THE JACQUET-LANGLANDS CORRESPONDENCE FOR $\mathrm{GL}_2(\mathbb{Q}_p)$

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Contents

1. Introduction

In these notes, all representations of groups are over the field $\mathbb{C}$ of complex numbers. Let p be a prime number, D a quaternion algebra over $\mathbb{Q}_p$. The topological group $D^\times$ is compact (even profinite) modulo its center $\mathbb{Q}_p^\times$.

Theorem 1.1 (Jacquet-Langlands)**.** *For any continuous irreducible finite-dimensional representation σ of $D^\times$, there is a unique* essentially square-integrable *irreducible smooth representation π of $\mathrm{GL}_2(\mathbb{Q}_p)$ such that for any $g \in D^\times \smallsetminus \mathbb{Q}_p^\times$ we have $\operatorname{tr}\sigma(g) = -\Theta_\pi(g')$, where $g' \in \mathrm{GL}_2(\mathbb{Q}_p)$ has the same trace and determinant as g and Θ_π is the Harish-Chandra character of π. Moreover any π corresponds to a unique σ.*

We will explain later what "essentially square-integrable" means. Let us simply mention that all these representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ have infinite dimension. In fact we will classify representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ as follows: principal series (quite explicit), "special" (also quite explicit, and essentially square-integrable), and supercuspidal (also essentially square-integrable). We will define the Harish-Chandra character Θ_π even later, it plays the role of the trace function of π, but since we are considering infinite-dimensional representations, defining the trace is not obvious.

We will see that supercuspidal representations are the most well-behaved representations of $\mathrm{GL}_2(\mathbb{Q}_p)$, i.e. that they behave much like representations of a compact group. Among irreducible smooth representations of $\mathrm{GL}_2(\mathbb{Q})$, they are however the least explicit and most mysterious ones. One can see the Jacquet-Langlands correspondence as a classification of all supercuspidal representations π of $\mathrm{GL}_2(\mathbb{Q}_p)$ by seemingly simpler finite-dimensional representations of $D^\times$.

However, this is not the true motivation for this theorem. The theorem should be seen as a consequence of the Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ and $D^\times$. The following theorem is the most difficult part of this correspondence (we will prove the easier part concerning reducible Galois representations, in the first part of this course).

Theorem 1.2 (Jacquet-Langlands, Gelfand-Graev, Tunnell, Kutzko)**.** *There is a "natural" bijection between isomorphism classes of irreducible representations of $D^\times$ of dimension > 1 (resp. irreducible supercuspidal representations of $\mathrm{GL}_2(\mathbb{Q}_p)$) and having central character $\mathbb{Q}_p^\times \to \mathbb{C}^\times$ of finite order, and isomorphism classes of continuous irreducible 2-dimensional representations of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$.*

Finite-dimensional continuous representations of $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ *over* $\mathbb{C}$ factor through $\mathrm{Gal}(F/\mathbb{Q}_p)$ for some finite Galois extension $F/\mathbb{Q}$, so the Galois representations occurring above are also relatively concrete objects. In fact for $p > 2$ it is not too difficult to explicitly classify all such Galois representations (essentially because the restriction to the wild ramification subgroup, which is a p-group, cannot be irreducible). Characterizing the correspondences (i.e. giving "natural" a precise meaning) is not straightforward: one has to introduce invariants on both sides (L-functions and ϵ factors), so the Langlands correspondence is not as obviously natural as the Jacquet-Langlands correspondence.

There is an analogous Langlands correspondence for representations of $D^\times$. Although it is probably possible to deduce the Jacquet-Langlands correspondence from the Langlands correspondences for both $\mathrm{GL}_2(\mathbb{Q}_p)$ and $D^\times$ after proving the latter, this is not the path that we will follow. The Jacquet-Langlands correspondence generalizes to smooth irreducible representations of $\mathrm{GL}_n(F)$ (F a local non-Archimedean field) and n-dimensional "Galois representations". One of the two known strategies to prove the local Langlands correspondence for $\mathrm{GL}_n(F)$ for $F/\mathbb{Q}$ ([HT01], later simplified in [Sch13]; see [Hen00] for a different proof) is *global* ("Ihara-Langlands-Kottwitz method", which is also the main method to find ℓ-adic representations of the absolute Galois group of a number field attached to an automorphic representation in the étale cohomology of a Shimura variety) and uses the Jacquet-Langlands correspondence as an input. An essential global ingredient that occurs in the proof of the Jacquet-Langlands correspondence ([JL70], [DKV84]) and in the Ihara-Langlands-Kottwitz method is the *Arthur-Selberg trace formula*. Following [JL70] and [DKV84], the goal of this course is to prove the Jacquet-Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ using the *simple* trace formula.

More generally, one can (try to) formulate local and global Langlands correspondences for arbitrary connected reductive groups G over local or global fields (conjectural in general). On the Galois side, these involve "Galois representations" taking values in the Langlands dual group ${}^LG$ (for split G, this is a *complex* reductive group whose Dynkin diagram is dual to that of G, see [Bor79] for a proper definition).

Assuming these conjectures, whenever we have two connected reductive groups G and H and a morphism ${}^LH \to {}^LG$, we have a relation between representations (automorphic in the global setting) of H and G. In many cases, this relation ("Langlands functoriality", although by no means functorial in the categorical sense!) can be formulated without referring to Galois representations, and in some cases it can even be proved. Some cases of Langlands functoriality can be proved using (some version of) the Arthur-Selberg trace formula. Such results are needed to construct Galois representations corresponding to automorphic representations.

Rough plan of the course:

(1) Basic representation theory of $\mathrm{GL}_2(\mathbb{Q}_p)$ and "classification" of representations,

(2) A bit of harmonic analysis for $\mathrm{GL}_2(\mathbb{Q}_p)$,

(3) Trace formulas,

(4) Application to the Jacquet-Langlands correspondence.

2. Smooth representations of $\mathrm{GL}_2(\mathbb{Q}_p)$

We begin the study of smooth representations of the locally profinite topological group $G = \mathrm{GL}_2(\mathbb{Q}_p)$. Many tools work just as well for general reductive groups, adding the (non-trivial) combinatorics of Weyl groups etc. References: [Cas], [Ren10].

2.1. **Decompositions.** Let $K_0 = \mathrm{GL}_2(\mathbb{Z}_p)$. Note that this is exactly the set of $g \in G$ such that $g(\mathbb{Z}_p^2) = \mathbb{Z}_p^2$. Recall that a *lattice* in $\mathbb{Q}_p^2$ is a finitely generated sub-$\mathbb{Z}_p$-module of rank 2 (a finite sub-$\mathbb{Z}_p$-module of $\mathbb{Q}_p^2$ is torsion-free so if it is finitely generated then it is free of finite rank). It is very easy to check that a family of vectors in $\mathbb{Q}_p^2$ is linearly independent over $\mathbb{Q}_p$ if and only if it is linearly independent over $\mathbb{Z}_p$. Because G acts transitively on the set of bases in $\mathbb{Q}_p^2$, it also acts transitively on the set of lattices in $\mathbb{Q}_p^2$. We obtain an identification of G/K_0 with the set of lattices in $\mathbb{Q}_p^2$. Note that the quotient topology on G/K_0 is the discrete topology because K_0 is open in G.

Lemma 2.1. *Let K be a compact subgroup of G. There exists $g \in G$ such that $K \subset gK_0g^{-1}$.*

Proof. The group G acts transitively on the set of lattices in $\mathbb{Q}_p^2$ (defined as sub-$\mathbb{Z}_p$-modules of finite type and maximal rank, that is rank 2). The statement is equivalent to the existence of a lattice $L \subset \mathbb{Q}_p^2$ such that for any $k \in K$, $k(L) = L$. Because K is compact, the image of $\det : K \to \mathbb{Q}_p^\times$ is compact, and so for any $k \in K$ such that $k(L) \subset L$ we actually have $k(L) = L$. So we have to show that there is a lattice L stable under K. Let L_0 be any lattice, for example $\mathbb{Z}_p^2$. There is an open subgroup $K' \subset K$ such that L_0 is stable under L_0 (if $L_0 = \mathbb{Z}_p^2$ we can take $K' = K \cap K_0$). Let $L = \sum_{g \in K/K'} g(L_0)$. It works! $\square$

Denote by N be the subgroup

$$\left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| x \in \mathbb{Q}_p \right\}$$

of $\mathrm{GL}_2(\mathbb{Q}_p)$. Denote by T the subgroup of diagonal matrices in G, and $B = TN \simeq T \ltimes N$ the Borel subgroup of upper triangular matrices.

Lemma 2.2. *We have the Iwasawa decomposition $G = BK_0$.*

Proof. We use the same interpretation of G/K_0 as in the previous lemma, namely as the set of lattices in $\mathbb{Z}_p^2$: the coset gK_0 is identified to the lattice in $\mathbb{Q}_p^2$ admitting the columns of g as a basis. The lemma is equivalent to the claim that for any lattice $L \subset \mathbb{Q}_p^2$, there is a basis (e_1, e_2) of L such that the second coordinate of e_1 is zero. Denote by (f_1, f_2) the standard basis of $\mathbb{Q}_p^2$. Pick a basis e_1 of $L \cap \mathbb{Q}_p f_1$. Pick a basis $\overline{e_2}$ of the lattice $L/(L \cap \mathbb{Q}_p f_1)$ in $\mathbb{Q}_p^2/\mathbb{Q}_p f_1 \simeq \mathbb{Q}_p f_2$. Let $e_2 \in L$ be any preimage of e_2. It works! $\square$

Note that the multiplication map $B \times K_0 \to G$ is not injective, since $B \cap K_0$ is an open (in particular non-trivial) subgroup of B.

Theorem 2.3. *We have the Cartan decomposition*

$$G = \bigsqcup_{\substack{a,b \in \mathbb{Z} \\ a \geq b}} K_0 \mathrm{diag}(p^a, p^b) K_0.$$

Proof. The group G acts transitively on G/K_0, so we have an identification

$$G \backslash (G/K_0 \times G/K_0) \simeq K_0 \backslash G/K_0,$$

and so we interpret $K_0 \backslash G/K_0$ as the set of orbits of G acting on the set of pairs of lattices in $\mathbb{Q}_p^2$. With this identification in mind the theorem is easily deduced from the following statement: for any lattices $L_1 = g_1(\mathbb{Z}_p^2)$ and $L_2 = g_2(\mathbb{Z}_p^2)$ in $\mathbb{Q}_p^2$, there is a unique pair of integers $a \geq b$ such that there exists a basis (e, f) of L_1 (the columns of $g_1 k_1$ for $k_1 \in K_0$) such that $(p^a e, p^b f)$ is a basis of L_2 (the columns of $g_2 k_2$ for some $k_2 \in K_0$). The "relative position" of L_1 and L_2 is given by the double coset $K_0 g_1^{-1} g_2 K_0$. This statement on lattices is a particular case of the structure theorem of finitely generated modules over principal ideal domains (applied to $L_2/p^{N+1}L_1$ where $N \in \mathbb{Z}$ is large enough so that $p^N L_1 \subset L_2$). □

Let $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in G$.

Lemma 2.4. *We have the Bruhat decomposition $G = B \sqcup BwN$, where the natural map $B \times N \to BwN$ is an isomorphism of algebraic varieties.*

Proof. We have

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b & -a \\ c & 0 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b & bd - a \\ c & cd \end{pmatrix}$$

and solving the equation is easy. □

It will sometimes be more convenient to translate the Bruhat decomposition on the right by w^{-1} so that the "big open cell", which is the complement of a single coset in $B\backslash G$, contains $1 \in G$. Let $\overline{N}$ be the subgroup $\begin{pmatrix} 1 & 0 \\ \mathbb{Q}_p & 1 \end{pmatrix}$ of G, so that $\overline{N} = wNw^{-1}$. Then we have $G = B\overline{N} \sqcup Bw^{-1}$.

These three decompositions generalize to general linear groups of arbitrary dimension (only the Bruhat decomposition requires a more clever proof), and even to connected reductive groups over $\mathbb{Q}_p$, but the choice of K_0 is delicate if the group is not reductive over $\mathbb{Z}_p$ and in general there is more than one conjugacy class of maximal compact subgroups.

2.2. Smooth representations of G.

Definition 2.5. *A smooth representation of G is a complex vector space V together with a group representation $\pi : G \to \mathrm{GL}(V)$ such that for any $v \in V$, the stabilizer of v in G is an open subgroup. Denote $\mathrm{Rep}(G)$ the category of smooth representations of G.*

A smooth representation (V, π) is admissible *if for any open subgroup K of G, the space V^K of K-invariants has finite dimension.*

The contragredient $(\widetilde{V}, \widetilde{\pi})$ of a smooth representation (V, π) of G is the space of linear forms $\widetilde{v} : V \to \mathbb{C}$ invariant under some open subgroup of G (which typically depends on $\widetilde{v}$), i.e. the space of smooth vectors in the dual representation in the algebraic sense. We denote the pairing between $\widetilde{v} \in \widetilde{V}$ and $v \in V$ by $\langle v, \widetilde{v} \rangle$.

The same definitions can be made for any closed subgroup H of G, and we similarly denote by $\mathrm{Rep}(H)$ the category of smooth representations of H.

By definition, we have $\langle \pi(g)v, \widetilde{\pi}(g)\widetilde{v} \rangle = \langle v, \widetilde{v} \rangle$ for any $v \in V$, $\widetilde{v} \in \widetilde{V}$ and $g \in G$.

A typical smooth representation of G has infinite dimension. As the example $g \mapsto \begin{pmatrix} 1 & \log|\det(g)| \\ 0 & 1 \end{pmatrix}$ shows, even finite-dimensional representations of G may fail to be semisimple. Later we shall see less obvious examples.

We do however have Schur's lemma for irreducible smooth representations of G.

Lemma 2.6. *Let (V, π) be an* irreducible *smooth representation of G, and let $\phi \in \mathrm{Hom}_G(V, V)$ be an endomorphism. Then ϕ is multiplication by a scalar.*

Proof. We claim first that the complex vector space $\mathrm{Hom}_G(V, V)$ has countable dimension. Let $v \in V \setminus \{0\}$, and let K be a compact open subgroup of G such that $v \in V^K$. Then $V = \sum_{g \in G/K} \mathbb{C}\pi(g)v$. The Cartan decomposition implies that G/K is countable, so that V has countable dimension, and any G-equivariant map $V \to V$ is determined by the image of V so $\mathrm{End}_G(V)$ also has countable dimension.

Now we claim that there exists $\lambda \in C$ such that $\varphi - \lambda \mathrm{Id}_V \notin \mathrm{GL}(V)$. Otherwise we would obtain a morphism of algebras $\mathbb{C}(X) \to \mathrm{End}_G(V)$, which would be injective since the source is a field, and since $\mathbb{C}(X)$ does not have countable dimension over $\mathbb{C}$ (e.g. the vectors $((X - \lambda)^{-1})_{\lambda \in \mathbb{C}}$ are linearly independent) this gives a contradiction.

So for some $\lambda \in \mathbb{C}$ we have $\ker(\phi - \lambda) \neq 0$ or $\mathrm{im}(\phi - \lambda) \neq V$, and $\ker(\phi - \lambda)$ and $\mathrm{im}(\phi - \lambda)$ are subrepresentations of G. By irreducibility of V, in the first case we have $\ker(\phi - \lambda) = V$ and in the second case we have $\mathrm{im}(\phi - \lambda) = 0$. ☐

There is an easier proof under the assumption that V is admissible. Later we will show that in fact any irreducible smooth representation of G is admissible.

Let $Z = \{\mathrm{diag}(x, x) | x \in \mathbb{Q}_p^\times\}$ be the center of G.

Corollary 2.7. *If (V, π) is an irreducible smooth representation of G then there exists a unique smooth character $\omega_\pi : Z \to \mathbb{C}^\times$ such that for any $z \in Z$ we have $\pi(z) = \omega_\pi(z)\mathrm{Id}_V$.*

Let (V, π) be a smooth representation of G. Let K be any compact open subgroup of G. Recall that any compact open subgroup of G has an open subgroup which is contained and distinguished in K. For τ an irreducible representation of K factoring through K/K' for some distinguished open subgroup K' of K (note that K/K' is a finite group and so τ has finite dimension), denote $V_\tau = \mathrm{Hom}_K(\tau, V) \otimes_{\mathbb{C}} \tau$. Using just representation theory of finite groups we see that we have a canonical isomorphism $\bigoplus_\tau V_\tau \simeq V$ of representations of K, where the sum is over isomorphism classes of irreducible representations τ of K factoring through K/K' as above. (Exercise: any continuous irreducible finite-dimensional representation of a profinite topological group factors through a quotient by a distinguished open subgroup. We will not need this fact, in fact we will not need the Peter-Weyl theorem or any result concerning *continuous* representations of profinite groups.)

For an open distinguished subgroup K' of K, the finite group K/K' has only finitely many isomorphism classes of irreducible representations. We deduce that as a representation of K, $\widetilde{V}$ is canonically identified with $\bigoplus_\tau \mathrm{Hom}_{\mathbb{C}}(V_\tau, \mathbb{C})$. The following lemma then follows easily (exercise).

Lemma 2.8. *Let (V, π) be a smooth representation of G. Let K be a compact open subgroup of G. The following are equivalent*

(1) *For any irreducible representation τ of K factoring through K/K' for some distinguished open subgroup of K we have* $\dim_{\mathbb{C}} \mathrm{Hom}_K(\tau, V) < \infty$ *(equivalently, V_τ has finite dimension).*

(2) *V is admissible,*

(3) *$\widetilde{V}$ is admissible,*

(4) *the natural (always injective) map $V \to \widetilde{\widetilde{V}}$ is surjective.*

We fix (until further notice) a left Haar measure on G. Recall that it is unique up to multiplication by $\mathbb{R}_{>0}$. Recall the Riesz-Markov-Kakutani representation theorem: for a locally compact Hausdorff topological space X, Radon measures correspond bijectively to positive linear functionals on $C_c(X)$, the space of continuous compactly supported functions on X. With this formulation of measure theory, in the case of a locally profinite topological space such as G, one can construct the Haar measure concretely as follows (see [Bou63, Chapitre 7, §1.6] for details in a more general setting):

- Choose $\mathrm{vol}(K_0) \in \mathbb{R}_{>0}$ arbitrarily [1]
- For any open subgroup $K \subset K_0$, define $\mathrm{vol}(K) = |K_0/K|^{-1} \mathrm{vol}(K_0)$.
- For any $f \in C_c^\infty(G)$, the space of smooth (this means locally constant) compactly supported functions on G, choose K as above such that f is right K-invariant, and let $\int_G f = \mathrm{vol}(K) \sum_{gK \in G/K} f(g)$. Exercise: this does not depend on the choice of K!
- Extend $\int_G$ to $C_c(G)$ by continuity (approximate any continuous compactly supported function by smooth ones).

This concrete definition can also be used to check that G is unimodular, i.e. this left Haar measure is also right-invariant [2]. Later we will give an explicit "differential" definition of the Haar measure on G, and this will show that G is unimodular.

One can define Haar measures on the closed (also locally profinite) subgroups B, T, N in the same way. The groups T and N are commutative and so also unimodular, but B is not unimodular.

Exercise 2.9. *Let dt and dn be Haar measures on T and N.*

(1) *There exists a unique left Haar measure db on B such that for any $f \in C_c(B)$ we have*

$$\int_B f(b)db = \int_{T \times N} f(tn)dtdn.$$

[1] choosing it in $\mathbb{Q}_{>0}$ allows one to extend many results to fields of coefficients having characteristic zero but with no naturally embedding in $\mathbb{C}$.

[2] Exercise: prove it. Hint: using the Cartan decomposition we are reduced to computing the modulus character at $\mathrm{diag}(p^a, p^b)$, which can be done by considering the subgroup $K_i = \begin{pmatrix} 1 + p^i\mathbb{Z}_p & p^i\mathbb{Z}_p \\ p^i\mathbb{Z}_p & 1 + p^i\mathbb{Z}_p \end{pmatrix}$ of K_0 for large enough i.

(2) *For* $t = \mathrm{diag}(t_1, t_2) \in T$ *and* $f \in C_c(N)$ *we have*

$$\int_N f(t^{-1}nt)dn = |t_1/t_2| \int_N f(n)dn \tag{2.1}$$

where $|x| = p^{-v_p(x)}$.

(3) *For any* $f \in C_c(B)$ *and* $a \in B$ *we have*

$$\int_B f(ba)db = \delta_B(a) \int_B f(b)db$$

where $\delta_B \begin{pmatrix} x & u \\ 0 & y \end{pmatrix} = |x/y|$. *This implies that* $\delta_B(b)db$ *is a* right *Haar measure on* B.

By default Haar measures will be *left* Haar measures in these notes.

Let $\mathcal{H}(G)$ be the space of smooth (this means locally constant in this setting) compactly supported functions $f : G \to \mathbb{C}$. Denote $\mathcal{H}(G) = C_c^\infty(G)$. This notation emphasizes the fact that we think of it as a Hecke algebra: $\mathcal{H}(G)$ is a convolution algebra, for the convolution product defined by the formula

$$(f * f')(x) = \int_G f(y)f'(y^{-1}x)dy = \int_G f(xy)f'(y^{-1})dy.$$

Checking that this product is associative is left as an exercise. This algebra is not commutative and has no unit (that would be a Dirac distribution ...), but it has lots of idempotents: for any compact open subgroup K of G, $e_K := \mathrm{vol}(K)^{-1}1_K$ is an idempotent. Similarly, it is easy to define an idempotent $e_{K,\tau}$ for any finite-dimensional irreducible representation τ of K/K' where K' is an open distinguished subgroup of K, but we shall not need this. Note that for any idempotent $e \in \mathcal{H}(G)$ we have a unital subalgebra $e\mathcal{H}(G)e$ of $\mathcal{H}(G)$. We denote $\mathcal{H}(G, K) = e_K\mathcal{H}(G)e_K$, which is identified with the space of functions $K\backslash G/K \to C$ having finite support.

If (V, π) is a smooth representation then $\mathcal{H}(G)$ acts on V by the formula

$$\pi(f)v := \int_G f(g)\pi(g)v \; dg$$

Note that there is no analysis and very little measure theory involved in this action: choose a compact open subgroup K of G such that $v \in V^K$ and f is right K-invariant, then we have $\pi(f)v = \sum_{g \in G/K} \mathrm{vol}(K)f(g)\pi(g)v$ where only finitely many terms are non-zero since f has compact support. In fact this could be taken as the definition, and one can check directly that this expression does not depend on the choice of K. For any compact open subgroup K of G we have $e_KV = V^K$ (this is clear using the discussion above on the structure of V as a representation of K).

Conversely, if V is an $\mathcal{H}(G)$-module such that $\sum_K e_KV = V$, one can endow V with a smooth action of G: for $v \in V^K$ and $g \in G$ let $\pi(g)v = \mathrm{vol}(K)^{-1}1_{gK}v$ (exercise: this does not depend on K and defines a group action). Thus smooth representations of G are equivalent to smooth $\mathcal{H}(G)$-modules.

Lemma 2.10. *Let K be a compact open subgroup of G. The functor $V \rightsquigarrow V^K = \pi(e_K)V$, from smooth representations of G to representations of $\mathcal{H}(G,K)$, induces a bijection between irreducible smooth representations V of G such that $V^K \neq 0$ and simple $\mathcal{H}(G,K)$-modules.*

Proof.

(1) First we check that if (V,π) is an irreducible smooth representation of G then V^K is a simple $\mathcal{H}(G,K)$-module. Let $M \subset V^K$ be a non-zero sub-$\mathcal{H}(G,K)$-module. By irreducibility we have $V = \sum_{g\in G} \pi(g)M = \mathcal{H}(G)M$ and so we have
$$V^K = (\mathcal{H}(G)M)^K = e_K\mathcal{H}(G)M = e_K\mathcal{H}(G)e_KM = \mathcal{H}(G,K)M = M.$$

(2) Let M be a $\mathcal{H}(G,K)$-module, consider the $\mathcal{H}(G)$-module $F_0(M) = \mathcal{H}(G)\otimes_{\mathcal{H}(G,K)} M$. From the smoothness of $\mathcal{H}(G)$ as a representation of G (on either side) we deduce that $F_0(M)$ is also a smooth representation of G. In general this smooth representation of G is too big (i.e. it happens that for some simple $\mathcal{H}(G,K)$-modules M, $F_0(M)$ is not irreducible). Consider the linear map $\phi_M : F_0(M) \to M$, $f\otimes m \mapsto e_Kfe_Km$, which restricts to a morphism of $\mathcal{H}(G,K)$-modules $F_0(M)^K \to M$ which is surjective because $1\otimes m$ maps to m. Let $W(M) = \{v \in F_0(M) | e_K\mathcal{H}(G)v = 0\}$, that is the largest sub-$\mathcal{H}(G)$-module W of $F_0(M)$ such that we have $e_KW = 0$. We have $W(M) \subset \ker\phi_M$ and $e_KW(M) = 0$, and because $F_0(M)$ is a semisimple representation of K the natural map $F_0(M)^K \to (F_0(M)/W(M))^K$ is an isomorphism. So we have a functor $F : M \rightsquigarrow F_0(M)/W(M)$ from the category of $\mathcal{H}(G,K)$-module to $\mathrm{Rep}(G)$ and a natural transformation $\phi_? : F(?)^K \to ?$. The map $\phi_M : F(M)^K \to M$ is clearly surjective, and it is also injective:
$$(\mathcal{H}(G)\otimes_{\mathcal{H}(G,K)} M)^K = (e_K\mathcal{H}(G))\otimes_{\mathcal{H}(G,K)} M = e_K\mathcal{H}(G)e_K\otimes_{\mathcal{H}(G,K)} M \simeq M.$$

(3) Now assume that M is a simple $\mathcal{H}(G,K)$-module. We claim that $F(M)$ is an irreducible representation of G. With notation as above, let V_1 be a subrepresentation of $F_0(M)$ such that $W(M) \subsetneq V_1$. By definition of $W(M)$ we have $V_1^K \neq 0$ so that $\phi_M(V_1^K) \neq 0$. Since M is simple this implies $\phi_M(V_1^K) = M$, and so V_1 contains $\mathcal{H}(G)(1\otimes M) = F_0(M)$.

