Paris, April 11th 2019.

In this note we check that the 4 Siegel modular forms F_g of weight 13, with $g = 8, 12, 16, 24$, discovered in [CheTaï19] all satisfy Böcherer's criterion [Boc89]. Their standard parameters ψ_g are the following:

$\Delta_{21,13}[4] \oplus [1], \Delta_{19,7}[6] \oplus [1], \Delta_{17}[8] \oplus [9] \oplus [7] \oplus [1]$ and $\Delta_{11}[12] \oplus [25]$.

The weights of ψ_g are $\pm(13 - i)$, $1 \leq i \leq g$, and 0. The forms F_g may thus be in ν-harmonic theta correspondence with:

– O_{24} for $\nu = 1$ (Archimedean component $\Lambda^g\, \mathbb{R}^{24}$) and each g,

– O_{16} for $\nu = 5$ and $g = 8$.

Böcherer's criterion, Theorem 5 in [Boc89], is in term of the standard L-function of F_g. We denote by m the rank of the orthogonal group, so $m = 16$ or 24, and by $k = m/2 + \nu$ the weight, so $k = 13$ and $\nu \geq 1$. Böcherer introduces the following quantities:

$$\Gamma_g(s) = \prod_{i=0}^{g-1} \Gamma(s - i/2).$$

$$\gamma_g^k(s) = \texttt{constant} \cdot 4^{-gs} \cdot \Gamma_g(k + s - (g+1)/2)/\Gamma_g(k+s).$$

$$\omega_g(s) = \zeta(s) \prod_{i=1}^{g} \zeta(2s - 2i).$$

$$\mathrm{C}_g(s) = \prod_{i=0}^{g-1} (s + i/2).$$

$$\mathcal{C}_g(s, \nu) = \texttt{constant} \cdot \prod_{i=0}^{\nu-1} \mathrm{C}_g(-s - i).$$

Böcherer's theorem asserts that F_g is in the image of the ν-harmonic theta correspondence for O_m if, and only if, we have

$$\frac{\gamma_g^k(s)\ \mathcal{C}_g(\frac{m}{2} + s, \nu)}{\omega_g(\frac{m}{2} + 2s)}\ \mathrm{L}(m/2 - g + 2s, \mathrm{F}_g, \mathrm{St})\ \Big|_{s=0} \neq 0.$$

We will check this holds in all of our 5 cases. For each g we write

$$\Lambda^A(s, \mathrm{F}_g, \mathrm{St}) \ =\ \mathrm{L}(s, \mathrm{F}_g, \mathrm{St})\, \Gamma^A(s, \mathrm{F}_g, \mathrm{St})$$

where Γ^A is the Archimedean Γ-factor given by Godement-Jacquet and the expression à la Arthur of ψ_g. (This is neither Böcherer's nor Langlands's one, just the "obvious" one given the shape of ψ_g and the GL_m-theory : see p.3 below for their concrete definition).

CASE $g = 8$ AND $\nu = 5$.

We first consider the case $g = 8$, $m = 16$, $\nu = 5$. We have

$$\psi_8 = \Delta_{21,13}[4] \oplus [1]$$

and thus

$$\Gamma^A(s, \mathrm{F}_g, \mathrm{St}) \;=\; \Gamma_{\mathbb{R}}(s) \prod_{|i|\leq 3/2,\; i \text{ half-integer}} \Gamma_{\mathbb{C}}(s + \frac{21}{2} + i)\Gamma_{\mathbb{C}}(s + \frac{13}{2} + i).$$

– The order of vanishing at $m/2 - g = 0$ of

$$\Lambda^A(s, \mathrm{F}_g, \mathrm{St}) = \xi(s) \prod_{|i|\leq 3/2} \Lambda(s + i, \Delta_{21,13})$$

(product over half integers, ξ is the complete Riemann ζ function) is thus -1, since we have $\Lambda(\frac{1}{2}, \Delta_{21,13}) \neq 0$ (Chenevier-Lannes). The valuation of $\Gamma^A(s, \mathrm{F}_g, \mathrm{St})$ at $s = 0$ is -1, hence $\mathrm{L}(0, \mathrm{F}_g, \mathrm{St})$ is nonzero.

– The order of vanishing of $\omega_g(s)$ at $m/2 = g$ is 0 (we recall $\zeta(0) \neq 0$).

– The order of vanishing of $\gamma_g^k(s)$ at $s = 0$ is also 0 since we have $k - (g + 1)/2 > \frac{g-1}{2}$, i.e. $k > g$, and $\Gamma_g(s)$ is nonzero for $s > \frac{g-1}{2}$ real.

– Moreover, $\mathcal{C}_g(s, \nu)$ cannot vanish for $s > \frac{g-1}{2}$ real.

We have thus proved that F_8 is in the image of the 5-harmonic theta correspondence for O_{16}.

CASES WITH $\nu = 1$.

