those authors call $n, m, \pi_n(m)^*$ what we call $g, k, \rho_k(g)$ respectively.

Indeed, the first one is the product of a alternating signs starting with ϵ; it only depends on a mod 4. The second follows from the first by replacing ϵ with $\epsilon(-1)^b$.

Case (I)

This corresponds to case (i) of [MR, Théorème 7.1]. By this theorem, we have $I_0 = \{i_0\}$ (a singleton), the weight 0 of π_{i_0} has multiplicity 1, $d_{i_0} = d_{\max} = 1$, $k - 1 > g - k$, and the g integers $w_i + \frac{d_i - 1}{2} - r_i$, where i is in I, w_i is a positive weight of π_i, and with $0 \leq r_i \leq d_i - 1$, fill the length g segment $[k - g, k - 1]$ (hence are distinct). The representation π_i is regular for each i, with weights $\leq k - 1$, hence satisfies $n_i d_i \equiv 0$ mod 4 for $i \neq i_0$ by §2.2. Moreover, for $i \neq i_0$ the representation π_i does not have 0 among its weights.

In this case we must have $I_0 = I_{\mathrm{odd}}$ so C_ψ is generated by the s_i with $i \neq i_0$. Fix such an i, necessarily in I_{even}. For any sign $s = \pm 1$ we define $\mathrm{e}_s(\pi_i)$ as the number of integers $1 \leq j \leq k - 1$ with $(-1)^j = s$ such that $k - j$ is a weight of π_i. The first assertion of Proposition 18.3 of [MR] (we are in case (1) of §18 *loc. cit.*), together with Formula (5.2.2), show that $\chi_{\rho_k(g)}(\iota(s_i))$ is given by the formula:

$$(5.2.3) \qquad \chi_{\rho_k(g)}(\iota(s_i)) \;=\; \begin{cases} (-1)^{\frac{n_i d_i}{4}} & \text{if } d_i \equiv 0 \text{ mod } 2, \\ (-1)^{\mathrm{e}_\delta(\pi_i)} & \text{otherwise.} \end{cases}$$

Indeed, consider the sequence of g alternating signs $\underline{\mathfrak{s}} = (\mathfrak{s}_{k-1}, \mathfrak{s}_{k-2}, \ldots, \mathfrak{s}_{k-g})$ starting with $\mathfrak{s}_{k-1} = \delta$, *i.e.* set $\mathfrak{s}_{k-i} = \delta(-1)^{i-1}$. Formulas (5.2.2) show that, for any i in I_{even} and any positive weight w of π_i, the sign $\epsilon(\mathrm{I}_{2w}[d_i])$ of Proposition [MR, Prop. 18.3] is given by the formula

$$(5.2.4) \qquad \epsilon(\mathrm{I}_{2w}[d_i]) \;=\; \prod_{w - \frac{d_i-1}{2} \leq j \leq w + \frac{d_i-1}{2}} \mathfrak{s}_j$$

When d_i is even, this sign is $(-1)^{d_i/2}$. When d_i is odd, and thus $w \in \mathbb{Z}$, it coincides with $\mathfrak{s}_w$. The formula $\mathfrak{s}_w = \delta(-1)^{k-w-1}$ shows that we have $\mathfrak{s}_w = -1$ if and only if $w = k - j$ with $(-1)^j = \delta$. Formula (5.2.3) follows, as for i in I_{even} the sign $\chi_{\rho_k(g)}(\iota(s_i))$ is by definition the product, over all the positive weights w of π_i, of $\epsilon(\mathrm{I}_{2w}[d_i])$.

Note that when d_i is odd we have $\mathrm{e}_\delta(\pi_i) + \mathrm{e}_{-\delta}(\pi_i) = n_i/2 \equiv 0$ mod 2, as π_i is regular and does not have the zero weight, so $\mathrm{e}_1(\pi_i) \equiv \mathrm{e}_{-1}(\pi_i)$ mod 2. As a consequence, Formula (5.2.3) does not depend on δ.

Case (H)

This corresponds to case (ii) in [MR, Théorème 7.1]. According to Theorem 7.2 *loc. cit.* there are two subcases:

(H1) There is i_0 in I_0 with $d_{i_0} = d_{\max} = 2(g - k) + 1$, and $\mathrm{L}((\pi_{i_0})_\infty)$ contains $\varepsilon_{\mathbb{C}/\mathbb{R}}^k$.

(H2) There is i_0 in I_0 with $d_{i_0} = d_{\max} = 2(g - k) + 3$, and $\mathrm{L}((\pi_{i_0})_\infty)$ contains $\varepsilon_{\mathbb{C}/\mathbb{R}}^{k-1}$.

Note that i_0 is not unique in general, so we fix any i_0 satisfying (H1) or (H2). We set $k' = k$ in case (H1) and $k' = k - 1$ in case (H2). An inspection of $\psi_\mathbb{R}$ shows that in case (H2) we must have $g - k + 1 \leq k - 1$, that is $g \leq 2k'$ (hence $k' \geq 1$). In both cases we may write

$$\psi_\mathbb{R} \simeq \varepsilon_{\mathbb{C}/\mathbb{R}}^{k'} [2(g - k') + 1] \oplus \psi'.$$

We have $\dim \psi' = 2k'$, $\det \psi' = \varepsilon_{\mathbb{C}/\mathbb{R}}^{k'}$ and the eigenvalues of ψ_∞ contributing to ψ' are the $\pm i$ for $i = 0, \ldots, k - 1$ in case (H1), and the same ones except $\pm(g - k + 1)$ in case (H2). It follows that ψ' is an Adams-Johnson parameter for the compact group $\mathrm{SO}(2k')$, and in particular, is multiplicity-free. This implies:

– π_i satisfies (R_k) for all i (see Definition 5.3), and is regular for $i \neq i_0$.

In particular, for $i \neq i_0$, we have $n_i d_i \equiv 0 \bmod 4$ if n_i is even, and $\mathrm{L}((\pi_i)_\infty)$ contains $\varepsilon_{\mathbb{C}/\mathbb{R}}^{(n_i - 1)/2}$ if n_i is odd (see §2.2). Moreover, either π_{i_0} is regular or we are in the case (H1) (see Remark 5.4) and in one of the two following situations:

– $n_{i_0} \equiv 2 \bmod 4$, 0 is a double weight of π_{i_0}, and $\mathrm{L}((\pi_{i_0})_\infty)$ contains $\varepsilon_{\mathbb{C}/\mathbb{R}}^k$ twice,

– n_{i_0} is odd, 0 is a triple weight of π_{i_0}, $d_{i_0} = 1$, $g = k$, and $\mathrm{L}((\pi_{i_0})_\infty)$ contains $\varepsilon_{\mathbb{C}/\mathbb{R}}^k$ twice and $\varepsilon_{\mathbb{C}/\mathbb{R}}^{k-1}$ once.

Remark 5.4. Assume we are in the case (H2). Then the weight 0 of π_{i_0} has multiplicity 1, since the eigenvalue $g - k + 1$ occurs with multiplicity 1 in the infinitesimal character of $\rho_k(g)$. In particular we have $i_0 \in I_{\mathrm{odd}}$ and π_{i_0} is regular. Moreover, we also have $k' > 1$. Indeed, $k' = 1$ implies $g = 2$ as $k \leq g \leq 2k'$, $\dim \pi_{i_0} = 1$ hence $\pi_{i_0} = 1$ as π_{i_0} has level one, which contradicts (H2).

We now describe the restriction of the character $\chi_{\rho_k(g)}$ to C_ψ. For any i in I and any sign $s = \pm 1$ we define an integer $\mathrm{e}_s(\pi_i)$ as follows. If we are in case (H1), then $\mathrm{e}_s(\pi_i)$ is the number of integers $1 \leq j \leq k - 1$ with $(-1)^j = s$ such that $k - j$ is a weight of π_i (as in case (I)). If we are in case (H2), we first consider the decreasing sequence $(w_1, w_2, \ldots, w_{k'-1}) = (k - 1, k - 2, \ldots, \widehat{g - k + 1}, \ldots, 1)$ where $g - k + 1$ is omitted (this makes sense as $1 \leq g - k + 1 \leq k - 1$ and $k' > 1$ by Remark 5.4), and rather define $\mathrm{e}_s(\pi_i)$ as the number of integers $1 \leq j \leq k' - 1$ with $(-1)^j = s$ such that w_j is a weight of π_i. In all cases we have by property (R_k):

$$(5.2.5) \qquad \mathrm{r}(\pi_i) + 2\,\mathrm{e}_1(\pi_i) + 2\,\mathrm{e}_{-1}(\pi_i) = n_i.$$

– Assume first we have $i \in I_{\mathrm{even}}$ and $i \neq i_0$. For $i \notin I_0$ we have:

$$(5.2.6) \qquad \chi_{\rho_k(g)}(\iota(s_i)) = \begin{cases} (-1)^{\frac{n_i d_i}{4}} & \text{if } d_i \equiv 0 \bmod 2, \\ (-1)^{\mathrm{e}_\delta(\pi_i)} & \text{otherwise.} \end{cases}$$

Indeed, this follows from [MR, Proposition 18.3] by a similar argument as in Case (I). The only difference is to replace in this argument the alternating sequence of signs $\mathfrak{s}$ defined in Case (I) by the length k' alternating sequence $\mathfrak{s} =$

$(\mathfrak{s}_{k-1}, \mathfrak{s}_{k-2}, \ldots, \mathfrak{s}_0)$ starting with δ but *with the index $g - k + 1$ omitted in case (H2)*; in other words, we still set $\mathfrak{s}_{k-i} = \delta(-1)^{i-1}$ in Case (H1), and in Case (H2) we set $\mathfrak{s}_{k-i} = \delta(-1)^{i-1}$ for $k - i > g - k + 1$ and $\mathfrak{s}_{k-i} = \delta(-1)^i$ for $k - i < g - k + 1$ (so $\mathfrak{s}_{g-k+1}$ is undefined in case (H2)). With this definition for $\mathfrak{s}$, the sign $\epsilon(I_{2w}[d_i])$ of [MR, Prop. 18.3] is still given for $w > 0$ by Formula (5.2.4). The same reasoning as in case (I) shows then Formula (5.2.6), as well as its independence on δ.