(4) The outstanding claim that we have to check is that for V an irreducible smooth representation of G such that $V^K \neq 0$, we have a (natural) isomorphism $F(V^K) \simeq V$. There is a natural morphism $\mathcal{H}(G)\otimes_{\mathcal{H}(G,K)} V^K \to V$, which is surjective since $V^K \neq 0$ and V is irreducible. It clearly factors through $F(V^K)$, and since we have shown that $F(V^K)$ is irreducible we get that the natural morphism $F(V^K) \to V$ is an isomorphism.

□

This lemma suggests that we restrict our study to admissible representations, so that we can study only finite-dimensional objects. Unfortunately, beyond a few cases (small index in K_0) it is difficult to describe $\mathcal{H}(G,K)$ in a useful manner, so this point of view is rather limited. We will come back to $\mathcal{H}(G,K)$ in special cases later.

2.3. **Parabolic induction and the Jacquet functor.** The easiest way to construct representations of G is to induce representations from smaller subgroups. For these to be "not too big" it is natural to induce from the cocompact (by the Iwasawa decomposition) subgroup B. We will be even more specific and induce only representations of B which are trivial on its distinguished subgroup N. Non-trivial characters of N are also very interesting, but this is another subject (Whittaker functionals).

Definition 2.11. *Let (W, μ) be a smooth representation of T, that we see as a representation of B via the projection $B \to B/N \simeq T$. Let $\mathrm{Ind}_B^G W$ (or $\mathrm{Ind}_B^G \mu$) be the space of smooth functions $f : G \to W$ satisfying*

$$f(bg) = \delta_B(b)^{1/2}\mu(b)f(g)$$

for all $b \in B$ and $g \in G$. It is endowed with a smooth action of G defined by

$$(x \cdot f)(g) = f(gx)$$

for $x, g \in G$.

If W has dimension one then $\mu = \mu_1 \otimes \mu_2$ is a smooth character of T, that is $\mu(\mathrm{diag}(x_1, x_2)) = \mu_1(x_1)\mu_2(x_2)$ where $\mu_i : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ is a smooth character. Note that $\mathrm{Ind}_B^G \mu$ admits $\mu_1\mu_2$ as a central character, using the obvious identification $\mathbb{Q}_p^\times \simeq Z$. Note also that for a smooth character $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ we have a natural isomorphism $\mathrm{Ind}_B^G(\chi\mu_1 \otimes \chi\mu_2) \simeq \mathrm{Ind}_B^G(\mu_1 \otimes \mu_2) \otimes (\chi \circ \det)$.

Lemma 2.12. *Let μ be a smooth character of T. The representation $\mathrm{Ind}_B^G \mu$ of G is admissible.*

Proof. Let K be a compact open subgroup of G. We may assume that $K \subset K_0$. Since $G = BK_0$ and K_0/K is finite, the set $B\backslash G/K$ is also finite, showing that $\dim_{\mathbb{C}}(\mathrm{Ind}_B^G \mu)^K < +\infty$. □

For $W \neq 0$ one can see that $\mathrm{Ind}_B^G W$ has infinite dimension by producing functions as follows. Pick $x \in G$ and $v \in W \smallsetminus \{0\}$. There exists a compact open subgroup K of G such that w is fixed by $xKx^{-1} \cap B$. Then there is a unique $f \in \mathrm{Ind}_B^G W$ supported on BxK such that $f(xk) = 1$ for $k \in K$.

The character $\delta_B^{1/2} : \mathrm{diag}(x_1, x_2) \mapsto |x_1/x_2|^{1/2}$ was introduced so that Ind_B^G preserves unitarity. Let us briefly explain this (our goal is Corollary 2.15 below). If μ is unitary character, in which case we may assume $W = \mathbb{C}$, for $f \in \mathrm{Ind}_B^G \mu$ the function $|f|^2 : G \to \mathbb{C}$ belongs to $C^\infty(G, B, \delta_B) = \mathrm{Ind}_B^G \delta_B^{1/2}$, the space of smooth functions $G \to \mathbb{C}$ such that $f(bg) = \delta_B(b)f(g)$ for all $b \in B$ and $g \in G$. We will construct a G-invariant "integration map" $C^\infty(G, B, \delta_B) \to \mathbb{C}$. In fact this is a special case of Example A.5 (which deals more generally with continuous functions, not just locally constant functions), but we can be more concrete and give an alternative argument, proving a formula that will turn out to be useful on several occasions. We have a map

$$\begin{aligned} \psi : C_c^\infty(G) &\longrightarrow C^\infty(G, B, \delta_B) \\ f &\longmapsto \left(g \mapsto \int_B f(bg)db\right). \end{aligned}$$

One can show that it is surjective [3]. We want to define an "integration map" $C^\infty(G, B, \delta_B) \to \mathbb{C}$ mapping $\psi(f)$ to $\int_G f(g)dg$, so we have to check that $\int_G f(g)dg = 0$ if $\psi(f) = 0$. This follows from the following integration formula for the Iwasawa decomposition, which has the benefit of giving a simple explicit formula for the sought-after linear form on $C^\infty(G, B, \delta_B)$.

Lemma 2.13. *Choose Haar measures on the unimodular groups G, T, N and K_0 so that $\mathrm{vol}_G(K_0) = \mathrm{vol}_B(B \cap K_0) \mathrm{vol}_{K_0}(K_0)$. Let f be either an integrable function $G \to \mathbb{C}$ or a measurable function $G \to \mathbb{R}_{\geq 0}$. Then we have*

$$\int_G f(g)\, dg = \int_{T\times N\times K_0} f(tnk)\, dt\, dn\, dk = \int_{N\times T\times K_0} f(ntk)\delta_B^{-1}(t)\, dn\, dt\, dk.$$

Proof. It is enough to prove the formulas for $f \in C_c(G)$, by density of $C_c(G)$ in $L^1(G)$ for the first case, and monotone convergence for the second case. Also note that via the homeomorphism $T \times N \simeq B$, $(t, n) \mapsto tn$, the product of Haar measures on T and N is a *left* Haar measure on B. So the second equality has nothing to do with G or K_0 and is a consequence of the definition of the modulus character for B since $dn\, dt$ is a right Haar measure on $B = NT$. So it is enough to prove the formula $\int_G f(g)\, dg = \int_{B\times K_0} f(bk)\, db\, dk$ where a left Haar measure on B is used.

Consider the map

$$\begin{aligned} \phi : C_c(B \times K_0) &\longrightarrow C_c(G) \\ F &\longmapsto \left(bk \mapsto \int_{B\cap K_0} F(bh, k^{-1}h)\, dh \right) \end{aligned}$$

where we have written an arbitrary element of G as bk for $b \in B$ and $k \in K_0$ using the Iwasawa decomposition. It is surjective because $B \cap K_0$ is compact, in fact it has a natural section onto the subspace of $C_c(B \times K_0)$ consisting of the functions F factoring through $(b, k) \mapsto bk^{-1}$. Moreover $F \mapsto \int_G \phi(F)(g)\, dg$ is left $B \times K_0$-invariant because the Haar measure on G is invariant under left multiplication by B and right multiplication by K_0, so it coincides with $F \mapsto \int_{B\times K_0} F(b, k)\, db\, dk$ up to a scalar. The scalar is computed by taking f to be the characteristic function of K_0. □

Corollary 2.14. *The linear map*

$$\begin{aligned} C^\infty(G, B, \delta_B) &\longrightarrow \mathbb{C} \\ f &\longmapsto \int_{K_0} f(k)dk \end{aligned}$$

is G-invariant (for the action by right translation on the source and the trivial action on the target).

Proof. Any $f \in C^\infty(G, B, \delta_B)$ can be written as $g \mapsto \int_B \alpha(bg)db$ for some $\alpha \in C_c^\infty(G)$, so

$$\int_{K_0} f(k)\, dk = \int_{K_0} \int_B \alpha(bk)\, db\, dk = \int_G \alpha(g)\, dg$$

[3] Exercise: check this, either by adapting the above argument exhibiting non-zero functions in $\mathrm{Ind}_B^G \mu$, or using right translates of the Bruhat decomposition.

and $(g \cdot f)(x) = \int_B \alpha(bxg)db$, so the assertion follows from the fact that the Haar measure on G is right G-invariant. $\square$

Corollary 2.15. *If μ is a unitary character then $\mathrm{Ind}_B^G \mu$ has a natural G-invariant Hermitian inner product, defined by $\|f\|^2 = \int_{K_0} |f(k)|^2\, dk$.*

Corollary 2.14 also shows that $\int_{K_0}$ gives an intertwining operator $\mathrm{Ind}_B^G \delta_B^{1/2} \to \mathbb{C}$ where the target is endowed with the trivial representation of G. It is easy to see that this linear form is non-zero: as explained above the source contains a positive function.

Definition 2.16. *The Steinberg representation* St *of G is the subrepresentation of $\mathrm{Ind}_B^G \delta_B^{1/2}$ consisting of all functions f satisfying $\int_{K_0} f(k)\, dk = 0$.*

Later we will prove that the Steinberg representation is irreducible. Dually (this duality will not be explained in these notes ...), we have an embedding $\mathbb{C} \to \mathrm{Ind}_B^G \delta_B^{-1/2}$ (constant functions) and we will see later that St is also realized as the cokernel of this map.

Definition 2.17. *Let (V, π) be a smooth representation of N. Let $V(N) = \sum_{n \in N}(\pi(n) - 1)V$. Let $V_N = V/V(N)$, the space of coinvariants under N, i.e. the largest quotient of V on which N acts trivially. This defines a functor* $\mathrm{Rep}(N) \to \mathrm{Vec}$. *Restricting to smooth representations of B, using that N is distinguished in B we obtain a functor* $\mathrm{Rep}(B) \to \mathrm{Rep}(T)$ *(exercise: check that for V a smooth representation of B, the representation V_N of T is smooth).*

Define the (normalized) Jacquet functor $\mathrm{Res}_B : \mathrm{Rep}(B) \to \mathrm{Rep}(T)$ *by* $\mathrm{Res}_B V = \delta_B^{-1/2} \otimes V_N$. *Composing with the forgetful functor* $\mathrm{Rep}(G) \to \mathrm{Rep}(B)$, *we obtain a functor* $\mathrm{Rep}(G) \to \mathrm{Rep}(T)$ *that we abusively also denote by* Res_B.

If (V, π) is smooth representation of G we say that it is supercuspidal if $\mathrm{Res}_B V = 0$ *(equivalently if $V_N = 0$).*

Note that any compact open subgroup of $N \simeq \mathbb{Q}_p$ is of the form

$$\left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| x \in p^n \mathbb{Z}_p \right\}$$

for some $n \in \mathbb{Z}$. In particular N is the union of its compact open subgroups. For N_c such a subgroup and (V, π) a smooth representation of N we let $V(N_c)$ be the space of $v \in V$ such that $\int_{N_c} \pi(n) v dn = 0$. Note that for $N_c \subset N_c'$ compact open subgroups of N we have $V(N_c) \subset V(N_c')$. This notation is justified by the following lemma.

Lemma 2.18. (1) *For any smooth representation of N, $V(N) = \bigcup_{N_c} V(N_c)$ where the union is over all compact open subgroups N_c of N.*

(2) *The functor* $\mathrm{Res}_B : \mathrm{Rep}(N) \to \mathrm{Vec}$ *is exact, i.e. for any short exact sequence* $0 \to V_1 \to V_2 \to V_3 \to 0$ *in* $\mathrm{Rep}(N)$, *the sequence* $0 \to \mathrm{Res}_B V_1 \to \mathrm{Res}_B V_2 \to \mathrm{Res}_B V_3 \to 0$ *is also exact.*

Proof. (1) For $n_1, \ldots, n_k \in N$ choose N_c containing all n_i's, then it is clear that for any $v_1, \ldots, v_k \in V$ we have $\int_{N_c} \pi(n) \sum_{i=1}^k (\pi(n_i) - 1) v_i dn = 0$.

Conversely suppose that $v' \in V(N_c)$. Let $N_c' \subset N_c$ be an open subgroup fixing v'. We have

$$0 = \int_{N_c} \pi(n)v'dn = \mathrm{vol}(N_c') \sum_{n \in N_c/N_c'} \pi(n)v' \equiv \mathrm{vol}(N_c)v' \mod V(N)$$

and we deduce $v' \in V(N)$.

(2) The only non-formal part of this statement is the injectivity of $\mathrm{Res}_B V_1 \to \mathrm{Res}_B V_2$, but this follows easily from the previous point.

□

The following theorem (Frobenius reciprocity) is easy to prove but fundamental.

Theorem 2.19. *The Jacquet functor* $\mathrm{Res}_B : \mathrm{Rep}(G) \to \mathrm{Rep}(T)$ *is left adjoint to* $\mathrm{Ind}_B^G : \mathrm{Rep}(T) \to \mathrm{Rep}(G)$. *More explicitly, for any smooth representation* (π, V) *(resp.* (σ, W)*) of* G *(resp.* T*) we have an identification*

$$\begin{aligned} \mathrm{Hom}_G(V, \mathrm{Ind}_B^G W) &\simeq \mathrm{Hom}_T(\mathrm{Res}_B V, W) \\ \alpha &\mapsto (\overline{v} \mapsto \alpha(v)(1)) \\ \left(v \mapsto (x \mapsto \beta(\overline{\pi(x)v}))\right) &\leftarrow\!\shortmid \beta \end{aligned}$$

that is functorial in both V *and* W.

Proof. The proof is completely formal. First check that for $\alpha : V \to \mathrm{Ind}_B^G W$ the map $v \mapsto \alpha(v)(1)$ vanishes on $V(N)$. Then check that for $\alpha : V \to \mathrm{Ind}_B^G W$ (resp. $\beta : \mathrm{Res}_B V \to W$) the map $\overline{v} \to \alpha(v)(1)$ (resp. $v \mapsto (x \mapsto \beta(\overline{\pi(x)v}))$) is T-equivariant (resp. G-equivariant). Finally check that the two compositions are equal to the identity. Details are left as an exercise. □

Corollary 2.20. *Let* (V, π) *be an irreducible smooth representation of* G. *Assume that* $\mathrm{Res}_B V \neq 0$ *(equivalently,* $V_N \neq 0$*). Then* V *embeds in a representation induced from a character of* T.

Proof. First we show that the representation $\mathrm{Res}_B V$ of T is finitely generated. Choose $v \in V \smallsetminus \{0\}$ and let K be a compact open subgroup of G fixing v. There exists a finite $R \subset G$ such that $G = BRK$. By irreducibility we have

$$V = \left\{ \sum_{i \in I} \lambda_i \pi(b_i r_i k_i) v \,\middle|\, I \text{ finite}, b_i \in B, r_i \in R \text{ and } k \in K \right\}$$

and we see that $\{\pi(r)v | r \in R\}$ generates $\mathrm{Res}_B V$.

Now assume that $\mathrm{Res}_B V \neq 0$. We claim that $\mathrm{Res}_B V$ admits an irreducible quotient. We may assume that $\mathrm{Res}_B V$ is not irreducible. We have shown that the representation $\mathrm{Res}_B V$ of T may be generated by a finite set F. Applying Zorn's lemma to the set of subrepresentations of $\mathrm{Res}_B V$ which do not contain F, we find an irreducible quotient of $\mathrm{Res}_B T$. By Schur's lemma a smooth irreducible representation of T is one-dimensional (on which T acts by a character, of course), and we conclude by applying the previous theorem. □

In particular any irreducible smooth non-supercuspidal representation is admissible.

The Bruhat decomposition is useful to study the restriction of $\mathrm{Ind}_B^G \mu$ to N.

Lemma 2.21. *Let μ be a smooth character of T. The morphism of $\mathbb{C}$-vector spaces*

$$\begin{aligned} \mathrm{Ind}_B^G \mu &\longrightarrow C^\infty(\mathbb{Q}_p, \mathbb{C}) \oplus \mathbb{C} \\ f &\longmapsto \left(x \mapsto f\left(w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right), f(1) \right) \end{aligned}$$

is injective and its image is the space of pairs (F, v) such that there exists a compact subset C of $\mathbb{Q}_p$ containing 0 such that for any $x \in \mathbb{Q}_p \smallsetminus C$ we have

$$F(x) = [\delta_B^{1/2}\mu](\mathrm{diag}(x^{-1}, x))\, v.$$

Proof. Injectivity is clear. To characterize the image, note that for $k >> 0$ we have $f\left(\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}\right) = f(1)$ for any $y \in p^k\mathbb{Z}_p$, and if $y \neq 0$ we have

$$\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} y^{-1} & 1 \\ 0 & y \end{pmatrix} w \begin{pmatrix} 1 & y^{-1} \\ 0 & 1 \end{pmatrix}$$

and take $x = y^{-1}$. Details are left to the reader. $\square$

2.4. **The geometric lemma.** Let V be a complex line on which T acts by a smooth character $\mu = \mu_1 \otimes \mu_2$. We now compute $W = \mathrm{Res}_B \mathrm{Ind}_B^G V$. Let W_1 be the subspace of functions supported on BwN, i.e. functions f such that $f(1) = 0$. We have an identification

$$\begin{aligned} W_1 &\longrightarrow C_c^\infty(N, V) \\ f &\longmapsto f|_N \end{aligned}$$

By Lemma 2.21 we have a short exact sequence of smooth representations of B

$$0 \to W_1 \to \mathrm{Ind}_B^G V \to W_2 \to 0$$

with $W_2 = \delta_B^{1/2} \otimes V$. Since N acts trivially on W_2 we have $W_2(N) = 0$ and $\mathrm{Res}_B W_2 = V$. Now $W_1(N)$ is the kernel of $\varphi : W_1 \to V$, $f \mapsto \int_N f(wn)dn$. This morphism is easily seen to be surjective, and for $f \in W_1$ and $b \in B$, writing $b = ut$ with $t \in T$ and $u \in N$ we have

$$\begin{aligned} \varphi(b \cdot f) &= \int_N f(wnb)dn \\ &= \int_N f(wtw^{-1}wt^{-1}nut)dn \\ &= \delta_B^w(t)^{1/2}\mu^w(t) \int_N f(wt^{-1}nt)dn \\ &= \delta_B(t)^{-1/2}\mu^w(t)\delta_B(t)\varphi(f) \\ &= \delta_B(t)^{1/2}\mu^w(t)\varphi(f) \end{aligned}$$

where $\mu^w(t) := \mu(wtw^{-1})$ and using Equation (2.1). So $\operatorname{Res}_B W_1$ has dimension one and T acts by $\mu^w = \mu_2 \otimes \mu_1$ on it. So we have a short exact sequence

$$0 \to \mu^w \to \operatorname{Res}_B \operatorname{Ind}_B^G \mu \to \mu \to 0.$$

The existence of this short exact sequence is the "geometric lemma" for G. See [BZ77] for the case of a general reductive group over a non-Archimedean local field.

We can be more precise and completely determine $\operatorname{Res}_B \operatorname{Ind}_B^G \mu$ (Proposition 2.22 below). Since T is commutative, if $\mu^w \neq \mu$ (i.e. if $\mu_1 \neq \mu_2$) this short exact sequence splits: choose $t \in T$ such that $\mu(t) \neq \mu^w(t)$ and consider $\ker(t - \mu(t) | \operatorname{Res}_B \operatorname{Ind}_B^G V)$.

We now consider the case $\mu^w = \mu$, and show that in this case the short exact sequence does *not* split. The sequence splits if and only if there exists $f \in \operatorname{Ind}_B^G V$ such that $f(1) \neq 0$ and for any $b \in B$, $b \cdot f - \delta_B^{1/2}(b)\mu(b) f \in (\operatorname{Ind}_B^G V)(N) = W_1(N) = \ker \varphi$. We can take $b = t \in T$, then

$$\begin{aligned}(t \cdot f)\left(w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) - \delta_B^{1/2}(t)\mu(t) f\left(w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \\ = \delta_B^w(t)^{1/2}\mu^w(t) F(x t_1^{-1} t_2) - \delta_B^{1/2}(t)\mu(t)F(x)\end{aligned}$$

It is clear that $b \cdot f - \delta_B^{1/2}(b)\mu(b) f \in W_1$, so for $k >> 0$ we have

$$\begin{aligned}\varphi(f) &= \int_{p^{-k}\mathbb{Z}_p} \delta_B^w(t)^{1/2}\mu^w(t)F(xt_1^{-1}t_2) - \delta_B^{1/2}(t)\mu(t)F(x)\ |dx| \\ &= \delta_B(t)^{1/2}\left(\mu^w(t)\int_{p^{-k}t_1^{-1}t_2\mathbb{Z}_p} F(x)\ |dx| - \mu(t)\int_{p^{-k}\mathbb{Z}_p} F(x)\ |dx|\right)\end{aligned}$$

where $|dx|$ denotes the Haar measure on $\mathbb{Q}_p$ giving $\mathbb{Z}_p$ volume 1 (notation to be clarified later in the course). Under our assumption that $\mu^w = \mu$, we see that this vanishes for any $t \in T$ if and only if for $k >> 0$ we have vanishing of $\int_{p^{-k}\mathbb{Z}_p^\times} F(x)\ |dx|$. Using Lemma 2.21 this equals

$$\int_{p^{-k}\mathbb{Z}_p^\times} |x|^{-1}\mu_1(x)^{-1}\mu_2(x) f(1)\ |dx| = \left(1 - \frac{1}{p}\right) f(1)$$

since $\mu_1 = \mu_2$.

Let us state what we have just proved.

Proposition 2.22. *Let μ be a smooth character of T. If $\mu \neq \mu^w$ then $\operatorname{Res}_B \operatorname{Ind}_B^G \mu \simeq \mu \oplus \mu^w$. If $\mu^w = \mu$ then we have a short exact sequence*

$$0 \to \mu \to \operatorname{Res}_B \operatorname{Ind}_B^G \mu \to \mu \to 0$$

which does not split.

Corollary 2.23. *Let $\mu = \mu_1 \otimes \mu_2$ be a smooth character of T. If μ_1/μ_2 is unitary, that is if $|\mu_1(p)/\mu_2(p)| = 1$, then $\operatorname{Ind}_B^G \mu$ is irreducible.*

Proof. Up to twisting by an unramified character we can assume that μ_1 and μ_2 are unitary. Then $\operatorname{Ind}_B^G \mu$ admits a G-invariant Hermitian inner product, and in particular it is semi-simple, so it is irreducible if and only if the vector space $\operatorname{Hom}_G(\operatorname{Ind}_B^G \mu, \operatorname{Ind}_B^G \mu)$ has dimension one. By Theorem 2.19 this space is isomorphic to $\operatorname{Hom}_T(\mathbb{C}(\mu), \operatorname{Res}_B \operatorname{Ind}_B^G \mu)$ and we can conclude with the above computation of $\operatorname{Res}_B \operatorname{Ind}_B^G \mu$. □

2.5. **Reducibility of parabolically induced representations in the non-unitary case: intertwining operators.** It remains to study reducibility of $\mathrm{Ind}_B^G \mu$ in the case where μ_1/μ_2 is not unitary, in particular $\mu^w \neq \mu$. Frobenius reciprocity tells us that the representation $\mathrm{Ind}_B^G \mu$ is indecomposable, so if it is not irreducible Corollary 2.50 tells us that it has a unique irreducible representation and a unique irreducible quotient.

Note that by Theorem 2.19 and the computation of $\mathrm{Res}_B \mathrm{Ind}_B^G \mu$ (Proposition 2.22) we have

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\mathrm{Ind}_B^G \mu, \mathrm{Ind}_B^G \mu^w) = 1$$

whether μ^w is equal to μ or not. We will construct a basis of this space, i.e. a non-zero intertwining operator, more explicitly. Start with $f \in \mathrm{Ind}_B^G \mu$. To produce a function which transform under left action of B by μ^w, in particular left invariant under N, it is natural to consider the integral

$$g \longmapsto \int_N f(wng)dn. \tag{2.2}$$

Indeed if we assume that it converges absolutely then for any $t \in T$ we have

$$\begin{aligned} \int_N f(wntg)dn &= \int_N f(wtw^{-1}wt^{-1}ntg)dn \\ &= \int_N \mu^w(t)\delta_B^w(t)^{1/2} f(wt^{-1}ntg)dn \\ &= \mu^w(t)\delta_B^{1/2}(t) \int_N f(wn'g)dn' \end{aligned} \tag{2.3}$$

where we let $n' = t^{-1}nt$ and used Formula (2.1).