In the remaining cases we have $m = 24$ and $\nu = 1$.
Consider first the term

$$\mathcal{C}_g(m/2 + s, \nu) = 2^g \mathrm{C}_g(-\frac{m}{2} - s) = 2^g \prod_{i=0}^{g-1} (\frac{i - m}{2} - s).$$

Its order of vanishing at $s = 0$ is 1 for $0 \leq m \leq g - 1$, 0 otherwise. We have $g - 1 < 24 = m$ in all cases, hence this order is 0 in all cases.

The $\zeta(s)$ function vanishes at the order 1 at $s = -2, -4, \ldots$ and nowhere else on $2\mathbb{Z}$. It follows that the order of vanishing of $\omega_g(m/2 + 2s)$ at $s = 0$ is the number of integers $1 \leq i \leq g$ with $m - 2i < 0$, i.e. $m/2 < i \leq g$. We obtain thus $g - m/2$ for $g \geq m/2$, and 0 otherwise.

Consider $\gamma_g^k(s)$ at $s = 0$. As g is even, the order of vanishing of $\gamma_g^k(s)$ at $s = 0$ is the sum of:

– the opposite of the number of odd integers $0 \leq i < g$ with $k - \frac{g+1}{2} \leq i/2$, *i.e.* with $i \geq 2k - g - 1$. This number is zero for $k > g$.

– the number of even integers $0 \leq i < g$ with $k \leq i/2$, *i.e.* with $i \geq 2k$. This is always 0 since we have $2k > g$ in all cases.

Consider the order of vanishing at $s = m/2 - g$ of $\mathrm{L}(s, \mathrm{F}_g, \mathrm{St})$. This is boring... The order of vanishing at this point of $\Lambda^A(s, \mathrm{F}_g, \mathrm{St})$ is dealt with case by case.

– ($g = 8$) We have $m/2 - g = 4$ and $\psi_8 = \Delta_{21,13}[4] \oplus [1]$, so this order is 0 (note $4 - \frac{3}{2} > \frac{1}{2}$).
– ($g = 12$) We have $m/2 - g = 0$ and $\psi_{12} = \Delta_{19,7}[6] \oplus [1]$, so this order is -1 as we know $\Lambda(1/2, \Delta_{19,7}) \neq 0$ by Chenevier-Lannes.
– ($g = 16$) We have $m/2 - g = -4$ and $\psi_{16} = \Delta_{17}[8] \oplus [9] \oplus [7] \oplus [1]$. The contribution of $[1] \oplus [7] \oplus [9]$ to the order of vanishing is -1. The contribution of $\Delta_{17}[8]$ is 0 as $-4 + \frac{7}{2} < \frac{1}{2}$. All in all we get -1.
– ($g = 24$) We have $m/2 - g = -12$ and $\psi_{24} = \Delta_{11}[12] \oplus [25]$, so this order is -1 given the shape of ψ_{24}.

Consider now the Γ factor $\Gamma^A(s, \mathrm{F}_g, \mathrm{St})$ at $s = m/2 - g$, case by case.
– ($g = 8$) This is $\Gamma_{\mathbb{R}}(s) \prod_{|i| \leq 3/2} \Gamma_{\mathbb{C}}(s + \frac{21}{2} + i)\Gamma_{\mathbb{C}}(s + \frac{13}{2} + i)$ (over half-integers), whose order of vanishing at $s = 4$ is 0.
– ($g = 12$) This is $\Gamma_{\mathbb{R}}(s) \prod_{|i| \leq 5/2} \Gamma_{\mathbb{C}}(s + \frac{19}{2} + i)\Gamma_{\mathbb{C}}(s + \frac{7}{2} + i)$, whose order of vanishing at $s = 0$ is -1.
– ($g = 16$) This is $\Gamma_{\mathbb{R}}(s) \prod_{|i| \leq 3} \Gamma_{\mathbb{R}}(s + i) \prod_{|i| \leq 4} \Gamma_{\mathbb{R}}(s + i) \prod_{|i| \leq 7/2} \Gamma_{\mathbb{C}}(s + \frac{17}{2} + i)$, whose order of vanishing at $s = -4$ is $-1 - 3 - 5 = -9$.
– ($g = 24$) This is $\prod_{|i| \leq 12} \Gamma_{\mathbb{R}}(s + i) \prod_{|i| \leq 11/2} \Gamma_{\mathbb{C}}(s + \frac{11}{2} + i)$, whose order of vanishing at $s = -12$ is $-13 - 12 = -25$.

In the end, we have the following table:

g	$\mathcal{C}$	$1/\omega$	γ	Λ^A	$1/\Gamma^A$	`ord`
8	0	0	0	0	0	0
12	0	0	0	-1	1	0
16	0	-4	-4	-1	9	0
24	0	-12	-12	-1	25	0

The order of vanishing `ord` (sum of the 5 other terms) is thus zero in all cases: the F_g are in the image of the 1-harmonic theta correspondence for O_{24}, by Böcherer's theorem. □

Gaëtan Chenevier and Olivier Taïbi

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