Assume now $i \in I_0$, so that 0 is a double weight of π_i as $i \in I_{\text{even}}$. This forces $d_i = 1$, because otherwise $\pi_i[d_i]$ would contribute an eigenvalue 1 with multiplicity at least 2 to the Adams-Johnson parameter ψ', a contradiction. We find

$$(5.2.7) \qquad \chi_{\rho_k(g)}(\iota(s_i)) = (-1)^{e_\delta(\pi_i)} \delta\,(-1)^{k'-1}.$$

Indeed, we are in the situation (2) of §18 *loc. cit.* and in the notations there we have $a = 1$ and $\epsilon_1 \epsilon_2 = (-1)^{\lfloor \delta(-1)^{k'-1}/2 \rfloor} = \delta(-1)^{k'-1} = \mathfrak{s}_0$ [MR, Remarque 18.4]. By definition, we have $\chi_{\rho_k(g)}(\iota(s_i)) = (\prod_w \epsilon(I_{2w}))\epsilon_1\epsilon_2$ in the notations of [MR, Prop. 18.3], the product in parenthesis being over the positive weights w of π_i, hence equal to $(-1)^{e_\delta(\pi_i)}$ as explained above. This proves Formula (5.2.7). The congruence $n_i \equiv 0 \bmod 4$, the equality $r(\pi_i) = 2$, and Formula (5.2.5) show that (5.2.7) does not depend on the sign δ.

– Assume $i = i_0$ is in I_{even}, so we are in case (H1) by Remark 5.4. The sign $\chi_{\rho_k(g)}(\iota(s_{i_0}))$ is the product of $(-1)^{e_\delta(\pi_i)}$ and of the sign $\epsilon_2\epsilon_3$ in Case (2) of [MR, Proposition 18.3] (with $a = g - k + 1$). Recall that we have $n_{i_0} \equiv 0 \bmod 4$ if and only if π_{i_0} is regular by §2.2. By *loc. cit.* we have thus $\epsilon_2\epsilon_3 = \delta\,(-1)^{k-1}$ if $n_{i_0} \equiv 0 \bmod 4$, and $\epsilon_2\epsilon_3 = 1$ otherwise, and we obtain:

$$(5.2.8) \qquad \chi_{\rho_k(g)}(\iota(s_{i_0})) = \begin{cases} (-1)^{e_\delta(\pi_{i_0})} \delta\,(-1)^{k-1} & \text{if } n_{i_0} \equiv 0 \bmod 4, \\ (-1)^{e_\delta(\pi_{i_0})} & \text{otherwise.} \end{cases}$$

Again, these two formulas do not depend on δ by Formula (5.2.5) and $r(\pi_{i_0}) = 2$.

– We are left to consider the case $|I_{\text{odd}}| > 1$. We must have $|I_{\text{odd}}| = 3$ and $I_{\text{odd}} = I_0$. We want to give the value of $\chi_{\rho_k(g)}(\iota(s_{ij}))$ for $i \neq j$ in I_{odd}. We have $\min_{i \in I_0} d_i = 1$, since otherwise the eigenvalue 1 would have multiplicity at least 3 in the infinitesimal character of $\rho_k(g)$. We may thus write $I_0 = \{i_0, i_1, i_2\}$ with $d_{i_1} = 1$ and set $d_{i_2} = 2a - 1$. The sign $\chi_{\rho_k(g)}(\iota(s_{i_1 i_2}))$ is the product of $(-1)^{e_\delta(\pi_{i_1})}(-1)^{e_\delta(\pi_{i_2})}$ and of the sign $\epsilon_1\epsilon_2 = (-1)^{\lfloor \delta(-1)^{k'-a} a/2 \rfloor}$ in Case (2) of [MR, Proposition 18.3] (see also Remark 18.4 (ii) *loc. cit.*). We obtain:

$$(5.2.9) \qquad \chi_{\rho_k(g)}(\iota(s_{i_1 i_2})) = (-1)^{e_\delta(\pi_{i_1}) + e_\delta(\pi_{i_2}) + \lfloor \delta(-1)^{k'-a} a/2 \rfloor}.$$

Observe that we have $\delta\,(-1)^{k'-a} = \mathfrak{s}_{a-1}$. Indeed, this holds trivially in case (H1), and in case (H2) we have $a - 1 < g - k + 1$ since the eigenvalues $\{0, 1, 2, \ldots, a - 1\}$ contribute to ψ' (as $d_{i_2} = 2a - 1$), so $\mathfrak{s}_{a-1} = \delta(-1)^{k-a+1} = \delta(-1)^{k'-a}$ again. By Formula (5.2.2), this shows the alternative expression

$$(5.2.10) \qquad \epsilon_1 \epsilon_2 = (-1)^{\lfloor \delta(-1)^{k'-a} a/2 \rfloor} = \mathfrak{s}_{a-1} \cdots \mathfrak{s}_1 \mathfrak{s}_0.$$

We finally check that (5.2.9) does not depend on δ. As π_{i_0} is regular of odd dimension, and $L((\pi_{i_0})_\infty)$ contains $\varepsilon_{\mathbb{C}/\mathbb{R}}^{k'}$, we have $n_{i_0} \equiv 2k' + 1 \bmod 4$ by §2.2. As $d_{i_0} = 2(g - k') + 1$, this implies the congruence $n_{i_0} d_{i_0} \equiv 2g + 1 \bmod 4$, and using $2g + 1 = \sum_{i \in I} n_i d_i$ and $n_i d_i \equiv 0 \bmod 4$ for $i \notin I_0$, we obtain

$$(5.2.11) \qquad n_{i_1} + n_{i_2}(2a - 1) \equiv 0 \bmod 4,$$

which may also be written $\frac{n_{i_1}-1}{2} + \frac{n_{i_2}-1}{2} \equiv a \bmod 2$. The relation $e_\delta(\pi_{i_l}) + e_{-\delta}(\pi_{i_l}) \equiv \frac{n_{i_l}-1}{2} \bmod 2$, deduced from (5.2.5), and the trivial identity $(-1)^{\lfloor -ea/2 \rfloor} = (-1)^a (-1)^{\lfloor ea/2 \rfloor}$ for $e = \pm 1$, show that the right-hand side of Formula (5.2.9) does not depend on δ.

We still assume $I_{\mathrm{odd}} = \{i_0, i_1, i_2\}$ as above. It only remains to give $\chi_{\rho_k(g)}(\iota(s_{i_0 i_2}))$. It will depend on the following property on $\{\pi_{i_0}, \pi_{i_2}\}$:

$$(P_{i_0,i_2}) \qquad L((\pi_{i_0})_\infty) \oplus L((\pi_{i_2})_\infty) \text{ contains } 1 \oplus \varepsilon_{\mathbb{C}/\mathbb{R}}.$$

Similarly to (5.2.8), we have $\chi_{\rho_k(g)}(\iota(s_{i_0 i_2})) = (-1)^{e_\delta(\pi_{i_0})}(-1)^{e_\delta(\pi_{i_2})}\epsilon_2\epsilon_3$ with $\epsilon_2\epsilon_3$ as in [MR, Proposition 18.3]. We are in case (2) *loc. cit.* if (H1) holds, and in case (3) otherwise. We obtain $\epsilon_2\epsilon_3 = \delta\,(-1)^{k-1}$ if (P_{i_0,i_2}) holds, and $\epsilon_2\epsilon_3 = (-1)^{k-k'}$ otherwise, hence:

$$(5.2.12) \qquad \chi_{\rho_k(g)}(\iota(s_{i_0 i_2})) = \begin{cases} (-1)^{e_\delta(\pi_{i_0})+e_\delta(\pi_{i_2})}\,\delta\,(-1)^{k-1} & \text{if } (P_{i_0,i_2}) \text{ holds}, \\ (-1)^{e_\delta(\pi_{i_0})+e_\delta(\pi_{i_2})}\,(-1)^{k-k'} & \text{otherwise}. \end{cases}$$

These values do not depend on δ by (5.2.5): the integer $n_{i_0} + n_{i_2}$ is $\equiv 0 \bmod 4$ if (P_{i_0,i_2}) holds, and $\equiv 2 \bmod 4$ otherwise.