Lemma 2.24. *If $|\mu_1(p)/\mu_2(p)| < 1$, the integral (2.2) converges absolutely.*

Proof. Write $n = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$ with $y \in \mathbb{Q}_p$. We have $wnw^{-1} = \begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix}$. We want to show that we have

$$\int_{\mathbb{Q}_p} \left| f\left(\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} wg \right) \right| \, |dy| < +\infty$$

and since the integrand is obviously a smooth function it is enough to show that we have

$$\int_{\mathbb{Q}_p \smallsetminus p^{-k}\mathbb{Z}_p} \left| f\left(\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} wg \right) \right| \, |dy| < +\infty \tag{2.4}$$

for some integer k. As in Lemma 2.21 we write, for $y \in \mathbb{Q}_p^\times$,

$$\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} y^{-1} & 1 \\ 0 & y \end{pmatrix} w \begin{pmatrix} 1 & y^{-1} \\ 0 & 1 \end{pmatrix}.$$

For k_0 large enough f is constant on $w \begin{pmatrix} 1 & p^{k_0}\mathbb{Z}_p \\ 0 & 1 \end{pmatrix} wg$, equal to $f(w^2 g) = f(-g)$. We note for future use that we may even take the same k_0 for any g in a given compact subset of G. Decomposing $\mathbb{Q}_p \smallsetminus p^{1-k_0}\mathbb{Z}_p$ as $\bigsqcup_{k \geq k_0} p^{-k}\mathbb{Z}_p^\times$, we obtain that (2.4) is equivalent to the convergence of

$$\begin{aligned}\sum_{k\geq k_0} \int_{p^{-k}\mathbb{Z}_p^\times} \left|\mu_1(y)^{-1}\mu_2(y)|y|^{-1}f(-g)\right| \ |dy| &= |f(-g)| \sum_{k\geq k_0} |\mu_1(p)/\mu_2(p)|^k \int_{\mathbb{Z}_p^\times} |z^{-1}dz| \\ &= (1-1/p)|f(-g)| \sum_{k\geq k_0} |\mu_1(p)/\mu_2(p)|^k.\end{aligned}$$

where we have used the change of variables $y = p^{-k}z$ and the fact that $|\mu_1| = |\mu_2| = 1$ on the compact group $\mathbb{Z}_p^\times$. □

Removing absolute values in the integral in the proof of the lemma, we also see that if $|\mu_1(p)/\mu_2(p)| < 1$, for k_0 large enough (depending on f and a compact subset of G in which g lies) we have

$$\begin{aligned}\int_N f(wng)dn = &\int_{p^{1-k_0}\mathbb{Z}_p} f\left(\begin{pmatrix}1 & 0\\ y & 1\end{pmatrix} wg\right) \ |dy| \\ &+ \frac{(\mu_1/\mu_2)(p)^{k_0}}{1-(\mu_1/\mu_2)(p)} f(-g) \int_{\mathbb{Z}_p^\times} \mu_1(y)^{-1}\mu_2(y) \ |dy|.\end{aligned} \tag{2.5}$$

Moreover by orthogonality of smooth characters of $\mathbb{Z}_p^\times$ we have

$$\int_{\mathbb{Z}_p^\times} \mu_1(y)^{-1}\mu_2(y) \ |dy| = \begin{cases} (1-1/p) & \text{if } (\mu_1/\mu_2)|_{\mathbb{Z}_p^\times} = 1 \\ 0 & \text{otherwise.}\end{cases} \tag{2.6}$$

These formulas, which are purely algebraic (the integrals now involve only smooth functions on compact spaces), motivate the following definition which "removes denominators" and makes no assumption on μ.

Definition 2.25. *Let $\mu = \mu_1 \otimes \mu_2$ be a smooth character of T. For $f \in \operatorname{Ind}_B^G \mu$ define $J_\mu(f) : G \to \mathbb{C}$ by*

$$\begin{aligned}J_\mu(f)(g) =&(1-(\mu_1/\mu_2)(p)) \int_{p^{1-k_0}\mathbb{Z}_p} f\left(\begin{pmatrix}1 & 0\\ y & 1\end{pmatrix} wg\right) \ |dy| \\ &+ (1-1/p)f(-g)(\mu_1/\mu_2)(p)^{k_0}\end{aligned}$$

if $(\mu_1/\mu_2)|_{\mathbb{Z}_p^\times} = 1$, and

$$J_\mu(f)(g) = \int_{p^{1-k_0}\mathbb{Z}_p} f\left(\begin{pmatrix}1 & 0\\ y & 1\end{pmatrix} wg\right) \ |dy|$$

otherwise, where in both cases $k_0 \in \mathbb{Z}$ is large enough so that f is constant on $w\begin{pmatrix}1 & p^{k_0}\mathbb{Z}_p\\ 0 & 1\end{pmatrix} wg$ (Exercise: check that the above definition does not depend on the choice of k_0).

Lemma 2.26. *Let $\mu = \mu_1 \otimes \mu_2$ be a smooth character of T. For any $f \in \operatorname{Ind}_B^G \mu$ the function $J_\mu(f)$ belongs to $\operatorname{Ind}_B^G \mu^w$. The linear map $J_\mu : \operatorname{Ind}_B^G \mu \to \operatorname{Ind}_B^G \mu^w$ is G-equivariant (in other words, it is an* intertwining operator*).*

Proof. As noted in the previous proof for a given $f \in \mathrm{Ind}_B^G$ we may choose a uniform k_0 in the formula defining $J_\mu(f)$ for g in a given compact subset of G, and so it is clear that $J_\mu(f)$ is also smooth. If $|(\mu_1/\mu_2)(p)| < 1$ then by (2.5), (2.6) and (2.3) we have $J_\mu(f) \in \mathrm{Ind}_B^G \mu^w$. It could be possible to prove this for $|(\mu_1/\mu_2)(p)| \geq 1$ using an unpleasant computation, but it is more elegant to use analytic (in fact, algebraic) continuation.

Because of the decomposition $G = BK_0$ we can identify $\mathrm{Ind}_B^G \mu$ with the space of functions $f : K_0 \to \mathbb{C}$ such that $f(bx) = \mu(b)f(x)$ for any $b \in B \cap K_0$ and $x \in K_0$. Note that this space can be defined purely in terms of $\mu|_{T_0}$, but that the action of G really depends on μ (and is rather complicated to write explicitly). We now replace the coefficient field $\mathbb{C}$ by $A := \mathbb{C}[X_1^{\pm 1}, X_2^{\pm 1}]$, and consider the character

$$\begin{aligned} \mu_X : T &\longrightarrow A^\times \\ t = \mathrm{diag}(t_1, t_2) &\longmapsto \mu_1(t_1) X_1^{v_p(t_1)} \mu_2(t_2) X_2^{v_p(t_2)} \end{aligned}$$

and the space $\mathrm{Ind}_B^G A(\mu_X)$ of smooth A-valued functions which transform under B on the left by $\delta_B^{1/2}\mu_X$. Let $f_X \in \mathrm{Ind}_B^G A(\mu_X)$ be the unique function such that $f_X|_{K_0} = f|_{K_0}$ (a smooth interpolation of f). Define $J_{\mu_X}(f_X) : G \to A$ as in Definition 2.25, so that $J_\mu(f)$ is its specialization at $X_1 = X_2 = 1$, i.e. the composition with the morphism of $\mathbb{C}$-algebras $A \to \mathbb{C}$ mapping X_1 and X_2 to 1. Consider, for $g \in G$ and $b \in B$, the element $P := J_{\mu_X}(f_X)(bg) - \mu_X(b)\delta_B(b)^{1/2} J_{\mu_X}(f_X)(g)$ of A. The set S of $(x_1, x_2) \in \mathbb{C}^\times$ such that $|(\mu_1/\mu_2)(p)x_1/x_2| < 1$ is Zariski-dense in $(\mathbb{C}^\times)^2$, for example because for any $x_1 \in \mathbb{C}^\times$ there are infinitely many $x_2 \in \mathbb{C}^\times$ such that this condition is satisfied. For any $(x_1, x_2) \in S$ we have $P(x_1, x_2) = 0$, so $P = 0$ and $P(1, 1) = 0$. □

Proposition 2.27. (1) *In the case $\mu|_{T_0} = 1$, let $f_\mu \in \mathrm{Ind}_B^G \mu$ be the unique function such that $f|_{K_0} = 1$. Then $J_\mu(f_\mu) = (1 - p^{-1}(\mu_1/\mu_2)(p)) f_{\mu^w}$.*

(2) *In any case the intertwining operator $J_\mu : \mathrm{Ind}_B^G \mu \to \mathrm{Ind}_B^G \mu^w$ is non-zero.*

Proof. (1) For $g \in K_0$ the formula in Definition 2.25 holds for $k_0 = 0$ or $k_0 = 1$, and we easily deduce the formula.

(2) If $\mu_1\mu_2^{-1}|_{\mathbb{Z}_p^\times} = 1$ the up to twisting μ_1 and μ_2 by the same smooth character of $\mathbb{Q}_p^\times$ we may assume that $\mu|_{T_0} = 1$, and if moreover $(\mu_1/\mu_2)(p) \neq p$, we are done by the previous point.

Otherwise take f supported on $B\begin{pmatrix} 1 & 0 \\ \mathbb{Z}_p & 1 \end{pmatrix}$ and constant non-zero on $\begin{pmatrix} 1 & 0 \\ \mathbb{Z}_p & 1 \end{pmatrix}$. Evaluate at $g = w^{-1}$, then the above formula holds with $k_0 = 1$ and the second term vanishes while the first one does not (one can also go back to the original integral and observe that the integrand is compactly supported). □

Lemma 2.28. *Let μ be a smooth character of T. Assume that we have $\mu \neq \mu^w$. The representation $\mathrm{Ind}_B^G \mu$ is reducible (i.e. has length 2 by Corollary 2.50) if and only if $J_{\mu^w} \circ J_\mu = 0$.*

Proof. Assume first that $\mathrm{Ind}_B^G \mu$ has length two, then it is not semisimple since we have computed $\mathrm{End}_G(\mathrm{Ind}_B^G \mu) = \mathbb{C}$, and so it has a unique irreducible quotient Q_μ, and a unique irreducible subrepresentation S_μ. By the geometric lemma and exactness of the functor Res_B, $(\mathrm{Res}_B Q_\mu, \mathrm{Res}_B S_\mu) \in \{(\mu, \mu^w), (\mu^w, \mu)\}$. If $\mathrm{Res}_B Q_\mu = \mu$ then Q_μ embeds in $\mathrm{Ind}_B^G \mu$ (see Corollary 2.20), but this implies $Q_\mu \simeq S_\mu$, which is a contradiction since they have distinct Jacquet modules. So we have $\mathrm{Res}_B Q_\mu \simeq \mu^w$ and Q_μ embeds in $\mathrm{Ind}_B^G \mu^w$, and because $\mathrm{Res}_B \mathrm{Ind}_B^G \mu^w \simeq \mu \oplus \mu^w$ we see that $\mathrm{Ind}_B^G \mu^w$ is not irreducible either, so it also has length 2 and $S_{\mu^w} \simeq Q_\mu$, and by symmetry $Q_{\mu^w} \simeq S_\mu$. We know that J_μ is not identically zero, and it cannot be an isomorphism, for example because S_μ is not isomorphic to S_{μ^w}. So we have $\ker J_\mu = S_\mu$, $\mathrm{im} J_\mu = S_{\mu^w}$ and $J_{\mu^w} \circ J_\mu = 0$.

Conversely, if $J_{\mu^w} \circ J_\mu = 0$ then since both J_μ and J_{μ^w} are non-zero we get that at least one of $\mathrm{Ind}_B^G \mu$ or $\mathrm{Ind}_B^G \mu^w$ is not irreducible, and the previous argument shows that in fact both are reducible. □

Lemma 2.29. *Let μ be a smooth character of T. The composition of intertwining operators $J_{\mu^w} \circ J_\mu$ is a scalar.*

Proof. This follows from our computation $\mathrm{End}_G(\mathrm{Ind}_B^G \mu) = \mathbb{C}$, thanks to the geometric lemma. □

Corollary 2.30. *Let $\mu = \mu_1 \otimes \mu_2$ be a smooth character of T. If $\mu_1/\mu_2|_{\mathbb{Z}_p^\times} = 1$ then $\mathrm{Ind}_B^G \mu$ is reducible if and only if $(\mu_1/\mu_2)(p) \in \{p, p^{-1}\}$.*

Proof. The case where μ_1/μ_2 is unitary is covered by Corollary 2.23, so we may assume that we have $\mu \neq \mu^w$. Up to twisting μ_1 and μ_2 by the same character, we may also assume that we have $\mu|_{T_0} = 1$. By Lemma 2.29 and the first point of Proposition 2.27 we have

$$J_{\mu^w} \circ J_\mu = (1 - p^{-1}(\mu_1/\mu_2)(p))(1 - p^{-1}(\mu_2/\mu_1)(p))\mathrm{Id}_{\mathrm{Ind}_B^G \mu},$$

and by Lemma 2.28 the statement follows. □

In the ramified case, we are left with a computation.

Proposition 2.31. *Assume that $\mu_1/\mu_2|_{\mathbb{Z}_p^\times} \neq 1$. Let $r \geq 1$ be the smallest integer satisfying $\mu_1/\mu_2|_{1+p^r\mathbb{Z}_p} = 1$. Then $J_{\mu^w} \circ J_\mu = p^{-r}(\mu_1/\mu_2)(-1)$. In particular, $\mathrm{Ind}_B^G \mu$ is irreducible.*

Proof. By Lemma 2.29 it is enough to compute $J_{\mu^w} \circ J_\mu$ on some non-zero $f \in \mathrm{Ind}_B^G \mu$. Again take f supported on $B \begin{pmatrix} 1 & 0 \\ \mathbb{Z}_p & 1 \end{pmatrix}$, constant on $\begin{pmatrix} 1 & 0 \\ \mathbb{Z}_p & 1 \end{pmatrix}$ with $f(1) = 1$. Since the second term in the formula defining $J_\mu(f)$ vanishes in the ramified case, we "simply" have

$$\begin{aligned}(J_{\mu^w} \circ J_\mu)(f)(g) &= \int_{p^{1-k_0}\mathbb{Z}_p} J_{\mu^w}(f)\left(\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} wg\right) |dz| \\ &= \int_{p^{1-k_0}\mathbb{Z}_p} \int_{p^{1-k_1}\mathbb{Z}_p} f\left(\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} w \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} wg\right) |dy|\,|dz|.\end{aligned}$$

for any large enough integer k_0, and any large enough integer k_1 (which may be taken uniformly on $z \in p^{1-k_0}\mathbb{Z}_p$ since this set is compact, but note that k_1 depends on k_0). We may and do assume that $k_1 > 0$. We compute

$$\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} w \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} w = \begin{pmatrix} -1 & z \\ -y & yz-1 \end{pmatrix}$$

and since it is enough to evaluate the above integral for $g = 1$, we want to write this matrix (assuming that it lies in the support of f) as

$$\begin{pmatrix} c^{-1} & b \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} = \begin{pmatrix} c^{-1}+bu & b \\ cu & c \end{pmatrix}.$$

We write the above integral in terms of the new variables (c, u). To this end we may restrict to $y \neq 0$, since the subvariety corresponding to $y = 0$ has measure 0. We introduce the change of variable

$$\begin{aligned} \phi : \mathbb{Q}_p^\times \times (\mathbb{Z}_p \smallsetminus \{0\}) &\longrightarrow \mathbb{Q}_p^\times \times \mathbb{Q}_p \\ (c, u) &\longmapsto (y = -cu, z = -(1 + c^{-1})u^{-1}) \end{aligned}$$

which is clearly injective. We will see that it has everywhere invertible differential as well, and describe $S := \phi^{-1}((p^{1-k_1}\mathbb{Z}_p \smallsetminus \{0\}) \times p^{1-k_0}\mathbb{Z}_p)$.

Let us first compute the Jacobian of the change of variables. We find

$$dy = -udc - cdu, \qquad dz = \frac{dc}{c^2 u} + \frac{(1+c)du}{cu^2}, \qquad dy \wedge dz = \frac{du \wedge dc}{u}.$$

By Theorem B.7 the above integral for $g = 1$ is equal to

$$\int_S \mu_1(c)^{-1}\mu_2(c)|c|^{-1}|u|^{-1}|du||dc|.$$

The condition $(c, u) \in S$ is equivalent to (A) $v(cu) \geq 1 - k_1$ and (B) $v(1 + c^{-1}) \geq 1 - k_0 + v(u)$. For a given $u \in \mathbb{Z}_p \smallsetminus \{0\}$ we consider the set of $c \in \mathbb{Q}_p^\times$ such that these two conditions are satisfied.

- If $0 \leq v(u) \leq k_0 - 1$ then condition (B) is equivalent to $v(c^{-1}) \geq v(u) + 1 - k_0$, so that conditions (A) and (B) together are equivalent to $k_0 - 1 - v(u) \geq v(c) \geq 1 - k_1 - v(u)$. For any $k \in \mathbb{Z}$ we have $\int_{p^k\mathbb{Z}_p^\times} \mu_1(c)^{-1}\mu_2(c)|c|^{-1}|dc| = 0$.
- If $v(u) > k_0 - 1$ then condition (B) is equivalent to $c \in -1 + p^{v(u)+1-k_0}\mathbb{Z}_p$, and so condition (A) reads $v(u) \geq 1 - k_1$, which is automatically satisfied since $k_1 > 0$. We have

$$\int_{-1+p^{v(u)+1-k_0}\mathbb{Z}_p} \mu_1(c)^{-1}\mu_2(c)|dc| = \begin{cases} 0 & \text{if } v(u) < k_0 - 1 + r \\ p^{k_0-1-v(u)}(\mu_1/\mu_2)(-1) & \text{if } v(u) \geq k_0 - 1 + r. \end{cases}$$

Thus

$$\begin{aligned} \int_S \mu_1(c)^{-1}\mu_2(c)|c|^{-1}|u|^{-1}|du||dc| &= \int_{v(u) \geq k_0-1+r} (\mu_1/\mu_2)(-1)p^{k_0-1}|du| \\ &= p^{-r}(\mu_1/\mu_2)(-1). \end{aligned}$$

$\square$

We can finally state the classification of non-supercuspidal representations of G.

Theorem 2.32. *Any irreducible smooth non-supercuspidal representation of G is isomorphic to exactly one of the following:*

- $\mathrm{Ind}_B^G \mu$ *for* $\mu_1/\mu_2 \notin \{|\cdot|^{\pm 1}\}$, *with* $\{\mu, \mu^w\}$ *uniquely determined,*
- $\chi \circ \det$ *for some continuous character* $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$,
- $(\chi \circ \det) \otimes \mathrm{St}$ *for some continuous character* $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$.

Proof. Existence follows from Corollary 2.20, Corollary 2.30 and Proposition 2.31. Uniqueness follows from consideration of the Jacquet module and the fact that J_μ is an isomorphism when $\mathrm{Ind}_B^G \mu$ is irreducible. □

Remark 2.33. *We can define the local Langlands correspondence in an ad hoc manner for principal series. One of the main results of local class field theory is the existence of a natural isomorphism* $\mathrm{rec} : (\mathrm{W}_{\mathbb{Q}_p})^{\mathrm{ab}} \simeq \mathbb{Q}_p^\times$ *where* $\mathrm{W}_{\mathbb{Q}_p} \subset \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ *is the Weil group of* $\mathbb{Q}_p$. *Therefore, it is natural to declare that an irreducible* $\mathrm{Ind}_B^G \mu$ *corresponds to the reducible semi-simple two-dimensional representation* $\mu_1 \circ \mathrm{rec} \oplus \mu_2 \circ \mathrm{rec}$. *It is not as clear what one should do for the one dimensional and Steinberg representations. It turns out that it is natural to associate* $(\chi|\cdot|^{1/2}) \circ \mathrm{rec} \oplus (\chi|\cdot|^{-1/2}) \circ \mathrm{rec}$ *to* $\chi \circ \det$. *Later we will prove that the Steinberg representation is square-integrable, although it is clearly not supercuspidal. It turns out that it is natural to introduce the* Weil-Deligne group $\mathrm{WD}_{\mathbb{Q}_p} := \mathrm{W}_{\mathbb{Q}_p} \times \mathrm{SL}_2(\mathbb{C})$. *Of course the above representations of* $\mathrm{W}_{\mathbb{Q}_p}$ *can simply be seen as representations of* $\mathrm{WD}_{\mathbb{Q}_p}$ *which are trivial on the second factor. The Langlands parameter of* $(\chi \circ \det) \otimes \mathrm{St}$ *is* $\chi \circ \mathrm{rec} \otimes \nu_2$ *where* ν_2 *is (by definition) the irreducible algebraic* 2*-dimensional representation of* $\mathrm{SL}_2(\mathbb{C})$.

More generally, to formulate the local Langlands correspondance for $\mathrm{GL}_n(F)$, F *a non-Archimedean local fields, one should consider* n*-dimensional semi-simple continuous representations* ρ *of* $\mathrm{WD}_F := \mathrm{W}_F \times \mathrm{SL}_2(\mathbb{C})$ *which are algebraic (i.e. polynomial or equivalently, holomorphic) on the factor* $\mathrm{SL}_2(\mathbb{C})$. *To such a* ρ *one can associate a pair* (ρ', N) *where* $\rho' : \mathrm{W}_F \to \mathrm{GL}_n(\mathbb{C})$ *is the semi-simple continuous representation which is the composition of* ρ *with the embedding*

$$\begin{aligned} \mathrm{W}_F &\longrightarrow \mathrm{WD}_F \\ w &\longmapsto (w, \mathrm{diag}(|w|^{1/2}, |w|^{-1/2})) \end{aligned}$$

and $N \in M_n(\mathbb{C})$ *is defined as* $d\rho \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. *This pair satisfies* $\rho'(w) N \rho'(w)^{-1} = |w| N$ *for any* $w \in \mathrm{W}_F$. *Conversely using the Jacobson-Morozov theorem we see that any such pair* (ρ', N) *arises from* ρ *as above, so the two formulations are equivalent. Such a pair* (ρ', N) *is called a Weil-Deligne representation, and is more natural in an arithmetic setting (see* [Tat79, 4.1 and 4.2]*).*

2.6. **Supercuspidal representations.**

Definition 2.34. *Let* I *be a compact open subgroup of* G. *We say that* I *has an* Iwahori factorization *if, denoting* $\overline{N}_I = \overline{N} \cap I$, $T_I = T \cap I$ *and* $N_I = N \cap I$, *the product map* $N_I \times T_I \times \overline{N}_I \to I$ *is surjective.*

Of course this map is always injective, and it is clear that it is always a homeomorphism onto an open subset of I.

Remark 2.35. (1) *Let T^- be the set of $t \in T$ such that $|t_1/t_2| \leq 1$. In* [Cas, §1.4] *there is an extra condition in the definition of Iwahori factorization, namely that for any $t \in T^-$ we have $tN_I t^{-1} \subset N_I$ and $t^{-1}\overline{N}_I t \subset \overline{N}_I$. It is easy to check that this condition is automatically satisfied in the particular case of $\mathrm{GL}_2(\mathbb{Q}_p)$, due to the particularly simple classification of closed subgroups of $\mathbb{Q}_p$.*

(2) *Taking inverses, we also have $\overline{N}_I \times T_I \times N_I \simeq I$.*

Example 2.36. (1) *The Iwahori subgroup $\begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}$ has an Iwahori factorization (and is the reason for this terminology).*

(2) *For any integer $i \geq 1$ the subgroup $K_i = \begin{pmatrix} 1+p^i\mathbb{Z}_p & p^i\mathbb{Z}_p \\ p^i\mathbb{Z}_p & 1+p^i\mathbb{Z}_p \end{pmatrix}$ also satisfies this condition.*

We will use a more compact notation for in the second case: $\overline{N}_i = K_i \cap \overline{N}$, $T_i = T \cap K_i$ and $N_i = N \cap K_i$. For $i < 0$ we can also define $\overline{N}_i = \begin{pmatrix} 1 & 0 \\ p^i\mathbb{Z}_p & 1 \end{pmatrix}$ and $N_i = \begin{pmatrix} 1 & p^i\mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$.

Note however that K does not admit an Iwahori factorization, in fact one can check that I is maximal among compact open subgroups admitting an Iwahori factorization. For our purpose in this section we will only use the fact that there are arbitrarily small compact open subgroups of G having an Iwahori factorization. We will come back to the Iwahori case later.

Lemma 2.37. *Let I be a compact open subgroup of G admitting an Iwahori factorization. For any $f \in C^\infty(I)$, we have the integration formula.*

$$\int_{N_I \times T_I \times \overline{N}_I} f(nt\overline{n})\,dn\,dt\,d\overline{n} = \int_I f(g)\,dg.$$

Proof. The proof is similar to the proof of the integration formula for the Iwasawa decomposition (Lemma 2.13), only simpler because the decomposition is a bijection. The pullback to $N_I \times T_I \times \overline{N}_I$ of the Haar measure on I (restriction of the Haar measure on G) is clearly left N_I-invariant and right $\overline{N}_I$-invariant. It is also left and right T_I-invariant because any $t \in T_I$ normalizes $\overline{N}_I$ and N_I and preserves their Haar measures because T_I is compact (see Equation (2.1)). $\square$

For $g \in G$ denote by $\dot{g}$ its image in G/Z.

Theorem 2.38. *Let (π, V) be a smooth representation of G. The following are equivalent:*

(1) *(π, V) is supercuspidal.*

(2) *For any* $v \in V$ *and* $\widetilde{v} \in \widetilde{V}$, *there is a compact subset* C *of* G/Z *such that* $\langle \pi(g)v, \widetilde{v}\rangle = 0$ *for all* $g \in G$ *such that* $\dot{g} \in G/Z \smallsetminus C$.

(3) *For any* $v \in V$ *and any compact open subgroup* K *of* G, *there is a compact subset* C *of* G/Z *such that* $\pi(e_K)\pi(g)v = 0$ *for all* $g \in G$ *such that* $\dot{g} \in G/Z \smallsetminus C$.

Proof. First we prove that (2) implies (3). Let $G' = \{g \in G |\det g \in \mathbb{Z}_p^\times\}$, an open subgroup of G which contains all compact subgroups of G. Note that ZG' has finite index (two) in G, and that $G' \to G/Z$ is proper. Let $v \in V$ and $K \subset G$ a compact open subgroup of G. Let W be the sub-vector space of V^K generated by $\pi(e_K)\pi(g)v$ for $g \in G'$. Let S be a subset of $K\backslash G'$ such that $(\pi(e_K)\pi(g)v)_{g\in S}$ is a basis of W. Recall that V is a semi-simple representation of K, in particular we have a canonical decomposition $V = V^K \oplus V'$ where V' is stable under K and $(V')^K = 0$, and so $\widetilde{V}^K \simeq \mathrm{Hom}_{\mathbb{C}}(V^K, \mathbb{C})$ surjects (by restriction) onto $\mathrm{Hom}_{\mathbb{C}}(W, \mathbb{C})$. We deduce that there exists $\widetilde{v} \in \widetilde{V}^K$ such that for any $g \in S$ we have $\langle \pi(e_K)\pi(g)v, \widetilde{v}\rangle = 1$. Observe that we have $\langle \pi(e_K)\pi(g)v, \widetilde{v}\rangle = \langle \pi(g)v, \widetilde{v}\rangle$ because $\widetilde{v}$ is fixed by K (decompose $\pi(g)v$ in $V^K \oplus V'$). By the assumption (2) applied to the pair $(v, \widetilde{v})$, the set S is finite, i.e. $\dim_{\mathbb{C}} W < +\infty$. There is a finite family $(\widetilde{v}_i)_{1\leq i\leq k}$ of elements of $\widetilde{V}^K$ such that we have

$$\{w \in W \,|\, \langle w, \widetilde{v}_1\rangle = \cdots = \langle w, \widetilde{v}_k\rangle = 0\} = \{0\}.$$

Now applying assumption (2) to all pairs $(v, \widetilde{v}_i)$ shows that the function $G' \to W$, $g \mapsto \pi(e_K)\pi(g)v$ has compact support. Let $t = \mathrm{diag}(p, 1) \in G$. Since no assumption was made on v in the above argument, it also applies to $\pi(t)v$ and the map $G' \to V$, $g \mapsto \pi(e_K)\pi(gt)v$ also has compact support. Since Z is central in G (Z being the center ...) and $G = ZG' \sqcup ZG't$ we get that $G \to V$, $g \mapsto \pi(e_K)\pi(g)v$ has compact support modulo Z, i.e. there exists C as in (3).

Now let us show that (3) implies (1). Let $v \in V$. Pick $i \geq 1$ such that $v \in V^{K_i}$. We still denote $t = \mathrm{diag}(p, 1)$, so that $t^{-1}\overline{N}_i t \subset \overline{N}_i$, in fact $t^{-1}\overline{N}_i t = \overline{N}_{i+1}$. By assumption and the Cartan decomposition, for $m >> 0$ we have $\pi(e_{K_i})\pi(t^m)v = 0$. By the integration formula (Lemma 2.37) and passing t^m to the left, for any $m \geq 0$ we have

$$\pi(e_{K_i})\pi(t^m)v = \pi(t^m)\pi(e_{t^{-m}N_it^m})\pi(e_{T_i})\pi(e_{t^{-m}\overline{N}_it^m})v.$$

For $m >> 0$, because v is fixed by $\overline{N}_i \supset t^{-m}\overline{N}_i t^m$ and by T_i we obtain $\pi(e_{t^{-m}N_it^m})v = 0$, i.e. $v \in V(N_{i-m})$.

Finally we show that (1) implies (2). Using the same formula as above, we see that for any $i \geq 1$ and $v \in V^{K_i}$ there exists $m_0 \geq 0$ such that for any $m \geq m_0$ we have $\pi(e_{K_i})\pi(t^m)v = 0$. This implies the following:

$$\forall i \geq i, \forall v \in V^{K_i}, \exists m_0 \geq 0, \forall m \geq m_0, \forall z \in Z, \forall \widetilde{v} \in \widetilde{V}^{K_i}, \langle \pi(zt^m)v, \widetilde{v}\rangle = 0. \tag{2.7}$$

Now let $v \in V$ and $\widetilde{v} \in \widetilde{V}$. There exists $i \geq 1$ such that v and $\widetilde{v}$ are both fixed by K_i. Let $R_i \subset K_0$ be a set of representatives for the finite quotient K_0/K_i. By the Cartan decomposition we have

$$G = \sqcup_{m\geq 0} K_0 t^m K_0 Z = \cup_{\substack{m\geq 0\\ r\in R_i\\ r'\in R_i}} r^{-1}K_i t^m K_i r' Z.$$

Now we apply (2.7) to the pairs $(\pi(r')v, \pi(r)\widetilde{v})$ for $r, r' \in R_i$. Note that there are only finitely many such pairs and that all these vectors are also fixed by K_i because it is distinguished in K_0. Also note that we have (and this is the whole point of rewriting the Cartan decomposition as above)

$$\langle \pi(zt^m)\pi(r')v, \pi(r)\widetilde{v}\rangle = \langle \pi(r^{-1}t^m r' z)v, \widetilde{v}\rangle.$$

We deduce that there exists C of the form $\cup_{m<m_0} K_0 t^m K_0 Z/Z$ such that we have $\langle \pi(g)v, \widetilde{v}\rangle = 0$ for all $g \in G$ satisfying $\dot{g} \in G/Z \smallsetminus C$. □

Corollary 2.39. *Any irreducible supercuspidal representation is admissible.*

Proof. Pick any $v \in V \smallsetminus \{0\}$ and a compact open sugroup K of G which fixes v. The sub-vector space $W \subset V^K$ generated by $\pi(e_K)\pi(g)v$ for $g \in G$ has finite dimension by (3) above (here Schur's lemma is used). Since V is irreducible we have $W = \pi(e_K)V = V^K$. □

Remark 2.40. *Together with Lemma 2.12 and Corollary 2.20 (and the fact that a subrepresentation of an admissible representation is admissible), this shows that any irreducible smooth representation of G is admissible.*

Exercise 2.41. *Let (V, π) be a smooth representation of G.*

(1) *Show that if V is not irreducible then $\widetilde{V}$ is not irreducible.*

(2) *Show that if V is irreducible then $\widetilde{V}$ is irreducible.*

Definition 2.42. *Let (V, π) be a smooth representation of G, and assume that it has a (unique and smooth) central character $\omega_\pi : Z \to \mathbb{C}^\times$. We say that π is* square-integrable *(or part of the* discrete series*) if ω_π is unitary and for any $v \in V$ and $\widetilde{v} \in \widetilde{V}$, $\int_{G/Z} |\langle \pi(g)v, \widetilde{v}\rangle|^2 dg < +\infty$.*

We say that π is essentially square-integrable *if there exists $s \in \mathbb{R}_{>0}$ (unique) such that $|\det|^s \otimes \pi$ is square-integrable.*

Any irreducible supercuspidal representation is essentially square-integrable.

Lemma 2.43. *If (V, π) is an irreducible smooth representation of G with unitary central character ω_π (Corollary 2.7) then it is square-integrable if and only if there exist non-zero $v_0 \in V$ and $\widetilde{v}_0 \in \widetilde{V}$ such that $\int_{G/Z} |\langle \pi(g)v_0, \widetilde{v}_0\rangle|^2 dg < +\infty$.*

Proof. The set of $v \in V$ such that $\dot{g} \mapsto |\langle \pi(g)v, \widetilde{v}_0\rangle|^2$ is integrable is stable under G (by right invariance of the Haar measure) and is a sub-vector space of V (using the Cauchy-Schwarz inequality). Since it contains $v_0 \neq 0$, it equals V. Repeating this argument for $\widetilde{V}$ allows to conclude. □

Exercise 2.44. *Let (V, π) be a smooth representation of G. Let $\overline{V} := \mathbb{C} \otimes_{\mathbb{C}} V$ where $\mathbb{C}$ on the left is considered as a $\mathbb{C}$-algebra via the conjugation map. This is naturally a smooth representation of V with the action defined by $\overline{\pi}(g)(\lambda \otimes v) = \lambda \otimes (\pi(g)v)$. Let $\mathrm{Hermi}_G(V)$ be the $\mathbb{C}$-vector space of G-invariant sesquilinear pairings on V. Our* *convention is that these are linear in the first variable and conjugate-linear in the second variable. Show that the linear map*

$$\begin{aligned} \mathrm{Hom}_G(\overline{V}, \widetilde{V}) &\longrightarrow \mathrm{Hermi}_G(V) \\ \varphi &\longmapsto ((v_1, v_2) \mapsto \langle v_1, \varphi(1 \otimes v_2)\rangle) \end{aligned}$$

is an isomorphism.

Lemma 2.45. *Any irreducible square-integrable representation is unitarizable, i.e. admits a G-invariant hermitian inner product. Moreover the G-invariant hermitian inner product is unique up to $\mathbb{R}_{>0}$.*

Proof. Let $L^2(G, \omega_\pi)$ be the space of measurable functions $G \to \mathbb{C}$ such that $f(zg) = \omega_\pi(z)f(g)$ for all $z \in Z$ and $g \in G$ and $\int_{G/Z} |f(g)|^2 d\dot{g} < +\infty$, quotiented (as usual) by the subspace of functions which vanish outside a set of measure 0. This is a (non-smooth!) representation of G, for the action defined by $(gf)(x) = f(xg)$, and it has a G-invariant Hermitian inner product defined by

$$(f_1, f_2) \mapsto \int_{G/Z} f_1(g)\overline{f_2(g)} d\dot{g}.$$

Pick $\widetilde{v} \in \widetilde{V} \smallsetminus \{0\}$. Then $v \mapsto (g \mapsto \langle \pi(g)v, \widetilde{v}\rangle)$ gives a G-equivariant linear map $V \to L^2(G, \omega_\pi)$. Because V is irreducible it is injective. Restricting the above G-invariant Hermitian inner product on $L^2(G, \omega_\pi)$ to V gives the sought-after inner product. Denote by H this G-invariant Hermitian inner product on V.

It is easy to check that $(\overline{V}, \overline{\pi})$ (see previous exercise) is also irreducible. By Exercise 2.41 the contragredient representation $\widetilde{V}$ is irreducible, so by Schur's lemma we have $\dim_{\mathbb{C}} \mathrm{Hom}_G(\overline{V}, \widetilde{V}) \leq 1$. Using the previous exercise this implies that any G-invariant sesquilinear pairing on V is equal to λH for some $\lambda \in \mathbb{C}$, which is unique because H does not vanish identically. Finally the pairing λH is an inner product if and only if $\lambda \in \mathbb{R}_{>0}$. □

Remark 2.46. *The same argument shows that an irreducible supercuspidal representation of G can be realized in $C_c^\infty(G, \omega_\pi)$, the space of smooth functions $f \to \mathbb{C}$ such that $f(zg) = \omega_\pi(z)f(g)$ and such that the support of f is compact modulo Z. In fact it is easy to check that the image is included in the space $C_{\mathrm{cusp}}^\infty(G, \omega_\pi)$ of cuspidal functions, that is the subspace of functions f satisfying $\int_N f(xny)dn = 0$ for all $x, y \in G$.*

Proposition 2.47. *Let (V, π) be an irreducible essentially square-integrable representation of G. There exists a unique $d_\pi \in \mathbb{R}_{>0}$, called the* formal degree *of π, such that for any $u, v \in V$ and $\widetilde{u}, \widetilde{v} \in \widetilde{V}$ we have*

$$\int_{G/Z} \langle \pi(g)u, \widetilde{u}\rangle \langle \pi(g^{-1})v, \widetilde{v}\rangle d\dot{g} = d_\pi^{-1} \langle u, \widetilde{v}\rangle \langle v, \widetilde{u}\rangle.$$

Observe that the integral is well-defined and converges absolutely.

Proof. Fix v and $\widetilde{u}$. Then the integral, seen as a function of $(u, \widetilde{v})$, defines a G-invariant pairing on $V \times \widetilde{V}$. By Schur's lemma it is a complex number times the canonical pairing. The same argument with u and $\widetilde{v}$ fixed shows that the integral equals $c_\pi \langle u, \widetilde{v}\rangle \langle v, \widetilde{u}\rangle$ for some $c_\pi \in \mathbb{C}$.

It remains to show that $c_\pi \in \mathbb{R}_{>0}$. Up to twisting by a character we can assume that π is square-integrable. Pick a G-invariant Hermitian inner product $(\cdot,\cdot)$ on V (see the previous lemma), which is equivalent to an isomorphism $\varphi : \overline{V} \simeq \widetilde{V}$ such that $(v, v) := \langle v, \varphi(1 \otimes v)\rangle \in \mathbb{R}_{>0}$ for all non-zero $v \in V$. Taking $\widetilde{v} = \varphi(1 \otimes u)$ and $\widetilde{u} = \varphi(1 \otimes v)$ for arbitrary $u, v \in V \smallsetminus \{0\}$, the integrand equals

$$(\pi(g)u, v)(\pi(g^{-1})v, u) = |(\pi(g)u, v)|^2$$

which is non-negative, smooth and not identically vanishing, and the right-hand side equals $c_\pi(u, u)(v, v)$, therefore $c_\pi \in \mathbb{R}_{>0}$. □

Remark 2.48. (1) *The constant d_π depends on the choice of Haar measure for G/Z. The Haar measure $d_\pi d\dot{g}$ is canonically associated to π.*

(2) *This Proposition (and its proof) are inspired by the same result for irreducible unitary representations of finite (or more generally compact) groups, which are all square-integrable and finite-dimensional. In this simpler case, taking the Haar measure to be a probability measure (and removing the quotient by Z) one can check that d_π is the dimension of π. Proposition 2.47 for compact groups has a better known coordinate-free analogue (orthonormality of characters of irreducible representations), but it is not as straightforward to generalize this to the infinite-dimensional case. We will prove an analogous theorem for G later.*

Corollary 2.49. *If (V, π) is an irreducible supercuspidal representation of G and (U, σ) is a smooth representation of G such that $\sigma(zg) = \omega_\pi(z)\sigma(g)$ for all $z \in Z$ and $g \in G$ then any non-zero morphism $P : U \to V$ admits a splitting.*

Proof. Pick $v_0 \in V$ and $\widetilde{v}_0 \in \widetilde{V}$ such that $\langle v_0, \widetilde{v}_0\rangle = d_\pi$. Pick $u_0 \in U$ mapping to v_0. Define $s : V \to U$ by $s(v) = \int_{G/Z} \langle \pi(g^{-1})v, \widetilde{v}_0\rangle \sigma(g) u_0 d\dot{g}$. The linear map s is G-equivariant: for $h \in G$, using the change of variable $x = h^{-1}g$,

$$\begin{aligned} s(\pi(h)v) &= \int_{G/Z} \langle \pi((h^{-1}g)^{-1})v, \widetilde{v}_0\rangle \sigma(g) u_0 \, d\dot{g} \\ &= \int_{G/Z} \langle \pi(x^{-1})v, \widetilde{v}_0\rangle \sigma(hx) u_0 \, d\dot{g} = \sigma(h)s(v). \end{aligned}$$

To compute the image of $s(v)$ in V, take any test vector $\widetilde{v} \in \widetilde{V}$. The previous proposition gives us $\langle P(s(v)), \widetilde{v}\rangle = \langle v, \widetilde{v}\rangle$ and so $P(s(v)) = v$. □

Corollary 2.50. *For any smooth character μ of T, the induced representation $\mathrm{Ind}_B^G \mu$ has finite length ≤ 2, and no constituent is supercuspidal.*

Proof. First we show that any irreducible subquotient of $\mathrm{Ind}_B^G \mu$ is not supercuspidal. Let W be a subrepresentation of $\mathrm{Ind}_B^G \mu$ and W' an irreducible quotient of W. If W'

is supercuspidal then by Corollary 2.49 we have a splitting $W' \to W$, so we can see W' as an irreducible supercuspidal subrepresentation of $\mathrm{Ind}_B^G \mu$. But this contradicts Theorem 2.19!

Now consider a finite chain $0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_k = \mathrm{Ind}_B^G \mu$ of representations of G. By Zorn's lemma any quotient W_i/W_{i-1} admits an irreducible subquotient, which is also a subquotient of V and so is not supercuspidal. By exactness of the Jacquet functor this implies that each $\mathrm{Res}_B(W_i/W_{i-1}) \neq 0$. Since we have computed that the representation $\mathrm{Res}_B \mathrm{Ind}_B^G \mu$ of T has length 2, we deduce $k \leq 2$, in particular $\mathrm{Ind}_B^G \mu$ has finite length. We may then assume that each constituent W_i/W_{i-1} is irreducible. We also get that each W_i/W_{i-1} is not supercuspidal, so that $\mathrm{Res}_B(W_i/W_{i-1})$ is either one-dimensional (with action of T by μ or μ^w) or two-dimensional equal to $\mathrm{Res}_B \mathrm{Ind}_B^G \mu$. The latter case occurs if and only if $\mathrm{Ind}_B^G \mu$ is irreducible. □

2.7. **The Iwahori-Hecke algebra and the Steinberg representation.** Recall that the Iwahori subgroup I of K_0 is the preimage under $K_0 \twoheadrightarrow \mathrm{GL}_2(\mathbb{F}_p)$ of the upper-triangular Borel subgroup. We now study the Iwahori-Hecke algebra $\mathcal{H}(G, I)$, which will be useful to study the Steinberg representation because we will see that $\mathrm{St}^I \neq 0$.

Denote $T_0 = T \cap K_0$. In particular $T_0 \subset I$. Let $\widetilde{W} = N_G(T)/T_0$ be the extended affine Weyl group, it surjects onto $W = N_G(T)/T = \{1, w\}$, with kernel $T/T_0 \simeq \mathbb{Z}^2$. The natural map $N_K(T_0)/T_0 \to W$ is an isomorphism and gives a splitting of $\widetilde{W} \to W$, realizing $\widetilde{W}$ as $T/T_0 \rtimes W$.

Proposition 2.51 (Affine Bruhat decomposition)**.** *$G = \bigsqcup_{x \in \widetilde{W}} IxI$.*

Proof. First note that G/I parametrizes pairs (L, D) where L is a lattice in $\mathbb{Q}_p^2$ and $D \subset L/pL$ is an $\mathbb{F}_p$-line: the coset gI corresponds to $(\mathbb{Z}_p e_1 \oplus \mathbb{Z}_p e_2, \langle e_1 \rangle)$ where e_1, e_2 are the columns of g.

So we have to show that for any (L_1, D_1) and (L_2, D_2) as above, there exists a basis (e, f) of L_1 and $a, b \in \mathbb{Z}$ such that $D_1 = \langle e \rangle$, $(p^a e, p^b f)$ is a basis of L_2 and $D_2 = \langle p^a e_1 \rangle$ or $\langle p^b e_2 \rangle$, and that the pair $(a, b) \in \mathbb{Z}^2$ is unique (this equivalent statement of unuiqueness requires a bit of head scratching ...). Thanks to the Cartan decomposition we know that there is a basis (e, f) of L_1 and (unique) integers $a \geq b$ such that $(p^a e, p^b f)$ is a basis of L_2. It is clear that we may substitute $e + \mu f$ for e, for any $\mu \in \mathbb{Z}_p$. Since any line in L_1/pL_1 is generated by f or by $e + \mu f$ for some $\mu \in \mathbb{F}_p$, we obtain that up to swapping e and f (which does not preserve the condition $a \geq b$!) we may assume that $D_1 = \langle e_1 \rangle$.

- If $b \leq a$, we may substitute $f + p^{a-b}\mu e$ for f where $\mu \in \mathbb{Z}_p$ is arbitrary, since $p^b(f + p^{a-b}\mu e) = p^b f + p^a \mu e$. By the same argument as above, if $D_2 \neq \langle p^a e \rangle$ we can reduce to $D_2 = \langle p^b f \rangle$.

- If $b > a$, we may substitute $e + p^{b-a}\mu f$ for e where $\mu \in \mathbb{Z}_p$ is arbitrary, and as in the previous case this allows us to achieve $D_2 = \langle p^a e \rangle$ if $D_2 \neq \langle p^b f \rangle$.

Uniqueness can be seen on this argument and is left as an exercise. □

We will use the following more precise description of the double cosets contained in K_0.

Proposition 2.52. *We have a decomposition*

$$K_0 = I \sqcup IwI = I \sqcup \bigsqcup_{y \in \mathbb{F}_p} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} wI.$$

Note that the last term makes sense!

Proof. Observe that $K_1 \subset I \subset K_0$ with K_1 distinguished in K_0 and $K_0/K_1 = \mathrm{GL}_2(\mathbb{F}_p)$. The assertion follows immediately from the Bruhat decomposition in $\mathrm{GL}_2(\mathbb{F}_p)$. □

Note that we have $IxI = \bigsqcup_k kxI$ where k ranges over representatives of $I/I \cap xIx^{-1}$ (there is a similar description into left I-cosets). In particular $\mathrm{vol}(IxI) = |I/I \cap xIx^{-1}| \,\mathrm{vol}(I)$. If $x = \mathrm{diag}(p^a, p^b)$ then we have

$$xIx^{-1} = \begin{pmatrix} \mathbb{Z}_p^\times & p^{a-b}\mathbb{Z}_p \\ p^{1+b-a}\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix} \text{ and } I \cap xIx^{-1} = \begin{pmatrix} \mathbb{Z}_p^\times & p^{\max(0,a-b)}\mathbb{Z}_p \\ p^{1+\max(b-a,0)}\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}.$$

We easily deduce $|I/I \cap xIx^{-1}| = p^{|a-b|}$.

Before we consider elements in $\widetilde{W} \smallsetminus T/T_0$, define $\widetilde{w} = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$. This element is interesting because we have $\widetilde{w}I\widetilde{w}^{-1} = I$. For $x = \mathrm{diag}(p^a, p^b)w \in \widetilde{W} \smallsetminus T/T_0$ we have $x \in \mathrm{diag}(p^a, p^{b-1})\widetilde{w}T_0$ and we conclude $|I/I \cap xIx^{-1}| = p^{|a-b+1|}$.

Definition 2.53. *Let $l : I\backslash G/I \to \mathbb{Z}_{\geq 0}$ be the length function on $I\backslash G/I \simeq \widetilde{W}$, defined by $|I/I \cap xIx^{-1}| = p^{l(x)}$.*

Denote by $[IxI] \in \mathcal{H}(G, I)$ the element supported on IxI mapping x to $\mathrm{vol}(I)^{-1}$. Note that the action of $[IxI]$ on any smooth representation of G does not depend on the choice of a Haar measure on G, which appears both in the definition of the action and in the definition of $[IxI]$. For example the element $[IxI]$ acts on the trivial (one-dimensional) representation of G by

$$\mathrm{vol}(IxI)/\,\mathrm{vol}(I) = |I/I \cap xIx^{-1}| = p^{l(x)}.$$

Lemma 2.54. *For $x, y \in G$ we have*

$$[IxI][IyI] = \sum_{z \in I\backslash G/I} c(x, y, z)[IzI]$$

where $c(x, y, z) \in \mathbb{Z}_{\geq 0}$ is non-zero for only finitely many z, and $c(x, y, xy) > 0$.

Proof. For $x, y, g \in G$ we have

$$\begin{aligned} (1_{xI} * 1_{Iy})(g) &= \int_{xI} 1_{Iy}(h^{-1}g)dh \\ &= \int_{xI} 1_{g(y)^{-1}I}(h)dh \\ &= \mathrm{vol}(I) 1_{xIy}(g) \end{aligned}$$

and so $1_{xI} * 1_{Iy} = \mathrm{vol}(I) 1_{xIy}$. Using this formula we compute

$$[IxI][IyI] = \mathrm{vol}(I)^{-2} \sum_{k \in I/I \cap xIx^{-1}} \sum_{k' \in I \cap y^{-1}Iy \backslash I} 1_{kxI} * 1_{Iyk'} \tag{2.8}$$
$$= \mathrm{vol}(I)^{-1} \sum_{k \in I/I \cap xIx^{-1}} \sum_{k' \in I \cap y^{-1}Iy \backslash I} 1_{kxIyk'}$$

and so $[IxI][IyI]$ takes values in $\mathrm{vol}(I)^{-1}\mathbb{Z}_{\geq 0}$. Because $[IxI][IyI]$ is also I-invariant on both sides, it is a finite sum of finitely many $[IzI]$, including $z = xy$ at least once (consider $k = k' = 1$ in the sum above). □

The previous lemma is completely general and uses nothing particular about I. It implies that if $\mathrm{vol}(IxI) \mathrm{vol}(IyI) = \mathrm{vol}(I) \mathrm{vol}(IxyI)$, i.e. if $l(xy) = l(x) + l(y)$, then we have $[IxI][IyI] = [IxyI]$ (consider the action on the trivial representation). We will use this observation to obtain the following description of the structure of $\mathcal{H}(G, I)$.