Remark 5.5. (1) The cases (I) and (H) are disjoint for $k \neq g$, and for $k = g$ case (I) is a special case of (H1), and the two formulas (5.2.3) and (5.2.6) for $\chi_{\rho_k(g)}$ are identical.

(2) The parameter ψ does not always determine k: when g is even, parameters of type (H1) in weight $k = g/2$ coincide with parameters of type (H2) in weight $k = g/2 + 1$.

5.3. **Proof of theorem 2.** Fix $g \geq 1$, $\underline{k} = (k_1, \ldots, k_g) \in \mathbb{Z}^g$ and assume either that $\underline{k}$ is scalar or $k_g > g$. Arthur's multiplicity formula, as well as the multiplicity one results of [AMR18] and [MR], show that two Hecke eigenforms in $S_{\underline{k}}(\Gamma_g)$ with same Hecke eigenvalues, or equivalently with the same standard parameter, are proportional. It follows that $\dim S_{\underline{k}}(\Gamma_g)$ is the number of possible standard parameters of Hecke eigenforms in $S_{\underline{k}}(\Gamma_g)$. In what follows we enumerate these parameters in the case $k_1 \leq 13$. We thus fix a Hecke eigenform in $S_{\underline{k}}(\Gamma_g)$ and denote by $\psi \in \Psi(\mathrm{Sp}_{2g})$ its standard parameter. As in §5.1 we write

$$\psi = \oplus_{i \in I} \pi_i[d_i].$$

NOTATION: – Assume $v_1 > v_2 > \cdots > v_r$ are positive odd (resp. even) integers and that there is a unique self-dual regular π in Π^{alg} with weights $\pm\frac{v_1}{2}, \pm\frac{v_2}{2}, \ldots, \pm\frac{v_r}{2}$,

then we shall denote by $\Delta_{v_1,v_2,\ldots,v_r}$ (resp. $\mathrm{O}^{\mathrm{e}}_{v_1,v_2,\ldots,v_r}$) this unique element π. Similarly, when $v_1 > v_2 > \cdots > v_r$ are even positive integers and there is a unique self-dual π in Π^{alg} with weights 0 and $\pm\frac{v_1}{2}, \pm\frac{v_2}{2}, \ldots, \pm\frac{v_r}{2}$, then we shall denote by $\mathrm{O}^{\mathrm{o}}_{v_1,v_2,\ldots,v_r}$ this element. The Δ's are symplectic and the O's are orthogonal. These notations are compatible with the ones introduced in §1.3, and we have for instance $\mathrm{O}^{\mathrm{o}}_{22} = \mathrm{Sym}^2\Delta_{11}$. We shall also denote by 1 the trivial representation of PGL_1, and by Δ^1_{23} and Δ^2_{23} the two cuspidal representations of PGL_2 generated by the two normalized eigenforms in $\mathrm{S}_{24}(\mathrm{SL}_2(\mathbb{Z}))$.

– We will denote by $\mathcal{L}_{24}$ the subset of Π^{alg} whose elements are either of motivic weight ≤ 22, or of motivic weight 23 with the weight 23 having multiplicity 1, or regular self-dual of motivic weight 24. By the classification theorems ([CL19, Thm. F], Theorems 3 and 5), there are $11+13+3 = 27$ elements in $\mathcal{L}_{24}$, all regular self-dual, and according to the notations above we have

$$\mathcal{L}_{24} = \{1, \Delta_{11}, \Delta_{15}, \Delta_{17}, \Delta_{19}, \Delta_{19,7}, \Delta_{21}, \Delta_{21,5}, \Delta_{21,9}, \Delta_{21,13}, \mathrm{Sym}^2\Delta_{11}, \Delta^1_{23}, \Delta^2_{23}, \Delta_{23,7}, \Delta_{23,9}, \Delta_{23,13},$$

$$\Delta_{23,13,5}, \Delta_{23,15,3}, \Delta_{23,15,7}, \Delta_{23,17,5}, \Delta_{23,19,3}, \Delta_{23,17,9}, \Delta_{23,19,11}, \Delta_{23,21,17,11,3}, \mathrm{O}^{\mathrm{o}}_{24,16,8}, \mathrm{O}^{\mathrm{e}}_{24,18,10,4}, \mathrm{O}^{\mathrm{e}}_{24,20,14,2}\}.$$

5.3.1. *Case $\underline{k} = (k_1, k_2, \ldots, k_g)$ with $k_1 \leq 13$ and $k_g > g$.* We have $1 \leq g \leq 12$. We apply §5.2.1. In this case, each π_i is regular of motivic weight $\leq 2(k_1 - 1) \leq 24$, there is a unique $i_0 \in I$ with n_{i_0} odd, and we have $d_{i_0} = 1$. In particular, all the π_i are in $\mathcal{L}_{24}$ and π_{i_0} is either 1, $\mathrm{Sym}^2\Delta_{11}$ or $\mathrm{O}^{\mathrm{o}}_{24,16,8}$. It is a boring but trivial exercise to enumerate all the ψ in $\Psi(\mathrm{Sp}_{2g})$ with these properties and such that the eigenvalues of ψ_∞ are distinct and ≤ 12. We find exactly 199 such parameters. The possible ψ are then exactly the ones in this list satisfying the Arthur multiplicity formula (5.1.4), using Formulas (5.2.1), (5.1.3) and (5.1.2). We find that only 59 of these 199 do satisfy this formula, and obtain Table 5, as well as the part of Table 6 concerning the case $k > g$. All those computations can be done easily with the help of a computer: see [CT19b] for a `PARI` code doing it. They can also be made by hand as follows.

We only treat the case $\pi_{i_0} = \mathrm{Sym}^2\Delta_{11}$, the two other ones being similar. Note that for $i \neq i_0$ such that π_i is symplectic, we have $w(\pi_i) \leq 19$, and either $\pi_i[d_i] = \Delta_{19,7}[2]$ or $\pi_i = \Delta_w[d]$ with $w + d - 1 \leq 20$. Assume first that π_i is symplectic for all $i \neq i_0$. We have $\epsilon(\Delta_{19,7} \times \mathrm{Sym}^2\Delta_{11}) = 1$, so if we have $\pi_i[d_i] = \Delta_{19,7}[2]$ then Arthur's multiplicity formula $\epsilon_\psi(s_i) = \chi_{\rho_{\underline{k}}}(\iota(s_i))$ simply reads $1 = 1$. If we have $\pi_i[d_i] = \Delta_w[d]$, it rather asserts $-(-1)^{(w+1)/2} = (-1)^{d/2}$, i.e. $w \equiv d + 1 \bmod 4$ (note $\epsilon(\Delta_w \times \mathrm{Sym}^2\Delta_{11}) = -\epsilon(\Delta_w)$ for $w < 22$). This justifies the existence of the 18 ψ in Tables 5 and 6 containing $\mathrm{Sym}^2\Delta_{11}$. Assume now there is $i_1 \neq i_0$, necessarily unique, such that π_{i_1} is orthogonal. We will show that this case cannot happen. We have either $\pi_{i_1} = \mathrm{O}^{\mathrm{e}}_{24,18,10,4}$ or $\pi_{i_1} = \mathrm{O}^{\mathrm{e}}_{24,20,14,2}$, and $d_{i_1} = 1$. If $I = \{i_0, i_1\}$ we have $\epsilon_\psi = 1$ and we compute $\chi_{\rho_{\underline{k}}}(\iota(s_{i_1})) = -1$, so there is $i \neq i_0, i_1$ in I. For weight reasons we must have $\pi_i[d_i] = \Delta_w[2]$, with $w \in \{17, 11\}$ if $\pi_{i_1} = \mathrm{O}^{\mathrm{e}}_{24,20,14,2}$, and $w = 15$ otherwise. This implies $\chi_{\rho_{\underline{k}}}(\iota(s_i)) = -1$. Note that

$\epsilon(\Delta_w \times \pi_{i_1})$ is -1 if π_{i_1} has an odd number of weights $> w/2$, and is 1 otherwise. This shows $\epsilon_\psi(s_i) = -\epsilon(\Delta_w)\epsilon(\Delta_w \times \pi_{i_1}) = 1$ for $w = 17, 11$ and $\pi_{i_1} = \mathrm{O}^{\mathrm{e}}_{24,20,14,2}$, a contradiction. We finally exclude the last possible case $\pi_{i_1} = \mathrm{O}^{\mathrm{e}}_{24,18,10,4}$ and $w = 15$ as we have $I = \{i_0, i_1, i\}$, $\chi_{\rho_{\underline{k}}}(\iota(s_{i_1})) = -1$ and $\epsilon_\psi(s_{i_1}) = \epsilon(\Delta_w \times \pi_{i_1}) = 1$.

5.3.2. *Scalar-valued case with $k \leq 13$ and $g \geq k$.* We apply §5.2.2. We are either in case (I), (H1) or (H2).