Proposition 2.55. *Consider the elements $S = [IwI]$ and $T = [I\widetilde{w}I]$ of $\mathcal{H}(G, I)$.*

(1) *T is invertible in $\mathcal{H}(G, I)$.*

(2) *We have $T^2S = ST^2$ and $(S - p)(S + 1) = 0$.*

(3) *S, T and T^{-1} generate the algebra $\mathcal{H}(G, I)$, in fact for any $x \in G$ the element $[IxI]$ of $\mathcal{H}(G, I)$ can be written as a product of elements in $\{S, T, T^{-1}\}$.*

Proof. Since $\widetilde{w}$ normalizes I we have $T^2 = [I\widetilde{w}^2I]$ and $\widetilde{w}^2 = \mathrm{diag}(p, p)$ is central in G. This shows both that T^2 is central in $\mathcal{H}(G, I)$ and that T is invertible in $\mathcal{H}(G, I)$, with $T^{-1} = T[I\mathrm{diag}(p^{-1}, p^{-1})I]$. To compute S^2 we first observe that S^2, like S, is supported on the subgroup $K_0 = I \sqcup IwI$, and so we have $S^2 = \lambda e_I + \mu S$ for some $\lambda, \mu \in \mathbb{Z}_{\geq 0}$. To avoid computations we deduce from the action on the trivial representation the equality $p^2 = \lambda + p\mu$. To pin down λ and μ we resort to a computation similar to (2.8). For $x, y, g \in G$ we have

$$\begin{aligned}(1_{Ix} * 1_{yI})(g) &= \int_{Ix} 1_{yI}(h^{-1}g)dh \\ &= \int_{Ix} 1_{gIy^{-1}}(h)dh \\ &= \mathrm{vol}(Ixy \cap gI)\end{aligned}$$

Note that this function of g is clearly I-invariant on both sides, vanishes outside of $IxyI$, and so we have

$$1_{Ix} * 1_{yI} = \mathrm{vol}(I \cap y^{-1}x^{-1}Ixy) 1_{IxyI}.$$

For $x, y \in G$ we have

$$[IxI][IyI] = \mathrm{vol}(I)^{-2} \sum_{k \in I \cap x^{-1}Ix \backslash I} \sum_{k' \in I/I \cap yIy^{-1}} 1_{Ixk} * 1_{k'yI}.$$

Taking $x = y = w$ and evaluating at 1 we obtain

$$\lambda = |\{(k,k') \in (I/I \cap wIw) \times (I \cap wIw \backslash I) \mid wkk'w \in I\}|.$$

The set of elements

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix},\ u \in \{0, \dots, p-1\}$$

(or any other set of representatives of $\mathbb{Z}/p\mathbb{Z}$) is a set of representatives for both $I/I \cap wIw$ and $I \cap wIw \backslash I$. This allows us to conclude:

$$\lambda = \left|\left\{(u,u') \in \mathbb{F}_p^2 \;\middle|\; u + u' = 0\right\}\right| = p$$

and so $\mu = p - 1$.

We prove that for any $x \in \widetilde{W}$ the element $[IxI]$ can be written as a product of $T^{\pm 1}$'s and S's by induction on $l(x)$. It is clear from our computations just before Definition 2.53 that the elements of length 0 are exactly $Z \sqcup Z\widetilde{w}$, and in this case $[IxI]$ is a power of T. If $x \in \widetilde{W}$ is such that $l(x) > 0$, we distinguish two cases.

- If $x = \mathrm{diag}(p^a, p^b)$ then $a \neq b$. If $a < b$ then $l(xw) = b - a - 1 = l(x) - 1$, so that $l(x) = l(xw^{-1}) + l(w)$. This implies $[IxI] = [IxwI]S$. If $a > b$ then $l(wx) = a - b - 1 = l(x) - 1$ and similarly $[IxI] = S[IwxI]$.
- If $x \in \widetilde{W} \setminus T/T_0$ we may multiply x by $\widetilde{w}$ to reduce to the previous case.

$\square$

Remark 2.56. *This proposition holds with coefficients $\mathbb{Z}$ instead of $\mathbb{C}$, i.e. $\mathcal{H}(G,I)$ is naturally the extension of scalars from $\mathbb{Z}$ to $\mathbb{C}$ of a $\mathbb{Z}$-algebra, namely $\bigoplus_{x \in I\backslash G/I} \mathbb{Z}[IxI]$, in which the proposition holds. Although we will not need it, one can show that the proposition gives a full presentation of $\mathcal{H}(G,I)$. For generalizations see the original paper [IM65], and a more modern exposition [HKP10] using a different approach and including many other results, such as Bernstein's presentation.*

Now consider the $\mathcal{H}(G,I)$-modules $(\mathrm{Ind}_B^G \mu)^I$. We see functions in $\mathrm{Ind}_B^G \mu$ as functions on K_0.

From the decomposition $K_0 = I \sqcup N_0 w I$ (i.e. the Bruhat decomposition for $\mathrm{GL}_2(\mathbb{F}_p)$) and the fact that $T_0 \subset I$ is normalized by w, it follows that $(\mathrm{Ind}_B^G \mu)^I = 0$ if $\mu|_{T_0} \neq 1$. If $\mu|_{T_0} = 1$ we get an isomorphism $(\mathrm{Ind}_B^G \mu)^I \to \mathbb{C}^2$, $f \mapsto (f(1), f(w))$. Let us compute the matrices of S and T in the corresponding basis $\mathcal{B} = (b_1, b_w)$ of $(\mathrm{Ind}_B^G \mu)^I$.

We have

$$(\widetilde{w}f)(1) = f\left(\begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}\right) = f\left(\begin{pmatrix} -1 & 0 \\ 0 & p \end{pmatrix} w\right) = \mu_2(p)p^{1/2} f(w)$$

and

$$(\widetilde{w}f)(w) = f\left(\begin{pmatrix} -p & 0 \\ 0 & 1 \end{pmatrix}\right) = \mu_1(p)p^{-1/2} f(1)$$

so the matrix of T is $\begin{pmatrix} 0 & \mu_2(p)p^{1/2} \\ \mu_1(p)p^{-1/2} & 0 \end{pmatrix}$.

We have
$$([IwI]f)(1) = \sum_{y\in\mathbb{F}_p} f\left(\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix} w\right) = pf(w)$$
and
$$([IwI]f)(w) = \sum_{y\in\mathbb{F}_p} f\left(w\begin{pmatrix}1 & y\\ 0 & 1\end{pmatrix} w\right) = \sum_{y\in\mathbb{F}_p} f\left(\begin{pmatrix}-1 & 0\\ y & -1\end{pmatrix}\right) = f(1) + (p-1)f(w)$$
since $\begin{pmatrix}-1 & 0\\ y & -1\end{pmatrix} \in K_0$ and it belongs to I if and only if $y \in p\mathbb{Z}_p$. So the matrix of S is $\begin{pmatrix}0 & p\\ 1 & p-1\end{pmatrix}$.

Let $Q = \begin{pmatrix}1 & -p\\ 1 & 1\end{pmatrix}$, chosen because we have $Q^{-1}\mathrm{Mat}(S,\mathcal{B})Q = \mathrm{diag}(p,-1)$. We compute
$$Q^{-1}\mathrm{Mat}(T,\mathcal{B})Q = \frac{p^{1/2}}{p+1}\begin{pmatrix}\mu_1(p)+\mu_2(p) & \mu_2(p)-p\mu_1(p)\\ p^{-1}\mu_1(p)-\mu_2(p) & -\mu_1(p)-\mu_2(p)\end{pmatrix}. \tag{2.9}$$
We recover the fact that $\mathrm{Ind}_B^G\,\mu$ is reducible if $(\mu_1/\mu_2)(p) \in \{p^{\pm 1}\}$. More importantly, specializing to $\mu_1(p) = p^{-1/2}$ and $\mu_2(p) = p^{1/2}$ we see that (2.9) specializes to $\begin{pmatrix}1 & 0\\ p^{-1}-1 & -1\end{pmatrix}$ and we obtain that St^I is a line on which S and T both act by -1.

Proposition 2.57. *The Steinberg representation* St *is square-integrable.*

Proof. By Lemma 2.43 it is enough to check that one non-zero matrix coefficient is square-integrable. Let $v \in \mathrm{St}^I$ and $\widetilde{v} \in \widetilde{\mathrm{St}}^I$ (both lines) be such that $\langle v, \tilde{v}\rangle = 1$. Then
$$\int_{G/Z} |\langle \mathrm{St}(g)v, \tilde{v}\rangle|^2 dg = \sum_{x\in\widetilde{W}/Z} |\langle \mathrm{St}(x)v, \widetilde{v}\rangle|^2 \,\mathrm{vol}(IxIZ/Z).$$
Note that we have $\mathrm{vol}_{G/Z}(IxIZ/Z) = \mathrm{vol}_Z(Z\cap I)^{-1}\,\mathrm{vol}_G(IxI)$ (exercise!). Moreover for any $k, k' \in I$ we have $\langle \mathrm{St}(kxk')v, \widetilde{v}\rangle = \langle \mathrm{St}(x)v, \widetilde{v}\rangle$ and so we have
$$\langle \mathrm{St}([IxI])v, \widetilde{v}\rangle = \mathrm{vol}(IxI)\,\mathrm{vol}(I)^{-1}\langle \mathrm{St}(x)v, \widetilde{v}\rangle.$$
This allows us to express the above integral as
$$\mathrm{vol}_Z(Z\cap I)^{-1}\,\mathrm{vol}(I) \sum_{x\in\widetilde{W}/Z} |\langle \mathrm{St}([IxI])v, \widetilde{v}\rangle|^2 p^{-l(x)}.$$

Using Proposition 2.55 we see that we have $\langle \mathrm{St}([IxI])v, \widetilde{v}\rangle \in \{\pm 1\}$ for any $x \in \widetilde{W}$. Using the fact that there are exactly two elements of $\widetilde{W}/Z$ of a given length in $\mathbb{Z}_{\geq 0}$, the above integral equals
$$2\,\mathrm{vol}_Z(Z\cap I)^{-1}\,\mathrm{vol}(I)\sum_{a\geq 0} p^{-a} < \infty.$$
∎

Remark 2.58. *A similar argument shows that the trivial irreducible representation of G is not square-integrable.*

Proposition 2.59. *If (V,π) is a supercuspidal representation of G then $V^I = 0$.*

Proof. We go back to the computation to prove (3) implies (1) in Theorem 2.38, this time with the Iwahori subgroup I instead of a very small K_i. Let $v \in V^I$, then for large enough $a \in \mathbb{Z}$ we have $\pi(e_I)\pi(\mathrm{diag}(p^a,1))v = 0$. The vector v is invariant under I so we have

$$\pi([I\mathrm{diag}(p^a,1)I])v = \frac{\mathrm{vol}(I\mathrm{diag}(p^a,1)I)}{\mathrm{vol}(I)}\pi(e_I)\pi(\mathrm{diag}(p^a,1))v = 0.$$

In $\mathcal{H}(G,I)$ we have $[I\mathrm{diag}(p^a,1)I] = (ST)^a$ as soon as $a \geq 0$, so we obtain that the action of ST on V^I is nilpotent. But ST is invertible in $\mathcal{H}(G,I)$: we already know that T is invertible, and $S(S-p+1) = pe_I$. □

Remark 2.60. *Pushing this argument further, one can prove Casselman's criterion for square-integrability (see* [Cas, Theorem 4.4.6] *or* [Ren10, Théorème VII.1.2]*): one can read whether a representation of G is square-integrable on its Jacquet module. This generalizes to arbitrary connected reductive groups as well (but classifying representations as in Theorem 2.32 becomes a very complicated combinatorial problem for general groups).*

2.8. The unramified Hecke algebra and the Satake isomorphism.

Lemma 2.61. *Fix Haar measures on G and B. The map*

$$\begin{aligned} \mathcal{H}(G,K_0) &\longrightarrow \mathcal{H}(B, B\cap K_0) \\ f &\longmapsto \frac{\mathrm{vol}(K_0)}{\mathrm{vol}(B\cap K_0)} f|_B \end{aligned}$$

is a morphism of Hecke algebras.

Proof. First note (exercise) that $\mathcal{H}(B)$ is indeed an associate algebra for the convolution product defined using a left Haar measure, even though B is not unimodular.

We use the BK_0 integration formula 2.13. For $f_1, f_2 \in \mathcal{H}(G,K_0)$ and $b \in B$,

$$\begin{aligned} (f_1 * f_2)(b) &= \int_G f_1(g) f_2(g^{-1}b)\,dg \\ &= \int_B \int_{K_0} f_1(ak) f_2(k^{-1}a^{-1}b)\,dk\,da \\ &= \mathrm{vol}_{K_0}(K_0) \int_B f_1(a) f_2(a^{-1}b)\,da \end{aligned}$$

where vol_{K_0} is the volume with respect to the chosen Haar measure on K_0. Recall that the Haar measures on G, K_0 and B are chosen to be compatible for the integration formula, and this compatibility is equivalent to $\mathrm{vol}_B(B\cap K_0)\,\mathrm{vol}_{K_0}(K_0) = \mathrm{vol}_G(K_0)$ (apply the integration formula to the characteristic function of K_0). Multiplying both sides of the above equation by $\mathrm{vol}_{K_0}(K_0)$ shows that the map $\mathcal{H}(G,K_0) \to \mathcal{H}(B, B\cap K_0)$ preserves $*$. □

Lemma 2.62. *Choose Haar measures on T and N, determining a left Haar measure on $B = TN$ (the product measure). Then the map $\phi : \mathcal{H}(B) \to \mathcal{H}(T)$, $f \mapsto \left(t \mapsto \int_N f(tn)dn\right)$ is a morphism of algebras.*

Proof. We compute $\phi(f_1 * f_2)(t)$ for $t \in T$:

$$\begin{aligned}\int_N \int_B f_1(b) f_2(b^{-1}tn)\, db\, dn &= \int_N \int_T \int_N f_1(xu) f_2(u^{-1}x^{-1}tn)\, du\, dx\, dn \\ &= \int_T \int_N \int_N f_1(xu) f_2(x^{-1}t(t^{-1}xu^{-1}x^{-1}t)n)\, dn\, du\, dx \\ &= \int_T \int_N f_1(xu)\phi(f_2)(x^{-1}t)\, du\, dx \\ &= \int_T \phi(f_1)(x)\phi(f_2)(x^{-1}t)\, dx.\end{aligned}$$

□

In particular we obtain by composition a morphism of unital algebras $\mathcal{H}(G, K_0) \to \mathcal{H}(T, T_0) \simeq \mathbb{C}[T/T_0]$ (a group algebra because T is commutative). For reasons explained below, it is useful to twist this morphism by $\delta_B^{1/2}$.

Definition 2.63. *Normalize the Haar measures on G, T and N so that K_0, T_0 and N_0 all have measure 1. The Satake transform* $\mathrm{Sat} : \mathcal{H}(G, K_0) \to \mathcal{H}(T, T_0) = \mathbb{C}[T/T_0]$ *is the morphism of unital algebras defined by* $\mathrm{Sat}(f)(t) = \delta_B^{1/2}(t) \int_N f(tn)dn$.

Theorem 2.64 (Satake)**.** *The Satake transform takes values in $\mathbb{C}[T/T_0]^W$ and induces an isomorphism $\mathcal{H}(G, K_0) \simeq \mathbb{C}[T/T_0]^W$.*

Proof. The fact that the image of Sat is contained in $\mathbb{C}[T/T_0]^W$ will be proved later (Lemma 3.4), since it is natural to use orbital integrals for this (of course there will be no circular argument ...). Note that this invariance property is the reason for the normalisation by $\delta_B^{1/2}$.

Granting this, we are left to show that the image of Sat contains $\mathbb{C}[T/T_0]^W$ and that Sat is injective. By the Cartan decomposition the characteristic functions $f_{a,b}$ of the sets $K_0 \mathrm{diag}(p^a, p^b) K_0$ with $a \geq b$ form a basis of $\mathcal{H}(G, K_0)$. Similarly we have a basis $e_{a,b} = [\mathrm{diag}(p^a, p^b)] + [\mathrm{diag}(p^b, p^b)]$ of $\mathbb{C}[T/T_0]$. Write $\mathrm{Sat}(f_{a,b}) = \sum_{a' \geq b'} \lambda(a, b, a', b') e_{a',b'}$. It is clear that $\lambda(a, b, a, b) \in \mathbb{R}_{>0}$, and that $\lambda(a, b, a', b') = 0$ if $a' + b' \neq a + b$ (consider determinants).

For $t = \mathrm{diag}(x, y)$ and $n \in N$ the product tn belongs to a unique double coset $K_0 \mathrm{diag}(p^a, p^b) K_0$ with $a \geq b$. By Lemma 2.65 below we have $a - b \geq |v(x) - v(y)|$. This implies that for any $d \in \mathbb{Z}$ and $c \in \mathbb{Z}_{\geq 0}$ we have

$$\mathrm{Sat}\Big(\bigoplus_{\substack{a \geq b \\ a+b=d \\ a-b \leq c}} \mathbb{C} f_{a,b} \Big) \subset \bigoplus_{\substack{a \geq b \\ a+b=d \\ a-b \leq c}} \mathbb{C} e_{a,b}. \tag{2.10}$$

With this inclusion and the observation that $\lambda(a, b, a, b) \neq 0$, one easily shows by induction on $c \geq 0$ that the inclusion (2.10) is an equality, showing that Sat is surjective. Comparing dimensions (or using a similar induction) we obtain that Sat is also injective. □

Lemma 2.65. *Let $t = \mathrm{diag}(x, y) \in T$ and $c \geq 0$, then for $n = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ we have $tn \in K_0 \mathrm{diag}(p^a, p^b) K_0$ for some a, b satisfying $b + c \geq a \geq b$ if and only if*

$$|v(x) - v(y)| \leq c \quad \text{and} \quad v(u) \geq \frac{-c - v(x) + v(y)}{2}.$$

Proof. Let (a, b) be the unique pair of integers satisfying

$$a \geq b \ \text{ and } \ tn \in K_0 \mathrm{diag}(p^a, p^b) K_0.$$

We have to prove the equality

$$a - b = \max\left(|v(x) - v(y)|, -2v(u) - v(x) + v(y)\right). \tag{2.11}$$

First we assume that we have $v(x) \geq v(y)$.

- If $v(u) \geq v(y) - v(x)$ then

  $$tn = \begin{pmatrix} 1 & uxy^{-1} \\ 0 & 1 \end{pmatrix} t$$

  belongs to $K_0 t$ and we have

  $$-2v(u) - v(x) + v(y) \leq v(x) - v(y)$$

  so the equality (2.11) holds with $a - b = v(x) - v(y)$.

- If $v(u) < v(y) - v(x)$ we write

  $$\begin{pmatrix} u^{-1}x^{-1}y & -1 \\ 1 & 0 \end{pmatrix} tn \begin{pmatrix} 1 & 0 \\ -u^{-1} & 1 \end{pmatrix} = \begin{pmatrix} u^{-1}y & 0 \\ 0 & ux \end{pmatrix}.$$

  We have $v(-u^{-1}) \geq v(u^{-1}x^{-1}y) > 0$ so $\mathrm{diag}(u^{-1}y, ux)$ belongs to $K_0 tn K_0$. Furthermore we have

  $$v(u^{-1}y) - v(ux) = -2v(u) + v(y) - v(x) > v(x) - v(y) \geq 0$$

  so we have found the double coset appearing in the Cartan decomposition in which tn lies and we have

  $$a - b = -2v(u) + v(y) - v(x).$$

  We have already observed above the inequality

  $$-2v(u) + v(y) - v(x) > v(x) - v(y)$$

  so the equality (2.11) holds.

We are left to prove (2.11) in the case where $v(x) < v(y)$. To avoid repeating computations we observe that the Cartan decomposition is stable under transposition and we write

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x & 0 \\ xu & y \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} y & xu \\ 0 & x \end{pmatrix} = \begin{pmatrix} x' & x'u' \\ 0 & y' \end{pmatrix}$$

where $x' = y$, $y' = x$ and $u' = y^{-1}xu$. Thanks to the equalities

$$\begin{aligned} |v(x) - v(y)| &= |v(x') - v(y')| \\ \text{and } \quad -2v(u) - v(x) + v(y) &= -2v(u') - v(x') + v(y') \end{aligned}$$

we are reduced to the previous case. □

Remark 2.66. (1) *In particular, $\mathcal{H}(G, K_0)$ is commutative. This can also be proved by observing that the anti-automorphism $g \mapsto {}^t g$ of G preserves the Cartan decomposition.*

(2) *One can check that the Satake isomorphism can be defined over $\mathbb{Z}[p^{1/2}]$, and is still an isomorphism over this ring.*

Definition 2.67. *We say that an irreducible smooth representation (V, π) is unramified if $V^{K_0} \neq 0$.*

The Satake isomorphism gives a simple description of all unramified representations of G: by Lemma 2.10 they correspond bijectively to $\mathbb{C}$-algebra morphisms $\mathcal{H}(G, K_0) \to \mathbb{C}$ (any simple finite-dimensional $\mathcal{H}(G, K_0)$-module is one-dimensional since $\mathcal{H}(G, K_0)$ is commutative). More precisely, writing $\mathbb{C}[T/T_0] = \mathbb{C}[X_1^{\pm 1}, X_2^{\pm 2}]$ where X_1 (resp. X_2) corresponds to $\mathrm{diag}(p, 1)$ (resp. $\mathrm{diag}(1.p)$), we have $\mathbb{C}[T/T_0]^W = \mathbb{C}[X_1 + X_2, (X_1 X_2)^{\pm 1}]$. Therefore characters of $\mathcal{H}(G, K_0)$ are parametrized by pairs $(x_1, x_2) \in (\mathbb{C}^\times)^2$ up to permutation $(x_1, x_2) \mapsto (x_2, x_1)$.

Proposition 2.68. *Any unramified representation of G is isomorphic to*

- $\chi \circ \det$ *for some unramified character $\mathbb{Q}_p^\times \to \mathbb{C}^\times$, or*
- $\mathrm{Ind}_B^G \mu$ *for some unramified character $\mu = \mu_1 \otimes \mu_2$ such that $\mu_1(p)/\mu_2(p) \notin \{p^{\pm 1}\}$.*

Proof. Since $I \subset K_0$, Proposition 2.59 implies that for any supercuspidal (V, π) we have $V^{K_0} = 0$. By the classification theorem 2.32, we are left to consider $(\mathrm{Ind}_B^G \mu)^{K_0}$. Using the Iwasawa decomposition, we see that this space vanishes if μ is ramified, and is one-dimensional if μ is unramified. □

The explicit comparison of the two classifications of unramified representations (i.e. the relation $x_i = \mu_i(p)$ up to the action of the Weyl group) is left as an exercise.

Remark 2.69. (1) *The definition of the Satake morphism and proof that it is an isomorphism are easier than the complete classification. This becomes even more true for groups more complicated than GL_2.*

(2) *The phenomenon that $\mathrm{Ind}_B^G \mu$ is reducible in exceptional cases is not visible on the Satake isomorphism.*

3. Harmonic analysis

We start doing harmonic analysis in the following sense: relating conjugacy classes in G (more precisely, orbital integrals of functions on G, defined below) to traces $\operatorname{tr}\pi(f)$ for π an admissible representation of G and $f \in \mathcal{H}(G)$ (which makes sense because the image of $\pi(f)$ has finite dimension).

3.1. **Conjugacy classes in G.** The classification of conjugacy classes in G is a special case of the classification of $\mathrm{GL}_n(k)$-orbits under conjugation on $\mathrm{M}_n(k)$ where k is a commutative field (deduced from the structure theorem for finitely generated $k[X]$-modules, in the special case of torsion modules). There are four types of conjugacy classes:

- central elements, i.e. Z,
- non-semisimple elements, i.e. elements conjugated to $z\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ for some (uniquely determined) $z \in Z$,
- *hyperbolic* (or *split*) semisimple regular elements, conjugated to $\operatorname{diag}(x, y) \in T$ for $x \neq y$, uniquely determined up to the action of $N_G(T)/T = W$, that is up to $(x, y) \mapsto (y, x)$,
- *elliptic* semisimple regular elements, determined by an irreducible characteristic polynomial $X^2 + aX + b \in \mathbb{Q}_p[X]$ with $b \neq 0$ (explicitly, take the companion matrix). These can be grouped according to the quadratic extension E of $\mathbb{Q}_p$ splitting this polynomial as follows. Choose an isomorphism of $\mathbb{Q}_p$-vector spaces $\psi : \mathbb{Q}_p^2 \simeq E$, then any $x \in E^\times$ defines $m_x \in \operatorname{Aut}_{\mathbb{Q}_p}(E)$ (multiplication by x), and $\psi^{-1} \circ m_x \circ \psi \in G$ is elliptic semisimple regular if and only if $x \in E \smallsetminus \mathbb{Q}_p$. The subgroup $T' = \{\psi^{-1} \circ m_x \circ \psi \,|\, x \in E\}$ of G is called an anisotropic (or elliptic) maximal torus of G. Note that $T'/Z \simeq E^\times/\mathbb{Q}_p^\times$ is compact. Denoting $\operatorname{Gal}(E/\mathbb{Q}_p) = \{1, \sigma\}$, it is easy to check that $N_G(T')/T' = \mathbb{Z}/2\mathbb{Z}$, the non-trivial element being represented by $\psi^{-1} \circ \sigma \circ \psi$.

We denote by G_{rs} the set of semisimple regular elements of G. For T' a maximal torus of G (elliptic or conjugated to our "standard" split torus T) we will denote by $T'_{G-\mathrm{reg}} = T' \smallsetminus Z$ the subset of regular elements.

Central or non-semisimple elements of G (i.e. $G \smallsetminus G_{\mathrm{rs}}$) form a closed subset of G (in fact, Zariski-closed because they are the solutions of the equation $\operatorname{tr}^2 = 4\det$) of measure 0. Indeed, Z is a sub-p-adic manifold of G of dimension $1 < 4 = \dim G$, and it is easy to check that the differential of $\operatorname{tr}^2 - 4\det$ does not vanish at any point of $G \smallsetminus (Z \cup G_{\mathrm{rs}})$ so this subset of G is a submanifold of dimension 3.

For $g \in G$ let $D(g) = 4 - \det(g)^{-1} \operatorname{tr}(g)^2$, so that $G \smallsetminus G_{\mathrm{rs}}$ is also the vanishing locus of D. It is not difficult to compute that for T' a maximal torus of G and $g \in T'$ we have

$$D(g) = \det\left(1 - \operatorname{Ad}(g) \,|\, \operatorname{Lie}(G)/\operatorname{Lie}(T')\right).$$

3.2. Orbital integrals.

Definition 3.1. *For $\gamma \in G$ and $f \in C_c^\infty(G)$, define the orbital integral of f at γ as*

$$O_\gamma(f) := \int_{G_\gamma \backslash G} f(g^{-1}\gamma g)\, d\dot{g}$$

where G_γ is the centralizer of γ in G, provided the integral converges absolutely.

Remark 3.2. (1) *We have a well-defined right G-invariant quotient measure on $G_\gamma \backslash G$ because G and G_γ are both unimodular (we will give a "differential" definition of this measure in the proof of Theorem 3.12). Note that the orbital integral depends on choices of Haar measures on G and G_γ. Via the bijection $G_\gamma\backslash G \simeq G/G_\gamma$, $G_\gamma g \mapsto g^{-1}G_\gamma$, the quotient measures are identified and so we also have $O_\gamma(f) = \int_{G/G_\gamma} f(g\gamma g^{-1})\, d\dot{g}$.*

(2) *For $g \in G$, denoting $f^g : h \mapsto f(ghg^{-1})$, we have $O_\gamma(f^g) = O_\gamma(f)$.*

(3) *If $h \in G$ then, using the isomorphism $\mathrm{Ad}(h) : G_{h^{-1}\gamma h} \to G_\gamma$ to match Haar measures on these two groups, we have $O_\gamma(f) = O_{h^{-1}\gamma h}(f)$: use the measure-preserving bijection $G_\gamma\backslash G \simeq G_{h^{-1}\gamma h}\backslash G$, $g \mapsto h^{-1}g$.*

The integrand in the definition of an orbital integral is clearly smooth. If γ is semisimple we will show that the integrand is also compactly supported (Lemma 3.5 below). More precisely, let K be a compact open subgroup such that f is bi-K-invariant. We will show that there are only finitely many double cosets $[g] = G_\gamma g K \subset G$ such that $g^{-1}\gamma g$ belongs to the support of f, and so the integrand in Definition 3.1 is smooth and compactly supported. By the calculation of the quotient measure in Example A.5, we have

$$O_\gamma(f) = \sum_{[g]\in G_\gamma\backslash G/K} f(g^{-1}\gamma g)\frac{\mathrm{vol}_G(K)}{\mathrm{vol}_{G_\gamma}(G_\gamma \cap gKg^{-1})}. \tag{3.1}$$

Note that these statements are trivial if γ is central, so we will consider semi-simple regular γ's.

First look at the case where γ is regular semisimple hyperbolic. Up to conjugacy we may assume that $\gamma \in T$, so that $G_\gamma = T$.

Lemma 3.3. *Let C_G be a compact subset of G. Let C_T be a compact subset of $T_{G-\mathrm{reg}}$.*

(1) *There exists a compact subset X of N such that for any $\gamma \in C_T$, $n \in N \smallsetminus X$ and $k \in K_0$ we have $(nk)^{-1}\gamma nk \notin C_G$.*

(2) *The set of $Tg \in T\backslash G$ such that there exists $\gamma \in C_T$ for which $g^{-1}\gamma g$ belongs to C_G is compact.*

Proof. (1) For $\gamma = \mathrm{diag}(x, y) \in C_T$ and $n = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \in N$ we have

$$n^{-1}\gamma n = \gamma \begin{pmatrix} 1 & (1-y/x)u \\ 0 & 1 \end{pmatrix}$$

so if $(nk)^{-1}\gamma nk \in C_G$ for some $k \in K_0$ we have

$$\begin{pmatrix} 1(1-y/x)u & \\ 0 & 1 \end{pmatrix} \in C_T^{-1} K_0 C_G K_0$$

and so $(1-y/x)u$ belongs to a compact subset of $\mathbb{Q}_p$. On C_T the valuation of $1-y/x$ is bounded (in fact it only takes finitely many values) and so u belongs to a compact subset of $\mathbb{Q}_p$.

(2) The set in question is easily seen to be a closed subset of $T\backslash G$. By the Iwasawa decomposition $G = TNK_0$ and the previous point it is contained in a compact subset of $T\backslash G$, namely the image of XK_0.

□

This suggests calculating the orbital integral in this case using the integration formula for the Iwasawa decomposition given in Lemma 2.13. Normalizing Haar measures as in this lemma, it is clear that the quotient measure on $T\backslash G$ (both groups are unimodular so this is a special case of Example A.5) can be computed as

$$F \in C_c(T\backslash G) \longmapsto \int_{N\times K_0} F(nk)dndk.$$

So for $\gamma \in T_{G-\mathrm{reg}}$ and $f \in C_c^\infty(G)$ we have

$$O_\gamma(f) = \int_{N\times K_0} f(k^{-1}n^{-1}\gamma nk)\ dn\, dk.$$

As above denote $\gamma = \mathrm{diag}(x, y)$ and $n = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$, so that we have

$$n^{-1}\gamma n = \gamma \begin{pmatrix} 1 & (1-y/x)u \\ 0 & 1 \end{pmatrix}.$$

Using the change of variable $u' = (1-y/x)u$ we obtain

$$\begin{aligned} O_\gamma(f) &= |1-y/x|^{-1} \int_{N\times K_0} f(k^{-1}\gamma n' k)\, dn'\, dk \\ &= |x/y|^{1/2}|x/y-1|^{-1/2}|y/x-1|^{-1/2} \int_{N\times K_0} f(k^{-1}\gamma nk)\, dn'\, dk \\ &= |D(\gamma)|^{-1/2}\delta_B^{1/2}(\gamma) \int_{N\times K_0} f(k^{-1}\gamma nk)\, dn'\, dk. \qquad (3.2) \end{aligned}$$

We recognize a generalization of the formula defining the Satake morphism. This allows us to prove the outstanding claim in Theorem 2.64.

Lemma 3.4. *For any $f \in \mathcal{H}(G, K_0)$, $\mathrm{Sat}(f) \in \mathcal{H}(T, T_0)$ is invariant under the Weyl group $W = \{1, w\}$ of T.*

Proof. Any coset in T/T_0 contains a regular element γ, and $\mathrm{Sat}(f)(\gamma) = |D(\gamma)|^{1/2}O_\gamma(f)$ by 3.2. By the third point in Remark 3.2, this is invariant under w. □

Lemma 3.5. *Let T' be a maximal torus of G. Let $C_{T'}$ be a compact subset of $T'_{G-\mathrm{reg}}$. Let C_G be a compact subset of G. The set of $g \in T'\backslash G$ such that there exists $\gamma \in C_{T'}$ for which $g^{-1}\gamma g$ belongs to C_G is compact.*

In particular for any $f \in C_c^\infty(G)$ we have:

- *for any semisimple $\gamma \in G$ the sum on the right-hand side of (3.1) has finitely many non-zero terms,*
- *for any maximal torus T' of G the map*

$$\begin{aligned} T'_{G-\mathrm{reg}} &\longrightarrow \mathbb{C} \\ \gamma &\longmapsto O_\gamma(f) \end{aligned}$$

  is smooth.

Proof. If T' is split then up to conjugating by an element of G we may assume that T' is the diagonal torus T, in which case the first statement is Lemma 3.3. So assume that T' is elliptic. There exists a quadratic field extension E of $\mathbb{Q}_p$ such that the characteristic polynomial of any element of T' splits over E. In fact E is unique up to isomorphism and can be taken to be the sub-$\mathbb{Q}_p$-algebra of $M_2(\mathbb{Q}_p)$ generated by the elements of T', but it will be clearer to keep it abstract. Let G_E be $\mathrm{GL}_2(E)$, which contains G as a subgroup. Let T'_E be the centralizer of T' in G_E, which is also the group of invertible elements in the sub-E-vector space of $M_2(E)$ generated by the elements of T'. Note that this vector space is two-dimensional and generated by 1 and any element of $T'_{G-\mathrm{reg}}$. There exists $h \in G_E$ such that hT'_Eh^{-1} is the subgroup of diagonal matrices in G_E. Let N'_E be the subgroup

$$h^{-1}\begin{pmatrix} 1 & E \\ 0 & 1 \end{pmatrix} h$$

of G_E. It is normalized by T'_E, and it depends on the choice of h but the choice will not matter for the argument (as long as h is fixed). Denote $\mathrm{Gal}(E/\mathbb{Q}_p) = \{1, \sigma\}$. We simply denote by σ the obvious action on G_E, which leaves G fixed pointwise. The choice of h gives an isomorphism $\phi : T'_E \simeq (E^\times)^2$: if $\phi(t) = (t_1, t_2)$ then $hth^{-1} = \mathrm{diag}(t_1, t_2)$. For $\gamma \in T'_{G-\mathrm{reg}}$ we have $\phi(\gamma) = (x, \sigma(x))$ for some $x \in E^\times \smallsetminus \mathbb{Q}_p^\times$. Using the above description of T'_E and the equality $\sigma(\gamma) = \gamma$ we see that for any $t \in T'_E$, denoting $\phi(t) = (t_1, t_2)$, we have

$$\phi(\sigma(t)) = (\sigma(t_2), \sigma(t_1)).$$

Let K_E be a maximal compact subgroup of G_E (for example $\mathrm{GL}_2(\mathcal{O}_E)$; the proof of Lemma 2.1 generalizes to G_E). The proof of Lemma 2.2 (Iwasawa decomposition) also generalizes, so we have $G_E = T'_E N'_E K_E$. The proof of Lemma 3.3 also generalizes: there is a compact subset X of N'_E such that for any $n \in N'_E \smallsetminus X$, $\gamma \in C_{T'}$ and $k \in K_E$ we have $(nk)^{-1}\gamma nk \notin C_G$.

Let P be the set of $(t, n, k) \in T'_E \times N'_E \times K_E$ such that

- $g = tnk$ belongs to G, i.e. it is fixed by σ, and
- $g^{-1}\gamma g$ belongs to C_G for some $\gamma \in C_{T'}$.

For $(t,n,k) \in P$ we have $n \in X$, and so

$$t^{-1}\sigma(t) = nk\sigma(k)^{-1}\sigma(n)^{-1}$$

belongs to

$$T'_E \cap \left(XK_E\sigma(K_E)^{-1}\sigma(X)^{-1}\right),$$

a compact subset of T'_E. There exists $c \in \mathbb{R}_{\geq 0}$ such that for any $\mathrm{diag}(x,y)$ in this compact subset of T'_E we have $|v(x)| \leq c$. For $t \in T'_E$, denoting $\phi(t) = (t_1, t_2)$ we have

$$\phi(t^{-1}\sigma(t)) = (t_1^{-1}\sigma(t_2), t_2^{-1}\sigma(t_1)).$$

In particular if $(t,n,k) \in P$ then we have $|v(t_1) - v(t_2)| \leq c$. In particular, because $v(E^\times)$ is a subgroup of $\frac{1}{2}\mathbb{Z}$, up to multiplying t on the left by an element of Z we may assume that we have

$$0 \leq v(t_1) \leq c + 1/2 \text{ and } 0 \leq v(t_2) \leq c + 1/2,$$

which defines a compact subset Y of T'_E. We have shown that for $g \in G$ such that $g^{-1}\gamma g$ belongs to C_G for some $\gamma \in C_{T'}$ there exists $z \in Z$ such that zg belongs to the compact subset $G \cap (YXK_E)$ of G. In particular the set

$$\left\{Zg \in Z\backslash G \,\middle|\, \exists \gamma \in C_{T'},\, g^{-1}\gamma g \in C_G\right\}$$

is relatively compact in $Z\backslash G$, and since it is clearly closed in $Z\backslash G$ it is simply compact.

The remaining claims in the lemma are simple consequences of the first claim and are left as an exercise. □

Remark 3.6. (1) *A similar argument works for semisimple elements in arbitrary reductive groups, see* [HC70, Lemma 19, p.52].

(2) *For non-semisimple elements it is not true that the right-hand side of* (3.1) *has finitely many non-vanishing terms, but the integral defining the orbital integral does converge absolutely. For* GL_2 *this is an exercise; for arbitrary reductive groups it is a theorem of Ranga Rao and Deligne (see* [RR72]*).*

(3) *For ω a smooth character of Z, the same arguments apply to orbital integrals of smooth ω-equivariant functions on G which have compact support modulo Z.*

Lemma 3.7. *Let $f \in C_c^\infty(G)$. Let T' be a maximal torus in G. Then the support of*

$$\begin{aligned} \phi : T'_{G-\mathrm{reg}} &\longrightarrow \mathbb{C} \\ t &\longmapsto O_t(f) \end{aligned}$$

is relatively compact in T'. If the support of f is contained in G_{rs} then the support of ϕ is compact.

Proof. The map $\Xi = (\mathrm{tr}, \det) : G \to \mathbb{Q}_p \times \mathbb{Q}_p^\times$ is continuous. Let us show that the restriction $\Xi|_{T'}$ of Ξ to T' is proper. If T' is elliptic then T' can be identified with $E^\times$ for some quadratic extension $E/\mathbb{Q}_p$ and via this identification the determinant $T' \to \mathbb{Q}_p^\times$ is given by the norm $N_{E/\mathbb{Q}_p} : E^\times \to \mathbb{Q}_p^\times$. This norm map is proper (consider valuations) so $\det|_{T'}$ is proper and $\Xi_{T'}$ is proper. If T' is split we have an identification $T' \simeq \mathbb{Q}_p^\times \times \mathbb{Q}_p^\times$ and the restriction of Ξ to T' is identified with the map

$$\begin{aligned} \mathbb{Q}_p^\times \times \mathbb{Q}_p^\times &\longrightarrow \mathbb{Q}_p \times \mathbb{Q}_p^\times \\ (x, y) &\longmapsto (x + y, xy). \end{aligned}$$

Any compact subset in $\mathbb{Q}_p \times \mathbb{Q}_p^\times$ is contained in the union of finitely many compact open subsets of the form $p^a\mathbb{Z}_p \times p^b\mathbb{Z}_p^\times$. If $x, y \in \mathbb{Q}_p^\times$ are such that $x + y \in p^a\mathbb{Z}_p$ and $xy \in p^b\mathbb{Z}_p^\times$ then either x and y both have valuation $b/2$, in which case they lie in the compact subset $p^{b/2}\mathbb{Z}_p^\times$ of $\mathbb{Q}_p^\times$, or they have distinct valuations and we have $a = v(x + y) = \min(v(x), v(y))$, which implies

$$a \leq \min(v(x), v(y)) \leq \max(v(x), v(y)) \leq b - a.$$

This shows that the closed subset $(\Xi|_{T'})^{-1}(p^a\mathbb{Z}_p \times p^b\mathbb{Z}_p^\times)$ of T is compact.

Now ϕ vanishes away from the compact subset $(\Xi_{T'})^{-1}(\Xi(\mathrm{supp} f))$ of T'. If f is supported on G_{rs} then this compact subset is contained in

$$\Xi^{-1}(\{(t, d) \,|\, t^2 \neq 4d\}) = G_{\mathrm{rs}}.$$

□

Orbital integrals will show up naturally in the trace formula. But right now we will compute these in a special case. This will allow us to estimate them in general, and such estimates will be useful when we study characters of admissible representations of G.

Let γ be a semisimple regular element of G. Assume that γ is compact, i.e. the sequence $(\gamma^n)_{n \in \mathbb{Z}}$ is bounded, equivalently its closure is compact. Equivalently, γ is conjugated to an element of K_0 (by Lemma 2.1). We will compute $O_\gamma(e_{K_0})$.

If γ is hyperbolic, i.e. if its eigenvalues are in $\mathbb{Q}_p$, then we may assume that $\gamma \in T_0$ and so $G_\gamma = T$. Since e_{K_0} is bi-K_0-invariant Formula (3.2) gives $O_\gamma(e_{K_0}) = |D(\gamma)|^{-1/2} \operatorname{vol}(T_0)^{-1}$.

Consider now the case where γ is elliptic, i.e. $E := \mathbb{Q}_p[\gamma] \subset M_2(\mathbb{Q}_p)$ is a quadratic extension of $\mathbb{Q}_p$. Recall that G_γ can be identified with $E^\times$. Notice that the set of $[g] \in G_\gamma \backslash G/K_0$ such that $g^{-1}\gamma g \in K_0$ maps (by $g \mapsto g^{-1}\gamma g \dots$) bijectively onto the set of K_0-conjugacy classes $[\gamma']$ in K_0 having the same characteristic polynomial as γ. Let $\mathcal{G}_1$ be the groupoid associated to the action of G_γ on G/K_0 by left multiplication, i.e. the objects of $\mathcal{G}_1$ are the elements of G/K_0 and the set of morphisms from xK_0 to yK_0 is the set of $h \in G_\gamma$ such that $hxK_0 = yK_0$ (equivalently, $h \in yK_0x^{-1}$). The group of automorphisms of an object xK_0 of $\mathcal{G}_1$ is $G_\gamma \cap xK_0x^{-1}$. We see that the groupoid $\mathcal{G}_1$ has additional structure: each automorphism group is endowed with a topology for which it is a topological group, and for any morphism between two objects, the resulting group isomorphism between automorphism groups is a homeomorphism. In fact each automorphism group is profinite. Furthermore each automorphism group $\mathrm{Aut}_{\mathcal{G}_1}(x)$ is endowed with a Haar measure μ_x, induced by the fixed Haar measure on $G_\gamma \simeq E^\times$. So $\mathcal{G}_1$ is naturally a "measured topological groupoid" (this terminology is not standard ...). Note that $O_\gamma(e_{K_0})$ can be interpreted as the *mass* of this measured topological groupoid: Formula (3.1) can be written abstractly

$$O_\gamma(e_{K_0}) = \sum_{[x]} \mu_x(\mathrm{Aut}_{\mathcal{G}_1}(x))^{-1}$$

where the sum is over isomorphism classes in $\mathcal{G}_1$. Let $\mathcal{G}_2$ be the groupoid of $\mathbb{Z}_p[\gamma]$-modules which are finite free of rank 2 over $\mathbb{Z}_p$ (equivalently, finite torsion-free $\mathbb{Z}_p[\gamma]$-modules which become one-dimensional over E after $E \otimes_{\mathbb{Z}_p[\gamma]} \cdot$). We have an obvious functor $\mathcal{G}_1 \to \mathcal{G}_2$ mapping the object gK_0 to the $\mathbb{Z}_p$-lattice $g(\mathbb{Z}_p^2)$ with the obvious action of γ, and mapping $h \in \mathrm{Mor}_{\mathcal{G}_1}(xK_0, yK_0) = G_\gamma \cap yK_0x^{-1}$ to the induced isomorphim of $\mathbb{Z}_p[\gamma]$-modules $x(\mathbb{Z}_p^2) \to y(\mathbb{Z}_p^2)$. This functor is easily seen to be an equivalence of categories. Note that for an object L of $\mathcal{G}_2$ the group of automorphisms of L contains $\mathbb{Z}_p[\gamma]^\times$ and is contained in $E^\times$, this inclusion being compatible with the above functor and the identification of G_γ with $E^\times$. By compactness, the group of automorphisms of L is even contained in $\mathcal{O}_E^\times$. We deduce

$$O_\gamma(e_{K_0}) = \sum_{[L]} \mathrm{vol}(\{\lambda \in \mathcal{O}_E^\times | \lambda L = L\})^{-1}$$

where the sum is over isomorphism classes in $\mathcal{G}_2$, and the volume is taken for the fixed Haar measure on $G_\gamma \simeq E^\times$. The following lemma gives an explicit representative in each isomorphism class.

Lemma 3.8. (1) *Let L be a $\mathbb{Z}_p[\gamma]$-lattice which is free of rank two over $\mathbb{Z}_p$. Then there is an isomorphism of $\mathbb{Z}_p[\gamma]$-modules $\phi : L \simeq \phi(L)$ with $\mathbb{Z}_p[\gamma] \subset \phi(L) \subset \mathcal{O}_E$, and $\phi(L)$ is uniquely determined by the isomorphism class of L.*

(2) *Let L be a $\mathbb{Z}_p$-submodule of $\mathcal{O}_E$ which contains $\mathbb{Z}_p[\gamma]$. Then L is a $\mathbb{Z}_p$-algebra (i.e. it is stable under multiplication), in particular it is a $\mathbb{Z}_p[\gamma]$-module, and the group of automorphisms of the $\mathbb{Z}_p[\gamma]$-module L is $L^\times$.*

Proof. Choose an isomorphism of E-vector spaces $\phi : \mathbb{Q}_p \otimes_{\mathbb{Z}_p} L \simeq E$. Since L is p-torsion free, L embeds in $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} L$ and so ϕ embeds L in E. The $\mathcal{O}_E$-module $\mathcal{O}_E\phi(L) \subset E$ is of the form $\varpi_E^i\mathcal{O}_E$ for some $i \in \mathbb{Z}$, where ϖ_E is a uniformizer of E. Up to composing ϕ with multiplication by ϖ_E^{-i}, we may assume that $\mathcal{O}_E\phi(L) = \mathcal{O}_E$. In particular $\phi(L)$ contains an element $u \in \mathcal{O}_E^\times$. Up to composing ϕ with multiplication by u^{-1} we can also assume that $1 \in \phi(L)$, and so $\mathbb{Z}_p[\gamma] \subset \phi(L) \subset \mathcal{O}_E$. This shows existence.

Let $x \in \mathcal{O}_E$ be such that $\mathcal{O}_E = \mathbb{Z}_p[x]$. Then any sub-$\mathbb{Z}_p$-module L of $\mathcal{O}_E$ of rank 2 and containing $\mathbb{Z}_p$ is of the form $\mathbb{Z}_p \oplus \mathbb{Z}_p p^n x$, where n is determined by the index $|\mathcal{O}_E/L| = p^n$. From this description it is clear that L is stable under multiplication.

Now if $\mathbb{Z}_p[\gamma] \subset L, L' \subset \mathcal{O}_E$ are $\mathbb{Z}_p[\gamma]$-modules, any isomorphism (of $\mathbb{Z}_p[\gamma]$-modules) $L \simeq L'$ is multiplication by some $t \in L'$ (t being the image of $1 \in L$), because such an isomorphism is determined by its restriction to $\mathbb{Z}_p[\gamma]$. Since L and L' both generate $\mathcal{O}_E$ as an $\mathcal{O}_E$-module we have $t \in \mathcal{O}_E^\times$, which implies $|\mathcal{O}_E/L| =$

$|\mathcal{O}_E/L'|$ and so $L = L'$. This shows uniqueness. Considering the inverse morphism we see that t^{-1} also belongs to L, and so the group of automorphisms of L is $L^\times$. □

We deduce

$$O_\gamma(e_{K_0}) = \sum_{\mathbb{Z}_p[\gamma] \subset L \subset \mathcal{O}_E} \mathrm{vol}(L^\times)^{-1} \tag{3.3}$$

where the sum is over $\mathbb{Z}_p$-modules. To obtain a simple explicit formula it remains to compute the index of each $L^\times$ in $\mathcal{O}_E^\times$.

Lemma 3.9. *Let L be an order for E, i.e. a sub-$\mathbb{Z}_p$-algebra of $\mathcal{O}_E$ which has rank 2 as a $\mathbb{Z}_p$-module. Let $n \in \mathbb{Z}_{\geq 0}$ be the integer defined by the equality $|\mathcal{O}_E/L| = p^n$. We have*

$$|\mathcal{O}_E^\times/L^\times| = \begin{cases} p^n & \text{if } E/\mathbb{Q}_p \text{ is ramified,,} \\ 1 & \text{if } L = \mathcal{O}_E, \\ p^n + p^{n-1} & \text{if } E/\mathbb{Q}_p \text{ is unramified and } L \subsetneq \mathcal{O}_E. \end{cases}$$

Proof. As in the proof of the previous lemma choose $x \in \mathcal{O}_E$ such that $\mathcal{O}_E = \mathbb{Z}_p[x]$. If $E/\mathbb{Q}_p$ is ramified, we may and do assume that x generates the maximal ideal of $\mathcal{O}_E$ (equivalently, $2v(x) = v(p)$).