Case (I). Assume first we are in case (I), in particular we have $I_0 = I_{\mathrm{odd}} = \{i_0\}$, $d_{i_0} = 1$ and the π_i are regular for all i in I. For each i in I, we have $\frac{w(\pi_j)+d_j-1}{2} \leq 12$, and in particular π_i has motivic weight ≤ 24: it belongs to the list $\mathcal{L}_{24}$. By inspection, π_{i_0} has thus to be $1, \mathrm{Sym}^2\Delta_{11}$ or $\mathrm{O}^{\mathrm{o}}_{24,16,8}$. As the weight 0 has multiplicity 3 in ψ_∞ and $d_{i_0} = 1$ we have $I \neq \{i_0\}$. There is then a unique j in $I - \{i_0\}$ such that, if a_i denotes the smallest positive weight of π_i, we have $a_j - \frac{d_i-1}{2} \leq 0$; we must have $a_j - \frac{d_j-1}{2} = k - g$. We have both $\frac{d_j-1}{2} \geq a_j$ and $\frac{d_j-1}{2} \leq 12 - w(\pi_j)/2$, which implies $a_j + w(\pi_j)/2 \leq 12$. By an inspection of $\mathcal{L}_{24}$, this forces $\pi_j = \Delta_{11}$ and $d_j = 12$. But this implies that the positive weights of the π_i with i in $I - \{j\}$ are ≥ 12. Only the trivial representation has this property in $\mathcal{L}_{24}$. This shows $\pi_{i_0} = 1$ and that the unique possibility for ψ is

$$\psi = \Delta_{11}[12] \oplus [1].$$

We recognize the standard parameter of the genus 12 Ikeda lift of Δ_{11} [Ike01], a well-known element of $\mathrm{S}_{12}(\Gamma_{12})$, hence ψ does exist. Alternatively, Arthur's multiplicity formula (5.1.4) is satisfied as we have $\epsilon_\psi(s_j) = \epsilon(\Delta_{11}) = 1 = \chi_{\rho_{12}(12)}(\iota(s_j))$ by (5.2.3), so ψ is indeed the standard parameter of an eigenform in $\mathrm{L}^2_{12}(\Gamma_{12})$. The shape of ψ and the Zharkovskaya relation[17] imply that this eigenform has to be cuspidal.

Case (H1). Assume we are in case (H1). We will show again $\psi = \Delta_{11}[12] \oplus [1]$. Write $\psi = \pi_{i_0}[2(g - k) + 1] \oplus \bigoplus_{i \neq i_0} \pi_i[d_i]$ as in the definition of (H1).

Lemma 5.6. *The representation π_i is in $\mathcal{L}_{24}$ for all $i \in I$, and we have $\pi_{i_0} = 1$.*

Proof. As a general fact, all the π_i satisfy condition (R) of §5.2.2. They have motivic weight $\leq 2(k - 1) \leq 24$ and their nonzero weights have multiplicity 1. Moreover π_i is regular for $i \neq i_0$. It follows that all the π_i are in $\mathcal{L}_{24}$, except perhaps π_{i_0} in the case $w(\pi_{i_0}) = 24$. But for each i and each positive weight λ of π_i we have $\lambda + \frac{d_i-1}{2} \leq k - 1 \leq 12$. Thus $w(\pi_{i_0}) = 24$ implies $k = 13$ and $d_{i_0} = 1 = 2(g - k) + 1$, so $g = k = 13$: this is absurd as there is no nonzero Siegel modular form for Γ_g with odd weight and genus. So π_{i_0} is in $\mathcal{L}_{24}$ with motivic weight < 24, and the unique possibilities are thus $\pi_{i_0} = 1$ or $\pi_{i_0} = \mathrm{Sym}^2\Delta_{11}$ since

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[17]This asserts that if F is an eigenform in $\mathrm{M}_k(\Gamma_g)$, and if $\Phi_g F$ in $\mathrm{M}_k(\Gamma_{g-1})$ is non zero, with Φ_g the Siegel operator, then $\Phi_g F$ is an eigenform and the standard L-function of F and $\Phi_g F$ satisfy $\mathrm{L}(s, \mathrm{St}, F) = \mathrm{L}(s, \mathrm{St}, \Phi_g F)\zeta(s - (g - k))\zeta(s + (g - k))$.

0 is a weight of π_{i_0}. Assume we have $\pi_{i_0} = \mathrm{Sym}^2\Delta_{11}$. Then $\mathrm{L}((\pi_{i_0})_\infty)$ contains $\varepsilon_{\mathbb{C}/\mathbb{R}}$ so k is odd by (H1), g is even, and we have $d_{i_0} \equiv 3 \bmod 4$. The inequality $11 + \frac{d_{i_0}-1}{2} \leq k-1 \leq 12$ implies then $k = 13$, $d_{i_0} = 3$ and $g = 14$. We have proved

$$\psi = \mathrm{Sym}^2\Delta_{11}[3] \oplus \psi'$$

with $\psi' = \oplus_{i\neq i_0}\pi_i[d_i]$, and ψ'_∞ has the eigenvalues $\pm 9, \pm 8, \ldots, \pm 1$ and 0 twice. But the π_i are in $\mathcal{L}_{24}$ with motivic weight ≤ 18, hence in $\{1, \Delta_{11}, \Delta_{17}\}$, and we have $d_i \leq 3$ for i in I_{odd}. The only possibility is thus $\psi' = \Delta_{11}[8] \oplus [3] \oplus [1]$. We have $I_{\mathrm{even}} = \{i\}$ with $\pi_i[d_i] = \Delta_{11}[8]$, $\epsilon_\psi(s_i) = \epsilon(\Delta_{11} \times \mathrm{Sym}^2\Delta_{11}) = -1$ but $\chi_{\rho_{13}(14)}(\iota(s_i)) = 1$ by Formula (5.2.6): the multiplicity formula is not satisfied. $\square$

Note that $\pi_{i_0} = 1$ implies $k \equiv 0 \bmod 2$ by (H1), hence $k \leq 12$. Write again

$$\psi = [2(g-k)+1] \oplus \psi'$$

where $\psi' = \oplus_{i\neq i_0}\pi_i[d_i]$ is a certain $2k$-dimensional parameter with weights $\pm(k-1), \pm(k-2), \ldots, \pm 1$ and 0 twice, and $k-1 \leq 11$. Each π_i with $i \neq i_0$ is then regular of motivic weight ≤ 22. The list of all possible ψ' with these properties is easy to determine: see Proposition 9.2.2 of [CL19]. For $k = 2, 4, 6$ we find $\psi' = [2k-1] \oplus [1]$. For $k = 8$ we have $[15] \oplus [1]$ and $\Delta_{11}[4] \oplus [7] \oplus [1]$. For $k = 10$ we have

$$[19] \oplus [1], \Delta_{11}[8] \oplus [3] \oplus [1], \Delta_{15}[4] \oplus [11] \oplus [1], \Delta_{17}[2] \oplus [15] \oplus [1], \Delta_{17}[2] \oplus \Delta_{11}[4] \oplus [7] \oplus [1].$$

For $k = 12$, we have 24 possibilities for ψ', namely the ones in [CL19, Thm. E].

Lemma 5.7. *We have $k \equiv 0 \bmod 4$.*

Proof. Assume that $k \equiv 2 \bmod 4$, then by inspection $|I_{\mathrm{odd}}| = 3$ and we denote $I_{\mathrm{odd}} = \{i_0, i_1, i_2\}$ so that $\pi_{i_1}[d_{i_1}] = [1]$ and $\pi_{i_2}[d_{i_2}] = [2a-1]$. For any $i \in I_{\mathrm{even}}$ we have $n_i d_i/2 \equiv 0 \bmod 2$ and so $a \equiv k \bmod 2$, i.e. a is even. Thus Arthur's multiplicity formula (5.1.4) implies

$$\chi_{\rho_k(g)}(\iota(s_{i_1 i_2})) = (-1)^{a/2} = \epsilon(i_1)\epsilon(i_2),$$

the first equality being (5.2.9). For $\psi' = [1] \oplus [2k-1]$ those epsilon factors are 1 and we have $a = k \equiv 2 \bmod 4$ so this formula does not hold. This rules out $k = 2$ and $k = 6$. The four other parameters for $k = 10$ are ruled out the same way using $\epsilon(\Delta_w) = (-1)^{(w+1)/2}$. $\square$

We will now prove that, apart from the case $\psi = \Delta_{11}[12] \oplus [1]$, none of the remaining ψ come from a *cuspidal* modular form. We will need first to recall some results on orthogonal automorphic forms and theta series. For each integer $n \equiv 0 \bmod 8$ we fix arbitrarily an even unimodular lattice of rank n and denote respectively by Ω_n and $\mathrm{S}\Omega_n$ its orthogonal and special orthogonal group schemes over $\mathbb{Z}$. We refer to [CL19, Sects. 4.4 & 6.4.7] for the basics of the theory of level 1 automorphic forms for Ω_n and $\mathrm{S}\Omega_n$ (beware that these group schemes are rather denoted by O_n and SO_n *loc. cit.*). By results of Arthur [Art13] and Taïbi [Taï19],

any discrete automorphic representation of $S\Omega_n$ or Ω_n has a standard parameter ψ in $\Psi(S\Omega_n)$, the latter being defined exactly as in the case of Sp_{2g} (see §5.1) but with the condition $\sum_{i \in I} n_i d_i = n$ instead of $\sum_{i \in I} n_i d_i = 2g + 1$.