If $E/\mathbb{Q}_p$ is unramified we have

$$|\mathcal{O}_E^\times/(1+p\mathcal{O}_E)| = |\mathbb{F}_{p^2}^\times| = p^2 - 1.$$

If $E/\mathbb{Q}_p$ is ramified we have

$$|\mathcal{O}_E^\times/(1+p\mathcal{O}_E)| = |\mathcal{O}_E^\times/(1+x\mathcal{O}_E)||(1+x\mathcal{O}_E)/(1+p\mathcal{O}_E)| = p|\mathbb{F}_p^\times| = p^2 - p.$$

For any $i \geq 1$ we have $(1+p^i\mathcal{O}_E)/(1+p^{i+1}\mathcal{O}_E) \simeq \mathcal{O}_E/p\mathcal{O}_E$. By induction on $i \geq 1$ we find

$$|\mathcal{O}_E^\times/(1+p^i\mathcal{O}_E)| = \begin{cases} p^{2i} - p^{2i-2} & \text{if } E/\mathbb{Q}_p \text{ is unramified,} \\ p^{2i} - p^{2i-1} & \text{if } E/\mathbb{Q}_p \text{ is ramified.} \end{cases}$$

For L and n as in the lemma we have $L = \mathbb{Z}_p \oplus \mathbb{Z}_p p^n x$. The case $n = 0$ is trivial so assume $n > 0$. In this case we have $L^\times = \mathbb{Z}_p^\times + \mathbb{Z}_p p^n x$ and so the morphism $\mathbb{Z}_p^\times \to L^\times/(1+p^n\mathcal{O}_E)$ is surjective. The kernel of this morphism is $1 + p^n\mathbb{Z}_p$, so we have

$$|L^\times/(1+p^n\mathcal{O}_E)| = p^n - p^{n-1}.$$

Decomposing

$$|\mathcal{O}_E^\times/L^\times| = \frac{|\mathcal{O}_E^\times/(1+p^n\mathcal{O}_E)|}{|L^\times/(1+p^n\mathcal{O}_E)|^{-1}}$$

gives the formula in the lemma. □

Let $m \in \mathbb{Z}_{\geq 0}$ be defined by the equality $|\mathcal{O}_E/\mathbb{Z}_p[\gamma]| = p^m$. Plugging the result of Lemma 3.9 into Formula (3.3), we finally obtain

$$O_\gamma(e_{K_0})\,\mathrm{vol}(\mathcal{O}_E^\times) = \begin{cases} (1+p+\cdots+p^m) = \frac{(p^{m+1}-1)}{(p-1)} & \text{if } E/\mathbb{Q}_p \text{ is ramified,} \\ (1+(1+p^{-1})(p+\cdots+p^m)) = \frac{p^{m+1}+p^m-2}{p-1} & \text{if } E/\mathbb{Q}_p \text{ is unramified.} \end{cases} \tag{3.4}$$

Proposition 3.10. *There are constants $C > c > 0$ such that for any $\gamma \in G_{\mathrm{rs}}$ we have $C|D(\gamma)|^{-1/2} \geq O_\gamma(e_{K_0}) \geq c|D(\gamma)|^{-1/2}$.*

Note that this makes sense even though the orbital integrals depend on choices of measures: there are finitely many conjugacy classes of maximal tori in G, and we may fix a Haar measure on each maximal torus, as well as a Haar measure on G. Different choices only affect the constants C and c.

Proof. For $\gamma \in G_{\mathrm{rs}}$ not conjugated to an element of K_0 it is clear that $O_\gamma(e_{K_0}) = 0$, so we may restrict to $\gamma \in G_{\mathrm{rs}} \cap K_0$. For γ split this is clear by the above computation $O_\gamma(e_{K_0}) = |D(\gamma)|^{-1/2}$. For γ elliptic it remains to relate the integer m appearing in Formula (3.4) to $|D(\gamma)|^{1/2}$. As above introduce $x \in \mathcal{O}_E$ such that we have $\mathcal{O}_E = \mathbb{Z}_p[x]$. Up to multiplying x by an element of $\mathbb{Z}_p^\times$ we may assume that we have $\gamma = p^m x + y$ with $y \in \mathbb{Z}_p$. Write σ for the non-trivial element of $\mathrm{Gal}(E/\mathbb{Q}_p)$. We have $D(\gamma) = (\gamma/\sigma(\gamma) - 1)(\sigma(\gamma)/\gamma - 1)$, and since γ is compact we have $v(D(\gamma)) = 2v(\gamma - \sigma(\gamma)) = 2(m + v(x - \sigma(x)))$. The estimate of the proposition is easily deduced from this equality and Formula (3.4). □

Corollary 3.11. *Let $f \in \mathcal{H}(G)$. Then there exists $C > 0$ such that for any $\gamma \in G_{\mathrm{rs}}$ we have $|O_\gamma(f)| \leq C|D(\gamma)|^{-1/2}$.*

As in Proposition 3.10 the precise constant depends not only on f, but also on choices of Haar measures. As before we fix Haar measures on maximal tori of G.

Proof. Let $X = \{g \in \mathrm{supp}(f) | D(g) = 0\}$, a compact subset of G. For any $g \in X$, there exists $z \in Z$ and $h \in G$ such that $g \in zhK_0h^{-1}$ (in fact K_0 could be replaced by any neighbourhood of 1 in G). By compactness of X there is a finite family $(z_i, h_i)_{i \in I}$ such that $X \subset \bigcup_{i \in I} z_i h_i K_0 h_i^{-1}$. Therefore there exists $c_1 > 0$ and $f_{\mathrm{rs}} \in C_c^\infty(G_{\mathrm{rs}}, \mathbb{R}_{\geq 0})$ (for example, supported on $\mathrm{supp}(f) \smallsetminus \bigcup_{i \in I} z_i h_i K_0 h_i^{-1}$) such that

$$|f| \leq f_{\mathrm{rs}} + \sum_{i \in I} c_1 \,\mathrm{vol}(K_0)^{-1} 1_{z_i h_i K_0 h_i^{-1}}.$$

By Lemma 3.5 and Lemma 3.7 for any maximal torus T' of G the function

$$\begin{aligned} F : T'_{G-\mathrm{reg}} &\longrightarrow \mathbb{C} \\ \gamma &\longmapsto O_\gamma(f_{\mathrm{rs}}) \end{aligned}$$

is smooth and compactly supported, whence bounded.

By Proposition 3.10 there exists a constant $C > 0$ such that for any $\gamma \in G_{\mathrm{rs}}$,

$$O_\gamma(\mathrm{vol}(K_0)^{-1} 1_{z_i h_i K_0 h_i^{-1}}) = O_{\gamma z_i^{-1}}(e_{K_0}) \leq C|D(\gamma)|^{-1/2}.$$

We obtain

$$|O_\gamma(f)| \leq \left(\sup_{\gamma' \in G_{\mathrm{rs}}} \left(|D(\gamma')|^{1/2} O_{\gamma'}(f_{\mathrm{rs}}) \right) + |I| c_1 C \right) |D(\gamma)|^{-1/2}.$$

□

3.3. **The Weyl integration formula.** For T' a maximal torus of G we have a map $\phi_{T'} : T'_{G-\mathrm{reg}} \times T'\backslash G \to G_{\mathrm{rs}}$, $(t, \dot{g}) \mapsto g^{-1}tg$. Then $(t, \dot{g})$ and $(s, \dot{h})$ map to the same point if and only if $hg^{-1} \in N_G(T')$ and $s = hg^{-1}tgh^{-1}$. Since $N_G(T')/T' = \mathbb{Z}/2\mathbb{Z}$ we get that each non-empty fiber of $\phi_{T'}$ has two elements. Let $\mathcal{T}$ be a set of representatives of maximal tori of G, under conjugation by G. Note that $\mathcal{T}$ is finite.

Theorem 3.12 (Weyl integration formula)**.** *Let f be a measurable function on G. Then*

$$\int_G f(g)\, dg = \sum_{T' \in \mathcal{T}} \frac{1}{2} \int_{T'_{G-\mathrm{reg}}} |D(t)| O_t(f)\, dt$$

if one side is absolutely convergent (i.e. convergent if we substitute $|f|$ for f).

Note that the complex Haar measure $O_t(f)dt$ on T' does not depend on the choice of Haar measure on T', since $O_t(f)$ is defined using a quotient measure.

Proof. Since $G \smallsetminus G_{\mathrm{rs}}$ has measure zero and $G_{\mathrm{rs}} = \bigsqcup_{T' \in \mathcal{T}} \mathrm{Im}(\phi_{T'})$, by linearity of the formula to be proved we may assume without loss of generality that there exists $T' \in \mathcal{T}$ such that f vanishes identically outside $\mathrm{Im}(\phi_{T'})$. We will apply Theorem B.7 to $\phi_{T'}$. We can write $dg = |\omega_G|$ where $\omega_G \in \Omega^{\dim G}(G)$ is left G-invariant and non-zero, and so ω_G corresponds to a non-zero element in $\bigwedge^{\dim G}(T_1 G)^* = \bigwedge^{\dim G} \mathrm{Lie}(G)^*$. By Example B.8, ω_G is also right G-invariant. Similarly, we can choose $\omega_{T'}$ corresponding to an element of $\bigwedge^{\dim T'} \mathrm{Lie}(T')^*$, inducing a Haar measure on T'. We will use these to define a right G-invariant $\omega_{T'\backslash G} \in \Omega^{\dim G - \dim T'}(T'\backslash G)$. For $g \in G$, differentiating the submersion $G \to T'\backslash G$, $h \mapsto T'hg$ gives a short exact sequence

$$0 \to \mathrm{Lie}(T') \to \mathrm{Lie}(G) \to T_{\dot{g}}(T'\backslash G) \to 0$$

whose dual gives an isomorphism

$$\iota_g : \bigwedge^{\dim G} \mathrm{Lie}(G)^* \simeq \bigwedge^{\dim T'} \mathrm{Lie}(T')^* \otimes_{\mathbb{Q}_p} \bigwedge^{\dim G - \dim T'} T_{\dot{g}}(T'\backslash G)^*.$$

In particular we have a basis $\omega_{T'\backslash G, g}$ of the $\mathbb{Q}_p$-line $\bigwedge^{\dim G - \dim T'} T_{\dot{g}}(T'\backslash G)^*$ such that $\iota_g(\omega_G) = \omega_{T'} \otimes \omega_{T'\backslash G, g}$. We have to check that it depends on g only via $g \mapsto \dot{g} = T'g$. If $g' = tg$ with $t \in T'$ then we have a commutative diagram

$$\begin{tikzcd}
G \arrow[rr, "R_{g'}"] \arrow[dr, "\mathrm{Ad}(t^{-1})"'] & & T'\backslash G \\
& G \arrow[ur, "R_g"'] &
\end{tikzcd}$$

where $\mathrm{Ad}(t^{-1}) : h \mapsto t^{-1}ht$ fixes $1 \in G$ and preserves T'. We have $\mathrm{Ad}(t^{-1})^*\omega_G = \det(\mathrm{Ad}(t^{-1}) \,|\, \mathrm{Lie}(G))$ and $\mathrm{Ad}(t^{-1})^*\omega_{T'} = \det(\mathrm{Ad}(t^{-1}) \,|\, \mathrm{Lie}(T'))$, and so $\omega_{T'\backslash G, g'} = \det(\mathrm{Ad}(t^{-1}) \,|\, \mathrm{Lie}(G)/\mathrm{Lie}(T'))\omega_{T'\backslash G, g}$. It is easy to check that

$$\det(\mathrm{Ad}(t^{-1}) \,|\, \mathrm{Lie}(G)/\mathrm{Lie}(T')) = 1$$

as this can be computed after tensoring with a finite extension of $\mathbb{Q}_p$ splitting T'. We claim that there is a unique right G-invariant $\omega_{T'\backslash G} \in \Omega^{\dim G - \dim T'}(T'\backslash G)$ which specializes to $\omega_{T'\backslash G,1}$ at $\dot{1} = T' \in T'\backslash G$. Around any point of $T'\backslash G$ there exists an open subset U of $T'\backslash G$ and a local section $s : U \to G$ (morphism of p-adic manifolds) of $G \to T'\backslash G$. Define $\omega_{T'\backslash G}|_U$ by pulling back $\omega_{T'\backslash G,1}$ along $R_{s(?)^{-1}} : T'\backslash G \to T'\backslash G$. The previous computation shows that this does not depend on the choice of s, and it is clear that this construction glues to $T'\backslash G$. Note that we recover the existence of the quotient measure $|\omega_{T'\backslash G}|$ on $T'\backslash G$ (this is not surprising since the trivial determinant above implies that the modulus characters of G and T' coincide on T').

Let $t \in T'$ and $g \in G$, defining $\dot{g} = T'g \in T'\backslash G$. We want to compute the dual of

$$\bigwedge^{\dim G} d_{(t,\dot{g})}\phi_{T'} : \left(\bigwedge^{\dim T'} T_t(T')\right) \otimes_{\mathbb{Q}_p} \left(\bigwedge^{\dim G - \dim T'} T_{\dot{g}}(T'\backslash G)\right) \longrightarrow \bigwedge^{\dim G} T_{g^{-1}tg}(G)$$

in the bases $\omega_{T'}$, $\omega_{T'\backslash G}$ and ω_G. We have a commutative diagram (vertical maps are isomorphisms)

$$\begin{tikzcd}
T' \times T'\backslash G \arrow[r, "\phi_{T'}"] & G \\
T' \times T'\backslash G \arrow[u, "R_t \times R_g"] \arrow[r, "\psi_t"] & G \arrow[u, "R_{tg} \circ L_{g^{-1}}"']
\end{tikzcd}$$

where R_a (resp. L_a) denotes right (resp. left) multiplication by a and $\psi_t(x, \dot{h}) = h^{-1}xtht^{-1}$. Taking differentials, we get a commutative diagram

$$\begin{tikzcd}
T_t(T') \times T_{\dot{g}}(T'\backslash G) \arrow[r, "d_{t,\dot{g}}(\phi_{T'})"] & T_{g^{-1}tg}G \\
\mathrm{Lie}(T') \oplus \mathrm{Lie}(G)/\mathrm{Lie}(T') \arrow[u, "d_1(R_t) \oplus d_{\dot{1}}(R_g)"] \arrow[r, "d_{1,\dot{1}}(\psi_t)"] & \mathrm{Lie}\, G \arrow[u, "d_1(R_{tg} \circ L_{g^{-1}})"']
\end{tikzcd}$$

Writing $x = \exp\delta = 1 + \epsilon + O(\delta^2)$ for $\delta \in \mathrm{Lie}\, T'$ and $h = \exp\epsilon = 1 + \epsilon + O(\epsilon^2)$ for ϵ in a complementary subspace of $\mathrm{Lie}\, T'$ in $\mathrm{Lie}\, G$, we compute $d_{1,\dot{1}}(\psi_t)(\delta, \epsilon) = \delta + (\mathrm{Ad}(t) - 1)(\epsilon)$. Since ω_G is invariant under left and right multiplication maps, $\omega_{T'}$ is also invariant under multiplication maps and $\omega_{T'\backslash G}$ is invariant under right multiplication maps, we obtain

$$(\phi_{T'}^* \omega_G)\,|_{t,\dot{g}} = \det\left(\mathrm{Ad}(t) - 1 \,|\, \mathrm{Lie}(G)/\mathrm{Lie}(T')\right) \times (\omega_{T'}|_t) \wedge \left(\omega_{T'\backslash G}|_{\dot{g}}\right).$$

The formula now follows from Theorem B.7 and Fubini's theorem. $\square$

Proposition 3.13. *Let $\epsilon > 0$. Any measurable function $G \to \mathbb{C}$ which coincides with $|D|^{-1+\epsilon}$ on G_{rs} is locally integrable.*

Proof. The function is locally smooth on G_{rs} so we only have to show that for any $x \in G \smallsetminus G_{\mathrm{rs}}$, there is a neighbourhood U of x in G such that $\int_{U \cap G_{\mathrm{rs}}} |D(g)|^{-1+\epsilon}\, dg < +\infty$. Recall that any element of $G \smallsetminus G_{\mathrm{rs}}$ is conjugated to an element of ZN (or even ZN_i for an arbitrary $i \in \mathbb{Z}$, using conjugation by T). The function D is invariant by conjugation and by multiplication by Z, so we may replace x by $zgxg^{-1}$ for some $g \in G$ and some $z \in Z$. This allows us to assume that x belongs to K_0. We simply take $U = K_0$. To show that $e_{K_0}|D|^{-1+\epsilon}$ is integrable, we apply the Weyl integration formula:

$$\mathrm{vol}(K_0)^{-1} \int_{K_0} |D(g)|^{-1+\epsilon}\, dg = \sum_{T' \in \mathcal{T}} \frac{1}{2} \int_{T'_{G-\mathrm{reg}}} |D(t)|^{\epsilon} O_t(e_{K_0})\, dt$$

and by Proposition 3.10 we are left to show that for any maximal torus T', the function $|D|^{-1/2+\epsilon}$ is locally integrable on T' (for the Haar measure on T'). For $T' = T$ this amounts to bounding

$$\int_{\mathbb{Z}_p^\times \smallsetminus \{1\}} |1-x|^{-1+2\epsilon}\, |dx| \leq \int_{\mathbb{Z}_p \smallsetminus \{0\}} |u|^{-1+2\epsilon}\, |du| = \mathrm{vol}(\mathbb{Z}_p^\times) \sum_{k \geq 0} p^{-k} p^{k-2\epsilon k} < +\infty.$$

For T' anisotropic, corresponding to a quadratic extension $E/\mathbb{Q}_p$, we have

$$\int_{\mathcal{O}_E^\times \smallsetminus \mathbb{Z}_p^\times} |x - \sigma(x)|^{-1+2\epsilon}\, dx \leq C \int_{\mathbb{Z}_p \smallsetminus \{0\}} |2ux_0|^{-1+2\epsilon}\, |du|$$

where $x_0 \in \mathcal{O}_E \smallsetminus \mathbb{Z}_p$ is such that $\sigma(x_0) = -x_0$, and we conclude as in the previous case. □

3.4. **Harish-Chandra characters.** We now begin the study of characters of representations of G. If (V, π) is an admissible representation of G (for example if it is an irreducible smooth representation of G) then for any $f \in \mathcal{H}(G)$ the operator $\pi(f) : V \to V$ has image contained in the finite-dimensional subspace V^K for any compact open subgroup K such that f is left K-invariant. Thus we can define $\mathrm{tr}\, \pi(f) = \mathrm{tr}\, (\pi(f) \,|\, \pi(f)(V))$, which also equals $\mathrm{tr}\, \left(\pi(f) \,\middle|\, V^K\right)$ for K as above by the following lemma applied to $W = V^K$ and $A = \pi(f)|_{V^K}$.

Lemma 3.14. *Let A be an endomorphism of a finite-dimensional vector space W over a field. Then $\mathrm{tr}\, A = \mathrm{tr}(A \,|\, A(W))$.*

Proof. Left as an exercise. □

Thanks to the theory of finite-dimensional representation of algebras, if (V_1, π_1), ..., (V_k, π_k) are non-isomorphic irreducible smooth representations of G then the linear forms $\mathrm{tr}\, \pi_i$ on $\mathcal{H}(G)$ are linearly independent (this follows from the existence of projection operators, see [Lan02, XVII Theorem 3.7]). In particular the trace of an admissible *semisimple* representation of G determines the isomorphism class of this representation (exercise ...).

Theorem 3.15. *Let (V, π) be an irreducible smooth representation of G. Then there is a unique smooth function $\Theta_\pi : G_{\mathrm{rs}} \to \mathbb{C}$ such that, extending Θ_π arbitrarily to G, Θ_π is locally integrable on G, and for any $f \in \mathcal{H}(G)$ we have*

$$\mathrm{tr}\, \pi(f) = \int_G f(g) \Theta_\pi(g)\, dg.$$

Moreover Θ_π is invariant under conjugation by G and $|D|^{1/2}\Theta_\pi$ is bounded on G_{rs}.

Remark 3.16. (1) *The function Θ_π does not depend on the choice of Haar measure (the measure occurs in the definition of $\operatorname{tr}\pi(f)$ as well).*

(2) *Using the Weyl integration formula the theorem also gives the expression*

$$\operatorname{tr}\pi(f) = \sum_{T'\in\mathcal{T}} \frac{1}{2}\int_{T'_{G-\mathrm{reg}}} |D(t)|$$

where $|D|^{1/2}\Theta_\pi$ and $t\mapsto |D|^{1/2}O_t(f)$ are both bounded (Corollary 3.11), and the support of the latter is relatively compact in T'.

Uniqueness and conjugation invariance in Theorem 3.15 are easier than existence and left as an exercise (use the fact that for any $x\in G_{\mathrm{rs}}$ there exists a compact open subgroup K of G such that xK is contained in G_{rs} and Θ_π is constant on xK, so that $\operatorname{tr}\pi(1_{xK}) = \mathrm{vol}(K)\Theta_\pi(x)$). The proof of existence is going to be quite long. First we handle the supercuspidal case.

Lemma 3.17. *Let (V,π) be an irreducible supercuspidal representation. Let $v\in V$ and $\widetilde{v}\in\widetilde{V}$ be such that $\langle v,\widetilde{v}\rangle = d_\pi$. Then for any $f\in\mathcal{H}(G)$,*

$$\dot{g}\mapsto \int_G f(h)\langle \pi(g^{-1}hg)v,\widetilde{v}\rangle\, dh$$

is a smooth compactly supported function on G/Z and we have

$$\operatorname{tr}\pi(f) = \int_{G/Z}\int_G f(h)\langle \pi(g^{-1}hg)v,\widetilde{v}\rangle\, dh\, d\dot{g}.$$

Proof. Let $(v_i)_i$ be a basis of V such that each v_i belongs to a K_0-isotypic component. Let $(\widetilde{v}_i)_i$ be the dual basis of $\widetilde{V}$ (this is well-defined by admissibility of V). Let $a_{i,j} = \langle \pi(f)v_i,\widetilde{v}_j\rangle$. By admissibility of π (and since f if bi-K-invariant for some open subgroup $K\subset K_0$) only finitely many of them are non-zero, and $\operatorname{tr}\pi(f) = \sum_i a_{i,i}$. Note that $\pi(f)v_i = \sum_j a_{i,j}v_j$, and so for any $w\in V$ we have $w = \sum_i\langle w,\widetilde{v}_i\rangle v_i$ and $\pi(f)w = \sum_i\langle w,\widetilde{v}_i\rangle\sum_j a_{i,j}v_j$, in particular (taking $w=\pi(g)v$)

$$\langle \pi(g^{-1})\pi(f)\pi(g)v,\widetilde{v}\rangle = \langle \pi(f)\pi(g)v,\widetilde{\pi}(g)\widetilde{v}\rangle = \sum_{i,j} a_{i,j}\langle \pi(g)v,\widetilde{v}_i\rangle\langle v_j,\widetilde{\pi}(g)\widetilde{v}\rangle.$$

This can also be written

$$\int_G f(h)\langle \pi(g^{-1}hg)v,\widetilde{v}\rangle dh = \sum_{i,j} a_{i,j}\langle \pi(g)v,\widetilde{v}_i\rangle\langle \pi(g^{-1})v_j,\widetilde{v}\rangle.$$

This function of $\dot{g}\in G/Z$ is clearly smooth. By Theorem 2.38 it is compactly supported. Integrating over $\dot{g}\in G/Z$ and using Schur orthogonality (Proposition 2.47) we get

$$\int_{G/Z}\int_G f(h)\langle \pi(g^{-1})hg)v,\widetilde{v}\rangle d\dot{g} = \sum_{i,j} a_{i,j}d_\pi^{-1}\langle v,\widetilde{v}\rangle\langle v_j,v_i\rangle = \sum_i a_{i,i} = \operatorname{tr}\pi(f).$$

$\square$

Now we would very much like to swap integral signs in this formula. This is not formal, and in fact wrong!

It is however justified if we restrict to g in the subset $G_{\mathrm{rs}}^{\mathrm{ell}}$ of *elliptic* regular semisimple elements.