For $n \equiv 0 \mod 8$, we denote by X_n the set of isomorphism classes of even unimodular lattices of rank n. The vector-space $\mathbb{C}[X_n]$ is in a natural way the dual of a space of level 1 automorphic forms for Ω_n. Any Hecke eigenform G in $\mathbb{C}[X_n]$ generates a discrete automorphic representation π_G of Ω_n (with trivial Archimedean component and $(\pi_F)_p^{\Omega_n(\mathbb{Z}_p)} \neq 0$ for each prime p), which has a standard parameter ψ_G in $\Psi(S\Omega_n)$. Moreover, Siegel theta series provide a linear map

$$(5.3.1) \qquad \vartheta_g : \mathbb{C}[X_n] \longrightarrow M_{n/2}(\Gamma_g)$$

for all $g \geq 0$ (see e.g. [CL19, §5.1], in particular for the conventions for $g = 0$), with $\Phi \circ \vartheta_g = \vartheta_{g-1}$ (here Φ denotes the Siegel operator). For G in $\mathbb{C}[X_n]$, the *degree* of G is the smallest integer $g_0 \geq 0$ with $\vartheta_{g_0}(G) \neq 0$; the form $\vartheta_{g_0}(G)$ is then cuspidal and we have $\vartheta_g(G) \neq 0$ for $g \geq g_0$. If G in $\mathbb{C}[X_n]$ is an eigenform with degree g_0, and for $g \geq g_0$, then the Eichler commutation relations show that $\vartheta_g(G)$ is an eigenform in $M_{n/2}(\Gamma_g)$, and there is a simple relation due to Rallis [Ral82, §6] between the Satake parameters of G and that of $F = \vartheta_g(G)$ (see [CL19, Sect 7.1]). Concretely, if F is square integrable (e.g. cuspidal), this relation is the equality $\psi_G = \psi_F \oplus [n - 2g_0 - 1]$ for $n > 2g_0 + 1$, $\psi_F = \psi_G \oplus [2g_0 + 1 - n]$ for $n < 2g_0 + 1$. Last but not least, we have the following result, a consequence of [Ral84, Thm. I.1.1] and [MW94, Lemme I.4.11] that we learnt from [MR, §16.2].

Lemma 5.8. *Let G be an eigenform in $\mathbb{C}[X_n]$ of degree g_0. If we have $g > g_0$ and $g > n - 1 - g_0$, then $F' = \vartheta_g(G)$ is square integrable and $\psi_{F'} = \psi_G \oplus [2g - n + 1]$.*

(Note that we have $2g + 1 > g + g_0 + 1 > n$, hence the last assertion.) We finally go back to our analysis of case (H1), setting $n = 2k$. The spaces $\mathbb{C}[X_8], \mathbb{C}[X_{16}]$ and $\mathbb{C}[X_{24}]$ have respective dimension $1, 2, 24$, and the standard parameters of their eigenforms turn out to be exactly the $1, 2$ and 24 parameters ψ' discussed above for $k = 4, 8$ and 12, by [CL19, Cor. 7.2.7 & Thm. E]. This reference determines as well the degree of each eigenform (see [CL19, Thm. 9.2.6], note that most of these degrees had already been found before by Nebe and Venkov): this is the smallest integer g_0 such that $[2k - 1 - 2g_0]$ is a summand of ψ' (hence $g_0 < k$), unless we have $\psi' = \Delta_{11}[12]$ and $g_0 = 12$. For $\psi' \neq \Delta_{11}[12]$ we have thus $g > g_0$ as well as $g > 2k - 1 - g_0$ by the necessary condition $d_{\max} = 2(g - k) + 1$ of (H1). By Lemma 5.8, the automorphic representation $\pi_{F'}$ is thus the (necessarily unique) discrete automorphic representation of Sp_{2g} with parameter $\psi = [2(g - k) + 1] \oplus \psi'$, and it is not cuspidal since we have $g > g_0$. In the remaining case we have $\psi = [2(g - k) + 1] \oplus \Delta_{11}[12]$, $k = 12$ and $g_0 = 12$, and again $\pi_{F'}$ is discrete but not cuspidal if we have $g > 12$. We conclude since for $g = 12$ we recover the form found in case (I).

Case (H2). We are going to show that there are exactly two Siegel eigenforms in this remaining case, both for $k = 13$, of respective genus 16 and 24, and parameters

$$\Delta_{17}[8] \oplus [9] \oplus [7] \oplus [1] \quad \text{and} \quad [25] \oplus \Delta_{11}[12].$$

Lemma 5.9. *We have $\pi_{i_0} = 1$, k odd and g even.*

Proof. As we are in case (H2) we must have $w(\pi_{i_0}) + d_{i_0} - 1 \leq 2(k-1) \leq 24$ and $d_{i_0} = 2(g-k) + 3 \geq 3$, and so $w(\pi_{i_0}) \leq 22$. Assume π_{i_0} is non trivial. We have $\pi_{i_0} = \mathrm{Sym}^2\Delta_{11}$ by the Chenevier-Lannes theorem, so $w(\pi_{i_0}) = 22$, $d_{i_0} = 3$ and $k = 13$ is odd. This contradicts the last condition of (H2). So π_{i_0} is trivial, k is odd by the last condition in (H2), hence g is even as we are in full level Γ_g. $\square$

Write $\psi = [2(g-k) + 3] \oplus \psi'$, with $\psi' = \bigoplus_{i \neq i_0} \pi_i[d_i]$. The eigenvalues of ψ_∞ corresponding to ψ' are the $2k - 2$ integers $\pm i$ with $0 \leq j \leq k - 1$, with the even number $j = g - k + 1$ omitted (we shall call those $2k - 2$ eigenvalues the "weights" of ψ' for short). Each π_i is regular algebraic of motivic weight ≤ 24, hence in the list $\mathcal{L}_{24}$. We are now led to do a simple enumeration exercise: for every odd $k \in \{1, \ldots, 13\}$, enumerate all possible ψ', with π_i in $\mathcal{L}_{24}$ for each i, and with weights $\pm 0, \ldots, \pm(k-1)$ where the even integer $\pm(g-k+1)$ is excluded and satisfies $k - 1 \geq g - k + 1 > 1$.

Lemma 5.10. *Assume $i \in I_0$ and $\pi_i \neq 1$, then we have $\pi_i = \mathrm{Sym}^2\Delta_{11}$, $d_i = 1$, $k = 13$ and $g = 24$, as well as $I_0 = I_{\mathrm{odd}} = \{i_0, i, j\}$ with $j \neq i_0, i$, $\pi_j = 1$ and $d_j \geq 5$.*

Proof. Assume first $\pi_i = \mathrm{O}^{\mathrm{o}}_{24,16,8}$. The only π in $\mathcal{L}_{24}$ with $w(\pi) \notin \{24, 23, 17, 15\}$, and having a weight $5 \leq \lambda \leq 7$, are Δ_{11} and $\Delta_{21,13}$. It follows that among the three consecutive integers $5, 6, 7$, either 7 or 5 is not a weight of ψ', hence must be $g - k + 1$: a contradiction as $g - k + 1$ is even.

An inspection of $\mathcal{L}_{24}$ shows then $\pi_i = \mathrm{Sym}^2\Delta_{11}$, $k - 1 \geq 11$ and $d_i \leq 3$. As k is odd we have $k = 13$. Assuming $d_i = 3$, $\pi_i[d_i]$ contributes to the weights $\pm 12, \pm 11, \pm 10$ and $\pm 1, 0$ of ψ'. There is thus a unique i_1 in I with $\pi_{i_1}[d_{i_1}] = [1]$, and for $j \neq i_0, i_1, i$ the representation π_j is symplectic with motivic weight ≤ 17. This shows $2 \leq g - k + 1 < 9$. But there is an odd number of integers $2 \leq n \leq 9$ with $n \neq g - k + 1$: a contradiction. We have proved $d_i = 1$.

As 12 is an eigenvalue of ψ_∞, we have either $g - k + 1 = 12$ or there exists some $j \in I$ with $w(\pi_j) = 24$ and not having the weight 22. The only remaining possibility in this latter case is $\pi_j = \mathrm{O}^{\mathrm{e}}_{24,18,10,4}$ for some j. But there is no π in $\mathcal{L}_{24}$ with $w(\pi) \notin \{24, 23, 19, 17, 11\}$ and having a weight $3 \leq \lambda \leq 4$: a contradiction with the presence of the weight 3 or 4 of ψ'. We have proved $g - k + 1 = 12$, i.e. $g = 24$.

The weights of ψ' are thus $\pm 11, \pm 10, \pm 9, \ldots, \pm 1$ and 0 twice. Those possible ψ' are easily determined (see [CL19, Prop. 9.2.2] or [CL19, Thm. E]): there are 10 possibilities, all of them containing some $[d]$ with $d \geq 5$, except

$$\psi' = \operatorname{Sym}^2 \Delta_{11} \oplus \Delta_{11}[10] \oplus [1].$$

We exclude this case using Arthur's multiplicity formula. We have $I_{\mathrm{odd}} = \{i_0, i, j\}$ with $\pi_j = \pi_{i_0} = 1$. We have $\epsilon_\psi(s_{i_0 j}) = 1$, but by Formula (5.2.12) we have $\chi_{\rho_k(g)}(\iota(s_{i_0 j})) = -1$: a contradiction. $\qquad\square$

We are now able to conclude the proof. Assume first that $\Delta_{11}[12]$ is a summand of ψ'. We must have

$$\psi = [25] \oplus \Delta_{11}[12],$$

which trivially satisfies Arthur's multiplicity formula by (5.2.6). In this case there is thus an eigenform $F \in \mathrm{L}^2_{13}(\Gamma_{24})$ with parameter ψ; this F is necessarily cuspidal as we have $\mathrm{M}_k(\Gamma_g) = \mathrm{S}_k(\Gamma_g)$ for k odd. As we will explain in §5.4, this F is actually the form constructed by Freitag in the last section of [Fre82].

So we may assume that $\Delta_{11}[12]$ is not a summand of ψ'. Consider the double weight 0 of ψ'. An inspection of $\mathcal{L}_{24}$ shows then that there are two elements i_1, i_2 in $I_{\mathrm{odd}} - \{i_0\}$, say with $1 = d_{i_1} \leq d_{i_2}$. Better, Lemma 5.10 implies that we have $\pi_{i_2} = 1$, $d_{i_2} \geq 3$, and either $\pi_{i_1} = 1$ or $\pi_{i_1} = \operatorname{Sym}^2 \Delta_{11}$. We apply Arthur's multiplicity formula at the element $s_{i_0 i_2}$. We have $\chi_{\rho_k(g)}(\iota(s_{i_0 i_2})) = -1$ by Formula (5.2.12). This implies $\epsilon_\psi(s_{i_0 i_2}) = -1$, which is equivalent to

$$\prod_{l \in L} \epsilon(\pi_l) = -1$$

where L is the set of elements l in I_{even} such that π_l is symplectic and with $d_{i_2} < d_l < d_{i_0}$. We have $d_l \geq 4$ for $l \in L$, as d_l is even and $d_{i_2} \geq 3$, which imposes $w(\pi_l) \leq 21$. Among the 9 symplectic representations with such motivic weight, only Δ_{17} and Δ_{21} have a negative epsilon factor. As a consequence, at least one summand of ψ' is among

(5.3.2) $\Delta_{17}[4], \Delta_{17}[8], \Delta_{21}[4].$

Observe that this implies $d_{i_2} < 8$ and that such a summand always contributes the weights 9 and 10 to ψ', so that we have $k \geq 11$.

Assume first $\pi_{i_1} = \operatorname{Sym}^2 \Delta_{11}$, hence $g = 24$, $k = 13$ and $d_{i_2} \geq 5$ by Lemma 5.10. Then $\Delta_{17}[8]$ is a summand of ψ', but the weight 11 occurs in both $\Delta_{17}[8]$ and $\operatorname{Sym}^2 \Delta_{11}$, a contradiction. We have proved $\pi_{i_1} = \pi_{i_2} = 1$. The congruence (5.2.11) implies then $d_{i_2} \equiv -1 \bmod 4$, which leaves only the two cases $d_{i_2} = 3$ or $d_{i_2} = 7$ by the inequality $d_{i_2} < 8$. In the case $d_{i_2} = 7$ the only possibility is thus that ψ' contains $\Delta_{17}[8] \oplus [7] \oplus [1]$, hence is equal to the latter for weights reasons, and

$$\psi = \Delta_{17}[8] \oplus [9] \oplus [7] \oplus [1].$$

Arthur's multiplicity formula is satisfied for this ψ by Formulas (5.2.6), (5.2.9) and (5.2.12). There is thus an eigenform in $L^2_{13}(\Gamma_{16})$ with parameter ψ, necessarily cuspidal as its weight is odd.

We are left to study the case $d_{i_2} = 3$. In this case we focus on the weight 3 of ψ'. It cannot come from a summand in the list (5.3.2). It must thus come from a summand $\pi_m[d_m]$ of ψ' which does not contribute to any weight in $\{0, 1, 9, 10\}$, in particular π_m does not have any weight in $\{21/2, 10, 19/2, 9, 17/2, 3/2, 1, 0\}$. If π_m has motivic weight < 23, an inspection of $\mathcal{L}_{24}$ shows $\pi_m[d_m] = \Delta_{11}[6]$, which leads to

$$\psi = \Delta_{21}[4] \oplus \Delta_{11}[6] \oplus [5] \oplus [3] \oplus [1].$$

If π_m has motivic weight 23, then we have $d_m = 2$, $\Delta_{17}[4]$ is a summand of ψ', so $15/2$ and $13/2$ are not weights of π_m, and the only possibility is $\pi_m = \Delta_{23,7}$ and

$$\psi = \Delta_{23,7}[2] \oplus \Delta_{17}[4] \oplus \Delta_{11}[2] \oplus [5] \oplus [3] \oplus [1].$$

In both cases, ψ does not satisfy the multiplicity formula at the element s_h with $\pi_h[d_h] = \Delta_{11}[d_h]$: we have $\chi_{\rho_k(g)}(\iota(s_h)) = (-1)^{d_h/2} = -1$ by (5.2.6) and $\epsilon_\psi(s_h) = 1$. This concludes the proof of Theorem 2.

5.4. Complements: theta series constructions.

Recall that for any integer $n \equiv 0 \bmod 8$ we denote by $\mathcal{L}_n$ the set of even unimodular lattices in the standard Euclidean space $\mathbb{R}^n$, and by $X_n = O(\mathbb{R}^n)\backslash\mathcal{L}_n$ the finite set of isometry classes of such lattices.

Our first complement concerns the question of the surjectivity of the linear map $\vartheta_g : \mathbb{C}[X_{2k}] \to M_k(\Gamma_g)$ of Formula (5.3.1), also called the *Eichler basis problem*, for $k \equiv 0 \bmod 4$. This surjectivity was proved in [CL19, §1.3] in the case $g \leq k \leq 12$.

Corollary 5.11. *Assume $k = 4, 8$ or 12. Then $\vartheta_g : \mathbb{C}[X_{2k}] \to M_k(\Gamma_g)$ is surjective for all g. In particular, the Siegel operator $\Phi_g : M_k(\Gamma_g) \to M_k(\Gamma_{g-1})$ is surjective as well for all $g \geq 1$.*

Proof. We have the relations $\Phi_g \circ \vartheta_g = \vartheta_{g-1}$ and $\operatorname{Ker} \Phi_g = S_k(\Gamma_g)$. The corollary follows thus from the case $g \leq k$, and from the vanishing $S_k(\Gamma_g) = 0$ for $g > k$ and $k \leq 12$, implied by Theorem 2. $\square$

In other words, the Eichler basis problem holds for all g for those three values of k. We stress that this is *not* a general phenomenon: as was observed in [KSM04, §3][18] the map ϑ_{20} is not surjective for $k = 16$.

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[18]We are grateful to Riccardo Salvati Manni for pointing out the reference [KSM04]. This reference was also unknown to the authors of [CL19], who independently observed with similar arguments that ϑ_{14} is not surjective for $k = 16$: see [CL19, Cor. 7.3.5].

Remark 5.12. For $k = 4, 8, 12$, the surjectivity of Φ_g and the determination of $\dim S_k(\Gamma_g)$ for all g by Table 6 allow to determine $\dim M_k(\Gamma_g)$ for all g. In particular, we have $\dim M_k(\Gamma_g) = |X_{2k}|$ for $k = 4$ and $g \geq 1$, for $k = 8$ and $g \geq 4$, and for $k = 12$ and $g \geq 12$.

Our second complement concerns the concrete construction via theta series of the four weight 13 Siegel modular eigenforms F_g of respective genus $g = 8, 12, 16$ and 24 given by Table 6. Consider again the standard Euclidean space $\mathbb{R}^n$ with $n \equiv 0 \bmod 8$. For any finite-dimensional continuous representation U of the compact orthogonal group $O(\mathbb{R}^n)$ over the complex number, we denote by $M_U(\Omega_n)$ the complex vector space of $O(\mathbb{R}^n)$-equivariant functions $\mathcal{L}_n \to U$; this is a space of automorphic forms for the orthogonal group scheme Ω_n introduced after Lemma 5.7 (see also [CL19, Sect. 4.4.4]). For any integers $g, \nu \geq 1$, we denote by $H_{\nu,g,n}$ the representation of $O(\mathbb{R}^n)$ on the space of harmonic polynomials of degree ν on $M_{n,g}(\mathbb{C})$ in the sense of [Bö89, §XI]. The construction of Siegel theta series with harmonic coefficients gives rise to a linear map (see [Bö89, §XI] and [CL19, Sect. 5.4.1])

$$(5.4.1) \qquad \vartheta_{\nu,g,n} : M_{H_{\nu,g,n}}(\Omega_n) \longrightarrow S_{\frac{n}{2}+\nu}(\Gamma_g),$$

mapping any Ω_n-eigenform to a Siegel eigenform (Eichler's *commutation relations*) or to zero. The following proposition is suggested by Rallis's theory [Ral82] and the fact that the standard parameters of the four weight 13 Siegel eigenforms F_g are respectively $\Delta_{21,13}[4]\oplus[1], \Delta_{19,7}[6]\oplus[1], \Delta_{17}[8]\oplus[9]\oplus[7]\oplus[1]$ and $\Delta_{11}[12]\oplus[25]$.

Proposition 5.13.   (i) *For each g, the form F_g is in the image of $\vartheta_{1,g,24}$.*

(ii) *The form F_8 is in the image of $\vartheta_{5,8,16}$.*

Proof.   Böcherer[19] gives in [Bö89, Thm. 5] a necessary and sufficient condition for these properties to hold in terms of the order of vanishing of the standard L-function $L(s, F_g, St)$ of F_g at $s = n/2 - g$, with $n = 24$ in case (i) and $n = 16$ in case (ii). A case-by-case analysis reveals that this criterion holds true in all five cases. We refer to [CT19b] for the details of this simple, but rather tedious, verification. The only non-trivial necessary ingredient is the non-vanishing at $1/2$ of the Godement-Jacquet L-function of $\Delta_{19,7}$ and $\Delta_{21,13}$, that was proved in [CL19, Prop. 9.3.39]. $\qquad\square$

In the companion paper [CT19a], we study the maps $\vartheta_{1,g,24}$ in a much more elementary way. Note that a harmonic polynomial of weight 1 on $M_{n,g}(\mathbb{C})$ is just the datum of a g-multilinear alternating form on $\mathbb{C}^n$. For any element f in $M_{H_{1,g,24}}(\Omega_{24})$, and any Niemier lattice Λ in $\mathcal{L}_{24}$, the g-multilinear alternating form $f(\Lambda)$ is invariant under the orthogonal group $O(\Lambda)$ of Λ. This actually forces f

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[19]Important contributions to this problem have been made by Siegel, Weissauer, Kudla-Rallis (*Siegel-Weil formula*), and also by Freitag, Garrett, Piatetski-Shapiro, Rallis and Waldspurger.

to vanish outside the $O(\mathbb{R}^{24})$-orbit of the Leech lattice by [CT19a, Prop. 4.1]. A curious consequence of Proposition 5.13 is thus that for $g = 8, 12, 16, 24$, there is a nonzero, O(Leech)-invariant, g-multilinear alternating form on the Leech lattice! A computation using the O(Leech)-page of the $\mathbb{ATLAS}$ fortunately confirms this property, and reveals furthermore that there is a unique such form of to scalar, and none for the other values of $g \geq 1$. (The existence of such a form for $g = 24$ is well-known, and follows from the fact that the Leech lattice is *orientable*, which means that any element in O(Leech) has determinant 1). In other words, $M_{H_{1,g,24}}(\Omega_{24})$ has dimension 1 for $g = 8, 12, 16, 24$ (and 0 otherwise). The main result of [CT19a] is a direct proof of the non-vanishing of the map $\vartheta_{1,g,24}$ for these four values of g. The non-vanishing of $\vartheta_{1,24,24}$, hence of $S_{13}(\Gamma_{24})$, and had already been observed in the past by Freitag, in the last section of [Fre82]. The Mathieu group M_{24} and certain oriented rank g sublattices of the Leech lattice play an important role in our argument for $g < 24$. We also prove differently *loc. cit.* that the standard parameter of the line of Siegel eigenforms in the image of $\vartheta_{1,g,24}$ is the one given in Table 13. All of this fully confirms Corollary 1 and Proposition 5.13 (i), and show the following.

Corollary 5.14. *The linear map $\vartheta_{1,g,24}$ in (5.4.1) is an isomorphism for all $g \geq 1$.*

Case (ii) of Proposition 5.13 also implies the nonvanishing of $M_{H_{5,8,16}}(\Omega_{16})$. Let us simply mention that we actually have $\dim M_{H_{5,8,16}}(\Omega_{16}) = 2$ using a computation similar to that of [CL19, Cor. 9.5.13]. The space $M_{H_{5,8,16}}(\Omega_{16})$ is actually generated by two Ω_{16}-eigenforms, with respective standard parameters $\Delta_{21,13}[4]$ and $\Delta_{17}[8]$.

5.5. **Remarks on the case $g \geq 2k$.** Let k and g be non-negative integers satisfying $g \geq 2k$. In this case we have $L_k^2(\Gamma_g) = M_k(\Gamma_g)$ by [Wei83, Satz 3]. We may thus apply Arthur's endoscopic classification to study $M_k(\Gamma_g)$.

Proposition 5.15. *We have $\dim M_k(\Gamma_g) = \dim M_k(\Gamma_{2k})$ whenever $g \geq 2k$, and this dimension vanishes unless k is divisible by 4.*

Proof. For any eigenform in $L_k^2(\Gamma_g)$ with standard parameter ψ, we are in case (H1) as $g - k \geq k$, and with $\pi_{i_0} = 1$. Indeed, we must have $n_{i_0} = 1$, otherwise $\pi_{i_0}[d_{i_0}]$ would contribute an eigenvalue greater than $g - k$ to the infinitesimal character of $\rho_k(g)$, and $n_{i_0} = 1$ forces $\pi_{i_0} = 1$ as π_{i_0} has level 1. In particular k is even, i_0 is in I_{odd}, and we have $\psi = \psi' \oplus [2(g - k) + 1]$ where ψ' is such that $\psi'_{\mathbb{R}}$ is an Adams-Johnson parameter for the compact group $SO(2k)$.

By the Arthur multiplicity formula, the characters ϵ_ψ and $\chi_{\rho_k(g)}$ coincide on C_ψ. Consider the element $s \in C_\psi$ defined as follows:

$$s = \begin{cases} \prod_{i \in I_{\mathrm{even}}} s_i & \text{if } I_{\mathrm{odd}} = \{i_0\}, \\ s_{i_1 i_2} \prod_{i \in I_{\mathrm{even}}} s_i & \text{if } I_{\mathrm{odd}} = \{i_0, i_1, i_2\}. \end{cases}$$

Formulas (5.2.6) and (5.2.9) imply $\chi_{\rho_k(g)}(s) = (-1)^{k/2}$. Indeed, one simpler way to argue is to use the interpretation of the signs *loc. cit.*, given by Formula (5.2.4) together with Formula (5.2.10) in the case $I_{\mathrm{odd}} = \{i_0, i_1, i_2\}$, to deduce the equality $\chi_{\rho_k(g)}(s) = \prod_{i=1}^{k} \mathfrak{s}_{k-i} = (-1)^{k/2}$. On the other hand, we have $\epsilon_\psi(s) = \prod_{i \neq i_0} \epsilon(\pi_i)^{\mathrm{Min}(d_i, 2(g-k)+1)}$. We claim that for $i \in I$ and any integer $g' \geq 2k$ we have the equality

$$(5.5.1) \qquad \epsilon(\pi_i)^{\mathrm{Min}(d_i, 2(g'-k)+1)} = 1.$$

Indeed, we may assume π_i symplectic (otherwise $\epsilon(\pi_i) = 1$), in which case we have $n_i \geq 2$ and $n_i d_i \leq 2k$ and thus $d_i \leq k < 2(g'-k)+1$, and we conclude as d_i is even. This shows in particular $\epsilon_\psi(s) = 1$, and together with $\chi_{\rho_k(g)}(s) = (-1)^{k/2}$, proves that k is divisible by 4 if $\mathrm{M}_k(\Gamma_g)$ is nonzero.

To prove the asserted equality of dimensions, it is enough to show that the fact that the multiplicity formula holds for ψ implies that for any genus $g' \geq 2k$ it also holds for the parameter $\psi_{g'} := \psi' \oplus [2(g'-k)+1]$, still in weight k and case (H1). We may index the summands of $\psi_{g'}$ with the same set I as for ψ, with the same π_i for $i \in I$, the same i_0, and the same d_i for $i \neq i_0$. There is an obvious bijection between C_ψ and $\mathrm{C}_{\psi_{g'}}$ matching all s_i and s_{ij}. Via this bijection the characters ϵ_ψ and $\epsilon_{\psi_{g'}}$ coincide, as for all $i \neq i_0$ we have $\epsilon(\pi_i)^{\mathrm{Min}(d_i, 2(g-k)+1)} = \epsilon(\pi_i)^{\mathrm{Min}(d_i, 2(g'-k)+1)} = 1$ by (5.5.1). We conclude as the characters $\chi_{\rho_k(g)}$ and $\chi_{\rho_k(g')}$ trivially coincide as well. $\square$

Of course this proposition is coherent with the known properties of Siegel modular forms for $g > 2k$:

$$(5.5.2) \qquad \mathrm{S}_k(\Gamma_g) = 0 \text{ and } \mathrm{M}_k(\Gamma_g) = \begin{cases} \mathrm{Im}\, \vartheta_g & \text{if } k \geq 0 \text{ and } k \equiv 0 \bmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

by [Res75], [Fre75] (first equality), [Fre77] (second equality, see also [How81]).

Corollary 5.16. *Assume $g \geq 2k$.*

(1) The Siegel operator $\Phi_{g+1} : \mathrm{M}_k(\Gamma_{g+1}) \to \mathrm{M}_k(\Gamma_g)$ is bijective.

(2) If $k \equiv 0 \bmod 4$, the linear map $\vartheta_g : \mathbb{C}[\mathrm{X}_{2k}] \to \mathrm{M}_k(\Gamma_g)$ is an isomorphism .

Proof. The first equality in (5.5.2) means that Φ_{g+1} is injective. By the equality of dimensions of Proposition 5.15, this implies that Φ_{g+1} is bijective. (In the case $g > 2k$, the surjectivity of Φ_{g+1} also follows from the second equality in (5.5.2), as $\mathrm{M}_k(\Gamma_g)$ is generated by theta series). This proves the first assertion.

For the second, it is obvious that ϑ_g is injective for $g = 2k$, hence for all $g \geq 2k$ as well by the relation $\Phi_{g+1} \circ \vartheta_{g+1} = \vartheta_g$. The surjectivity of ϑ_g follows from the second equality of (5.5.2) for $g > 2k$. The surjectivity of ϑ_{2k+1}, and the surjectivity of Φ_{2k+1} proved in (1), implies the remaining surjectivity of ϑ_{2k}. $\square$

This corollary seems to be new for $g = 2k$.

6. Tables

ψ	g	$\underline{k}$
$\mathrm{Sym}^2\Delta_{11} \oplus \Delta_{11}[2]$	3	$(12, 8, 8)$
$\mathrm{Sym}^2\Delta_{11} \oplus \Delta_{15}[2]$	3	$(12, 10, 10)$
$\mathrm{O}^{\mathrm{o}}_{24,16,8}$	3	$(13, 10, 7)$
$\Delta_{19,7}[2] \oplus [1]$	4	$(11, 11, 7, 7)$
$\Delta_{21,5}[2] \oplus [1]$	4	$(12, 12, 6, 6)$
$\Delta_{21,9}[2] \oplus [1]$	4	$(12, 12, 8, 8)$
$\Delta_{21,13}[2] \oplus [1]$	4	$(12, 12, 10, 10)$
$\mathrm{O}^{\mathrm{e}}_{24,18,10,4} \oplus [1]$	4	$(13, 11, 8, 6)$
$\mathrm{O}^{\mathrm{e}}_{24,20,14,2} \oplus [1]$	4	$(13, 12, 10, 5)$
$\Delta_{23,7}[2] \oplus [1]$	4	$(13, 13, 7, 7)$
$\mathrm{Sym}^2\Delta_{11} \oplus \Delta_{15}[2] \oplus \Delta_{11}[2]$	5	$(12, 10, 10, 10, 10)$
$\mathrm{Sym}^2\Delta_{11} \oplus \Delta_{19,7}[2]$	5	$(12, 12, 12, 8, 8)$
$\mathrm{Sym}^2\Delta_{11} \oplus \Delta_{19}[2] \oplus \Delta_{11}[2]$	5	$(12, 12, 12, 10, 10)$
$\mathrm{O}^{\mathrm{o}}_{24,16,8} \oplus \Delta_{19}[2]$	5	$(13, 12, 12, 12, 9)$
$\Delta_{21}[2] \oplus \Delta_{11}[4] \oplus [1]$	6	$(12, 12, 10, 10, 10, 10)$
$\Delta_{21,5}[2] \oplus \Delta_{17}[2] \oplus [1]$	6	$(12, 12, 12, 12, 8, 8)$
$\Delta_{21,9}[2] \oplus \Delta_{17}[2] \oplus [1]$	6	$(12, 12, 12, 12, 10, 10)$
$\mathrm{O}^{\mathrm{e}}_{24,20,14,2} \oplus \Delta_{17}[2] \oplus [1]$	6	$(13, 12, 12, 12, 12, 7)$
$\Delta_{23,7}[2] \oplus \Delta_{17}[2] \oplus [1]$	6	$(13, 13, 12, 12, 9, 9)$
$\mathrm{Sym}^2\Delta_{11} \oplus \Delta_{11}[6]$	7	$(12, 10, 10, 10, 10, 10, 10)$
$\mathrm{Sym}^2\Delta_{11} \oplus \Delta_{19,7}[2] \oplus \Delta_{11}[2]$	7	$(12, 12, 12, 10, 10, 10, 10)$
$\mathrm{Sym}^2\Delta_{11} \oplus \Delta_{19,7}[2] \oplus \Delta_{15}[2]$	7	$(12, 12, 12, 12, 12, 10, 10)$
$\mathrm{O}^{\mathrm{o}}_{24,16,8} \oplus \Delta_{21,5}[2]$	7	$(13, 13, 13, 12, 9, 9, 9)$
$\Delta_{21,5}[2] \oplus \Delta_{11}[4] \oplus [1]$	8	$(12, 12, 10, 10, 10, 10, 10, 10)$
$\Delta_{19,7}[4] \oplus [1]$	8	$(12, 12, 12, 12, 10, 10, 10, 10)$
$\Delta_{21,5}[2] \oplus \Delta_{15}[4] \oplus [1]$	8	$(12, 12, 12, 12, 12, 12, 10, 10)$
$\Delta_{23,7}[2] \oplus \Delta_{15}[4] \oplus [1]$	8	$(13, 13, 12, 12, 12, 12, 11, 11)$
$\Delta_{21,5}[4] \oplus [1]$	8	$(13, 13, 13, 13, 9, 9, 9, 9)$
$\Delta_{21,9}[4] \oplus [1]$	8	$(13, 13, 13, 13, 11, 11, 11, 11)$

TABLE 5. Standard parameters ψ of the non scalar-valued cuspidal Siegel modular eigenforms of weight $\underline{k} = (k_1, \ldots, k_g)$ and genus g with $k_1 \le 13$ and $k_1 \ge k_2 \ge \cdots \ge k_g > g \ge 1$.

ψ	g	k
$\mathrm{Sym}^2\Delta_{11}$	1	12
$\Delta_{17}[2] \oplus [1]$	2	10
$\Delta_{21}[2] \oplus [1]$	2	12
$\mathrm{Sym}^2\Delta_{11} \oplus \Delta_{19}[2]$	3	12
$\Delta_{11}[4] \oplus [1]$	4	8
$\Delta_{15}[4] \oplus [1]$	4	10
$\Delta_{19}[4] \oplus [1]$	4	12
$\Delta_{21}[2] \oplus \Delta_{17}[2] \oplus [1]$	4	12
$\mathrm{Sym}^2\Delta_{11} \oplus \Delta_{17}[4]$	5	12
$\mathrm{Sym}^2\Delta_{11} \oplus \Delta_{19}[2] \oplus \Delta_{15}[2]$	5	12
$\Delta_{17}[2] \oplus \Delta_{11}[4] \oplus [1]$	6	10
$\Delta_{17}[6] \oplus [1]$	6	12
$\Delta_{21}[2] \oplus \Delta_{15}[4] \oplus [1]$	6	12
$\Delta_{21,13}[2] \oplus \Delta_{17}[2] \oplus [1]$	6	12
$\mathrm{Sym}^2\Delta_{11} \oplus \Delta_{15}[6]$	7	12
$\mathrm{Sym}^2\Delta_{11} \oplus \Delta_{17}[4] \oplus \Delta_{11}[2]$	7	12
$\mathrm{Sym}^2\Delta_{11} \oplus \Delta_{19}[2] \oplus \Delta_{15}[2] \oplus \Delta_{11}[2]$	7	12
$\Delta_{11}[8] \oplus [1]$	8	10
$\Delta_{15}[8] \oplus [1]$	8	12
$\Delta_{19}[4] \oplus \Delta_{11}[4] \oplus [1]$	8	12
$\Delta_{21}[2] \oplus \Delta_{17}[2] \oplus \Delta_{11}[4] \oplus [1]$	8	12
$\Delta_{21,9}[2] \oplus \Delta_{15}[4] \oplus [1]$	8	12
$\Delta_{21,13}[4] \oplus [1]$	8	13
$\mathrm{Sym}^2\Delta_{11} \oplus \Delta_{19}[2] \oplus \Delta_{11}[6]$	9	12
$\mathrm{Sym}^2\Delta_{11} \oplus \Delta_{19,7}[2] \oplus \Delta_{15}[2] \oplus \Delta_{11}[2]$	9	12
$\Delta_{21}[2] \oplus \Delta_{11}[8] \oplus [1]$	10	12
$\Delta_{21,5}[2] \oplus \Delta_{17}[2] \oplus \Delta_{11}[4] \oplus [1]$	10	12
$\mathrm{Sym}^2\Delta_{11} \oplus \Delta_{11}[10]$	11	12
$\Delta_{11}[12] \oplus [1]$	12	12
$\Delta_{19,7}[6] \oplus [1]$	12	13
$\Delta_{17}[8] \oplus [9] \oplus [7] \oplus [1]$	16	13
$[25] \oplus \Delta_{11}[12]$	24	13

TABLE 6. Standard parameters ψ of the scalar-valued cuspidal Siegel modular eigenforms of weight $k \leq 13$ and arbitrary genus $g \geq 1$.

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