Lemma 3.18. *Let $f \in C_c^\infty(G)$ and $\psi \in C_c^\infty(G,\omega)$ for some smooth character $\omega : Z \to \mathbb{C}^\times$. Then the function $(\dot{g}, h) \mapsto f(h)\psi(g^{-1}hg)$ is integrable on $G/Z \times G_{\mathrm{rs}}^{\mathrm{ell}}$ and*

$$\int_{G/Z \times G_{\mathrm{rs}}^{\mathrm{ell}}} f(h)\psi(g^{-1}hg)\, d\dot{g}\, dh = \int_{G_{\mathrm{rs}}^{\mathrm{ell}}} f(h) \operatorname{vol}(G_h/Z) O_h(\psi) dh.$$

Proof. We can assume that f, ψ take values in $\mathbb{R}_{\geq 0}$. Let $\mathcal{T}^{\mathrm{ell}}$ be a set of representatives for the G-conjugacy classes of elliptic maximal tori in G. Now for $h \in G_{\mathrm{rs}}^{\mathrm{ell}}$ we have $\int_{G/Z} \psi(g^{-1}hg) d\dot{g} = \operatorname{vol}(G_h/Z) O_h(\psi)$ because G_h/Z is compact. Now by Corollary 3.11 and Proposition 3.13 the function $h \mapsto O_h(\psi)$, extended by zero on $G \smallsetminus G_{\mathrm{rs}}^{\mathrm{ell}}$, is locally integrable on G. Therefore

$$\int_{G_{\mathrm{rs}}^{\mathrm{ell}}} f(h) \int_{G/Z} \psi(g^{-1}hg)\, d\dot{g}\, dh = \int_{G_{\mathrm{rs}}^{\mathrm{ell}}} f(h) \operatorname{vol}(G_h/Z) O_h(\psi)\, dh < \infty.$$

□

This argument does not work for $h \in G_{\mathrm{rs}} \smallsetminus G_{\mathrm{rs}}^{\mathrm{ell}}$ because then $\operatorname{vol}(G_h/Z) = +\infty$. But recall from Remark 2.46 that the matrix coefficient ψ is not just any element of $C_c^\infty(G, \omega_\pi)$: it belongs to the subspace $C_{\mathrm{cusp}}^\infty(G, \omega_\pi)$.

Lemma 3.19 (Selberg's principle)**.** *Let $\omega : Z \to \mathbb{C}^\times$ be a smooth character. Let $\psi \in C_{\mathrm{cusp}}^\infty(G, \omega)$, i.e. for any $x, y \in G$ we have $\int_N \psi(xny)\, dn = 0$. Then for any $h \in G_{\mathrm{rs}} \smallsetminus G_{\mathrm{rs}}^{\mathrm{ell}}$ we have $O_h(\psi) = 0$.*

Proof. It is enough to consider $h \in T_{G-\mathrm{reg}}$. We computed (see (3.2))

$$O_h(\psi) = |D(h)|^{-1/2} \delta_B^{1/2}(h) \int_{K_0 \times N} \psi(k^{-1}hnk)\, dk\, dn = 0.$$

□

To "swap the two $\int$ signs" in the formula given in Lemma 3.17, we will write the outer integral as a limit over a particular increasing and exhaustive sequence of compact subsets of G/Z. For $c \geq 0$ an integer define $X_c = \bigsqcup_{m \leq c} K_0 \operatorname{diag}(p^m, 1) K_0 Z$, so that X_c/Z is a bi-K_0-invariant compact subset of G/Z.

Lemma 3.20. *Let $\omega : Z \to \mathbb{C}^\times$ be a smooth character and $\psi \in C_{\mathrm{cusp}}^\infty(G, \omega)$. The sequence of functions on $G_{\mathrm{rs}}^{\mathrm{hyp}}$*

$$\left(\Theta_{\psi,c} : h \mapsto \int_{X_c/Z} \psi(g^{-1}hg) d\dot{g} \right)_{c \geq 0}$$

converges pointwise as $c \to +\infty$ to a smooth function Θ_ψ which is invariant under conjugation by G. Moreover there exists $\kappa > 0$ such that for any $c \geq 0$ we have $|\Theta_{\psi,c}| \leq \kappa |D|^{-1/2}$ on $G_{\mathrm{rs}}^{\mathrm{hyp}}$. For $h \in T_{G-\mathrm{reg}}$ we have

$$\Theta_\psi(h) = \int_{N \times K_0} \psi(k^{-1}n^{-1}hnk) \min(0, 2v(n))\, dn\, dk$$

where we have denoted $v(n) = v(u)$ *for* $n = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \in N$.

Note that this formula for $\Theta_\psi(h)$ differs from the formula for $O_h(\psi)$ only by the factor $\min(0, 2v(n))$ in the integrand. This expression is called a *weighted* orbital integral.

Proof. We can replace ψ by $h \mapsto \int_{K_0} \psi(k^{-1}hk)\,dk$ and assume that ψ is invariant under conjugation by K_0. Let $h \in G^{\mathrm{hyp}}_{\mathrm{rs}}$, then we can write $h = \alpha^{-1}\mathrm{diag}(a,b)\alpha$ for some $\alpha \in G$ and $a, b \in \mathbb{Q}_p^\times$ satisfying $a \neq b$. For future use we note that up to conjugating by w we can assume that we have $\delta_B(\mathrm{diag}(a,b)) \leq 1$. We can write $\alpha \in TNK_0$. Since the sets X_c are left K_0-invariant the function $h \mapsto \int_{X_c/Z} \psi(g^{-1}hg)\,dg$ is invariant under conjugation by K_0 and we can reduce to $\alpha \in TN$, and so $h \in TN$. Since any element of T centralizes $\mathrm{diag}(a,b)$ we can even assume $\alpha \in N$. Let $\dot{g} \in G$ and write $g = \mathrm{diag}(x,1)\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} k$ with $x \in \mathbb{Q}_p^\times$, $u \in \mathbb{Q}_p$ and $k \in K_0$. Recall from Lemma 2.65 that $g \in X_c$ if and only if

$$|v(x)| \leq c \text{ and } v(u) \geq (-v(x) - c)/2.$$

Let Y_c be the compact open subset of TN/Z consisting of elements satisfying these two conditions. Using the integration formula for the Iwasawa decomposition (Lemma 2.13) and invariance under K_0-conjugation of ψ, we have

$$\int_{X_c/Z} \psi(g^{-1}hg)\,dg = \int_{Y_c/Z} \psi(n^{-1}t^{-1}htn)\,dt\,dn.$$

Writing $h = \mathrm{diag}(a,b)\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ we have

$$n^{-1}t^{-1}htn = \mathrm{diag}(a,b)\begin{pmatrix} 1 & x^{-1}z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & (1-a^{-1}b)u \\ 0 & 1 \end{pmatrix}.$$

The set of $(\dot{g_1}, \dot{g_2}) \in (G/Z)^2$ such that g_1 and g_1g_2 belong to $\mathrm{supp}(\psi)$ is compact. The subgroup ZN of G is closed so N is closed in G/Z, and so the set of $(\dot{g}, n) \in (G/Z) \times N$ such that $\dot{g}$ and $\dot{g}n$ belong to the support of ψ is also compact. Therefore its projection on N is contained in the compact subgroup $N_{-d(\psi)}$ of N for some integer $d(\psi) \geq 0$. Let $\beta \in G$ and $i \geq d(\psi)$. If βN_{-i} meets the support of ψ then we have

$$\mathrm{supp}(\psi) \cap \beta N \subset \beta N_{-i} N_{-d(\psi)} = \beta N_{-i}.$$

By cuspidality of ψ, this implies

$$\int_{p^{-i}\mathbb{Z}_p} \psi\left(\beta \begin{pmatrix} 1 & u' \\ 0 & 1 \end{pmatrix}\right) |du'| = 0. \tag{3.5}$$

(either the integrand vanishes identically or the integral is equal to the same integral over $\mathbb{Q}_p$). We will apply this to $\beta(h,x) = \mathrm{diag}(a,b)\begin{pmatrix} 1 & x^{-1}z \\ 0 & 1 \end{pmatrix}$.

For fixed $x \in \mathbb{Q}_p^\times$ such that $|v(x)| \leq c$ we integrate over $u \in \mathbb{Q}_p$ satisfying $v(u) \geq (-v(x)-c)/2$. Using the change of variables $u' = (1-a^{-1}b)u$ we compute

$$\int_{v(u)\geq(-v(x)-c)/2} \psi\left(\beta(h,x)\begin{pmatrix}1 & (1-a^{-1}b)u \\ 0 & 1\end{pmatrix}\right)|du| = \\ |1-a^{-1}b|^{-1}\int_{v(u')\geq v(1-a^{-1}b)+(-v(x)-c)/2} \psi\left(\beta(h,x)\begin{pmatrix}1 & u' \\ 0 & 1\end{pmatrix}\right)|du'|$$

If $v(1-a^{-1}b)+(-v(x)-c)/2 \leq -d(\psi)$ this vanishes. Otherwise, that is if $-c \leq v(x) < -c+2v(1-a^{-1}b)+2d(\psi)$, denoting $e = 2v(1-a^{-1}b)-v(x)-c > -2d(\psi)$ we have

$$\left|\int_{v(u')\geq e/2} \psi\left(\beta(h,x)\begin{pmatrix}1 & u' \\ 0 & 1\end{pmatrix}\right)|du'|\right| \leq \|\psi\|_\infty p^{-e/2}.$$

Therefore, summing over possible values for $v(x)$,

$$\begin{aligned}\left|\int_{X_c/Z}\psi(g^{-1}hg)\,d\dot{g}\right| &\leq |1-a^{-1}b|^{-1}\times\|\psi\|_\infty\sum_{e>-2d(\psi)}p^{-e/2}\\ &\leq |1-a^{-1}b|^{-1}\times p^{-d(\psi)}(1-p^{-1/2})^{-1}\|\psi\|_\infty.\end{aligned}$$

Recall (see (3.2)) that $|1-a^{-1}b|^{-1} = |D(h)|^{-1/2}\delta_B^{1/2}(\mathrm{diag}(a,b))$. As noted above we can assume that we have $\delta_B(\mathrm{diag}(a,b)) \leq 1$, so we have $|\Theta_{\psi,c}| \leq \kappa|D|^{-1/2}$ on $G_{\mathrm{rs}}^{\mathrm{hyp}}$ with

$$\kappa = p^{-d(\psi)}(1-p^{-1/2})^{-1}\|\psi\|.$$

Thanks to Proposition 3.13 (with $\epsilon = 1/2$) this last function is locally integrable on the closure of $G_{\mathrm{rs}}^{\mathrm{hyp}}$ in G.

We are left to compute, for a fixed h, the limit of $\Theta_{\psi,c}(h)$ as $c \to +\infty$. As observed above we can restrict to $x \in \mathbb{Q}_p^\times$ satisfying $-c \leq v(x) < -c+2v(1-a^{-1}b)+2d(\psi)$ and so $\beta(h,x) \to \mathrm{diag}(a,b)$ uniformly in x. Therefore the limit exists [4] and is given by

$$\begin{aligned}\Theta_\psi(h) :=& \lim_{c\to+\infty}\int_{X_c/Z}\psi(g^{-1}hg)\,d\dot{g}\\ =&|1-a^{-1}b|^{-1}\sum_{e=-2d(\psi)+1}^{2v(1-a^{-1}b)}\int_{v(u')\geq e/2}\psi\left(\mathrm{diag}(a,b)\begin{pmatrix}1 & u' \\ 0 & 1\end{pmatrix}\right)|du'|\\ =&2|1-a^{-1}b|^{-1}\sum_{k=-d(\psi)+1}^{v(1-a^{-1}b)}\int_{v(u')\geq k}\psi\left(\mathrm{diag}(a,b)\begin{pmatrix}1 & u' \\ 0 & 1\end{pmatrix}\right)|du'|\end{aligned}$$

where we have grouped the terms for $e = 2k-1$ and $e = 2k$. By Formula (3.5) above (cuspidality of ψ) we have

$$\int_{v(u')\geq k}\psi\left(\mathrm{diag}(a,b)\begin{pmatrix}1 & u' \\ 0 & 1\end{pmatrix}\right)|du'| = -\int_{k>v(u')\geq -d(\psi)}\psi\left(\mathrm{diag}(a,b)\begin{pmatrix}1 & u' \\ 0 & 1\end{pmatrix}\right)|du'|$$

[4]the sequence $(\Theta_{\psi,c}(h))_c$ is even stationary!

and so $|1-a^{-1}b|\Theta_{\psi,c}(h)$ is equal to

$$\begin{aligned}
&2\sum_{k=-d(\psi)+1}^{v(1-a^{-1}b)}\int_{v(u')\geq k}\psi\left(\mathrm{diag}(a,b)\begin{pmatrix}1 & u'\\ 0 & 1\end{pmatrix}\right)|du'|\\
=&-2\sum_{k=-d(\psi)+1}^{v(1-a^{-1}b)}\sum_{i=-d(\psi)}^{k-1}\int_{v(u')=i}\psi\left(\mathrm{diag}(a,b)\begin{pmatrix}1 & u'\\ 0 & 1\end{pmatrix}\right)|du'|\\
=&-2\sum_{i=-d(\psi)}^{v(1-a^{-1}b)-1}\sum_{k=i+1}^{v(1-a^{-1}b)}\int_{v(u')=i}\psi\left(\mathrm{diag}(a,b)\begin{pmatrix}1 & u'\\ 0 & 1\end{pmatrix}\right)|du'|\\
=&2\sum_{i=-d(\psi)}^{v(1-a^{-1}b)-1}(i-v(1-a^{-1}b))\int_{v(u')=i}\psi\left(\mathrm{diag}(a,b)\begin{pmatrix}1 & u'\\ 0 & 1\end{pmatrix}\right)|du'|.
\end{aligned}$$

Reverting the change of variable $u'=(1-a^{-1}b)u$, we compute

$$\begin{aligned}
&(i-v(1-a^{-1}b))\int_{v(u')=i}\psi\left(\mathrm{diag}(a,b)\begin{pmatrix}1 & u'\\ 0 & 1\end{pmatrix}\right)|du'|\\
=&(i-v(1-a^{-1}b))|1-a^{-1}b|\int_{v(u)=i-v(1-a^{-1}b)}\psi\left(\mathrm{diag}(a,b)\begin{pmatrix}1 & (1-a^{-1}b)u\\ 0 & 1\end{pmatrix}\right)|du|\\
=&|1-a^{-1}b|\int_{v(u)=i-v(1-a^{-1}b)}\psi\left(\mathrm{diag}(a,b)\begin{pmatrix}1 & (1-a^{-1}b)u\\ 0 & 1\end{pmatrix}\right)v(u)\,|du|
\end{aligned}$$

and we deduce the simplification

$$\begin{aligned}
&\Theta_\psi(h)\\
=&\int_{0>v(u)\geq -d(\psi)-v(1-a^{-1}b)}\psi\left(\mathrm{diag}(a,b)\begin{pmatrix}1 & (1-a^{-1}b)u\\ 0 & 1\end{pmatrix}\right)2v(u)\,|du|\\
=&|1-a^{-1}b|\int_{v(u)\geq -d(\psi)-v(1-a^{-1}b)}\psi\left(\mathrm{diag}(a,b)\begin{pmatrix}1 & (1-a^{-1}b)u\\ 0 & 1\end{pmatrix}\right)\min(0,2v(u))\,|du|.
\end{aligned}$$

Observe that substracting Formula (3.5) for two consecutive values of i gives the vanishing of

$$\int_{v(u)=i}\psi\left(\mathrm{diag}(a,b)\begin{pmatrix}1 & (1-a^{-1}b)u\\ 0 & 1\end{pmatrix}\right)|du|$$

for $i<-d(\psi)-v(1-a^{-1}b)$. We finally obtain, still under the assumption that ψ is invariant under conjugation by K_0:

$$\Theta_\psi(h)=\int_N\psi(n^{-1}\mathrm{diag}(a,b)n)\min(0,2v(n))\,dn.$$

Recall that we can reduce to this case by averaging over K_0, and the formula given in the Lemma follows. The smoothness and invariance by conjugation of ψ follow. $\square$

Corollary 3.21. *If (V,π) is an irreducible supercuspidal representation of G then Theorem 3.15 holds for π.*

Proof. Let $\psi \in C^\infty_{\mathrm{cusp}}(G, \omega_\pi)$ be the matrix coefficient defined by

$$\psi(g) = \langle \pi(g) v, \widetilde{v} \rangle$$

for $v \in V$ and $\widetilde{v} \in \widetilde{V}$ as in Lemma 3.17, i.e. satisfying $\langle v, \widetilde{v} \rangle = d_\pi$. By Lemma 3.17 we have, for c large enough,

$$\begin{aligned}
\operatorname{tr} \pi(f) &= \int_{G/Z} \int_G f(h) \psi(g^{-1} h g) \, dh \, d\dot{g} \\
&= \int_{X_c/Z} \int_G f(h) \psi(g^{-1} h g) \, dh \, d\dot{g} \\
&= \int_G f(h) \int_{X_c/Z} \psi(g^{-1} h g) \, d\dot{g} \, dh \\
&= \int_{G_{\mathrm{rs}}} f(h) \int_{X_c/Z} \psi(g^{-1} h g) \, d\dot{g} \, dh.
\end{aligned}$$

Here we are simply integrating a smooth function on a compact set (G could be replaced by the support of f), so swapping the integrals is justified. We split this last integral as two integrals over $G_{\mathrm{rs}}^{\mathrm{ell}}$ and $G_{\mathrm{rs}}^{\mathrm{hyp}}$, and take the limit as c goes to $+\infty$.

By Lemma 3.18 we have

$$\begin{aligned}
\lim_{c \to +\infty} \int_{G_{\mathrm{rs}}^{\mathrm{ell}}} \int_{X_c/Z} f(h) \psi(g^{-1} h g) \, d\dot{g} \, dh &= \int_{G_{\mathrm{rs}}^{\mathrm{ell}}} \int_{G/Z} f(h) \psi(g^{-1} h g) \, d\dot{g} \, dh \\
&= \int_{G_{\mathrm{rs}}^{\mathrm{ell}}} f(h) \operatorname{vol}(G_h/Z) O_h(\psi) dh
\end{aligned}$$

because the integrand is absolutely integrable on $G/Z \times G_{\mathrm{rs}}^{\mathrm{ell}}$.

By Lemma 3.20, Proposition 3.13 and the dominated convergence theorem we have

$$\lim_{c \to +\infty} \int_{G_{\mathrm{rs}}^{\mathrm{hyp}}} f(h) \int_{X_c/Z} \psi(g^{-1} h g) \, d\dot{g} \, dh = \int_{G_{\mathrm{rs}}^{\mathrm{hyp}}} f(h) \Theta_\psi(h) \, dh.$$

This concludes the proof of Theorem 3.15, with

$$\Theta_\pi(h) = \begin{cases} \operatorname{vol}(G_h/Z) O_h(\psi) & \text{if } h \text{ is elliptic,} \\ \Theta_\psi(h) & \text{if } h \text{ is hyperbolic.} \end{cases}$$

□

Remark 3.22. *This generalizes to arbitrary connected reductive groups: the Harish-Chandra character of an irreducible supercuspidal representation is given by the* weighted orbital integral *of any matrix coefficient whose value at* 1 *is the formal degree. See* [Art87].

To conclude the proof of Theorem 3.15 we are left to consider non-supercuspidal representations.

Proposition 3.23. *Let $\mu : T \to \mathbb{C}^\times$ be a smooth character, and consider $(V, \pi) = \mathrm{Ind}_B^G \mu$. Then Theorem 3.15 holds for π, and Θ_π is the unique G-invariant function on G_{rs} which vanishes identically on $G_{\mathrm{rs}}^{\mathrm{ell}}$ and such that for any $t \in T_{G-\mathrm{reg}}$ we have $\Theta_\pi(t) = |D(t)|^{-1/2}(\mu(t) + \mu^w(t))$.*

Note that since $\mathrm{Ind}_B^G \mu$ may be reducible, it is not strictly speaking Theorem 3.15 that we prove for $\mathrm{Ind}_B^G \mu$, but the statement makes sense.

Proof. Recall that we can realize $\mathrm{Ind}_B^G \mu$ as the space of smooth functions $\phi : K_0 \to \mathbb{C}$ such that $\phi(bk) = \mu(b)\phi(k)$ for any $b \in B_0$. For such a ϕ, $f \in \mathcal{H}(G)$ and $k_1 \in K_0$ we have

$$\pi(f)(\phi)(k_1) = \int_G f(g)\phi(k_1 g)dg = \int_G \phi(g) f(k_1^{-1} g)dg.$$

Using the integration formula for the Iwasawa decomposition this also equals

$$\begin{aligned}\int_{K_0\times B} \phi(bk_2) f(k_1^{-1}bk_2)\, dk_2\, db &= \int_{K_0} \phi(k_2) \int_B \mu(b)\delta_B^{1/2}(b) f(k_1^{-1}bk_2)\, db\, dk_2 \\ &= \int_{K_0} \phi(k_2)\psi(k_1, k_2)\, dk_2\end{aligned}$$

with $\psi(k_1, k_2) = \int_B \mu(b)\delta_B^{1/2}(b) f(k_1^{-1}bk_2)\, db$ (as usual, using a left Haar measure on B). Note that ψ is a smooth function on $K_0 \times K_0$. The operator

$$I(\psi) : \phi \longmapsto \left(k_1 \mapsto \int_{K_0} \phi(k_2)\psi(k_1, k_2)\, dk_2\right)$$

is defined for $\phi \in C^\infty(K_0)$, not just on the subspace $\mathrm{Ind}_B^G \mu$ of B_0-equivariant functions for μ. For $k_1, k_2 \in K_0$ and $x \in B_0$ we have

$$\begin{aligned}\psi(xk_1, k_2) &= \int_B \mu(b)\delta_B^{1/2}(b) f(k_1^{-1}x^{-1}bk_2)\, db \\ &= \int_B \mu(xb')\delta_B^{1/2}(xb') f(k_1^{-1}b'k_2)\, db' \\ &= \mu(x)\psi(k_1, k_2)\end{aligned}$$

using the change of variable $b' = x^{-1}b$. Therefore $I(\psi)$ maps $C^\infty(K_0)$ to $\mathrm{Ind}_B^G \mu$ and coincides with $\pi(f)$ on $\mathrm{Ind}_B^G \mu \subset C^\infty(K_0)$, and so $\mathrm{tr}\, \pi(f) = \mathrm{tr}\, I(\psi)$. Since ψ is smooth it is left and right K_i-invariant for some $i \geq 1$, and so the image of $I(\psi)$ is contained in $C^\infty(K_0)^{K_i}$ and we can compute $\mathrm{tr}\, I(\psi)$ on this finite-dimensional subspace. For this we consider the basis of characteristic functions of cosets of K_i in K_0: for $k_1 \in K_0$ we have

$$I(\psi)(1_{k_1 K_i})(k_1) = \int_{k_1 K_i} \psi(k_1, k_2) dk_2$$

and summing over all $k_1 K_i \in K_0/K_i$ we obtain

$$\mathrm{tr}\, I(\psi) = \int_{K_0} \psi(k, k)\, dk.$$

Going back to the definition of ψ we find

$$\operatorname{tr}\pi(f)=\int_T\int_N\int_{K_0}\mu(t)\delta_B^{1/2}(t)f(k^{-1}tnk)\,dk\,dn\,dt=\int_T\mu(t)|D(t)|^{1/2}O_t(f)\,dt.$$

This last integral is also equal to

$$\int_T\mu^w(t)|D(t)|^{1/2}O_t(f)\,dt$$

because the automorphism $\mathrm{Ad}(w)$ of T preserves the Haar measure and leaves the function $t\mapsto|D(t)|^{1/2}O_t(f)$ invariant. We conclude

$$\operatorname{tr}\pi(f)=\int_{G_{\mathrm{rs}}}\Theta_\pi(g)f(g)\,dg=\frac{1}{2}\int_T|D(t)|(\mu(t)+\mu^w(t))|D(t)|^{-1/2}O_t(f)\,dt$$

where Θ_π is defined as in the Proposition, thanks to the Weyl integration formula. □

Remark 3.24. *This generalizes to arbitrary connected reductive groups, see* [vD72].

Corollary 3.25. *Theorem 3.15 holds for the Steinberg representation, and for any elliptic maximal torus T' of G and any $t\in T'_{G-\mathrm{reg}}$ we have $\Theta_{\mathrm{St}}(t)=-1$.*

Proof. For $\mu=\delta_B^{\pm1/2}$ the semi-simplification of $\mathrm{Ind}_B^G\,\mu$ is isomorphic to $1\oplus\mathrm{St}$ (where 1 denotes the trivial one-dimensional representation of G). Obviously the trivial representation satisfies Theorem 3.15 and $\Theta_1=1$, so we deduce Theorem 3.15 sor the Steinberg representation and the relation $\Theta_{\mathrm{St}}=\Theta_{\mathrm{Ind}_B^G\delta_B^{1/2}}-\Theta_1$ on G_{rs}. □

Of course the Proposition also allows us to compute Θ_{St} on the split maximal torus T.

We have just proved a special case of the Jacquet-Langlands correspondence: the Steinberg representation of $G=\mathrm{GL}_2(\mathbb{Q}_p)$ will correspond to the trivial representation of $D^\times$.

Remark 3.26. *This strategy of reduction to the supercuspidal case was not successful for arbitrary reductive groups (in general we do not have enough "obvious" cases like the trivial representation). Harish-Chandra* [HC99] *proved the general case by passing to the Lie algebra instead. This uses the exponential map, so this argument does not apply over positive characteristic local fields.*

Remark 3.27. *Any ω^{-1}-equivariant smooth function with compact support modulo Z can be written as $g\mapsto\int_Z\omega(z)^{-1}f(zg)dz$ for some $f\in C_c^\infty(G)$ (this can be shown using local sections of $G\to G/Z$, for example $(\mathrm{SL}_2(\mathbb{Q}_p)\cap K_2)\times Z$ is isomorphic via the multiplication map to a neighbourhood of 1 in G). This implies (exercise) that for any $f\in\mathcal{H}(G,\omega_\pi^{-1})$, $\operatorname{tr}\pi(f)=\int_{G/Z}f(g)\Theta_\pi(g)\,d\dot{g}$.*

3.5. **Coefficients and pseudo-coefficients.** We push further the argument used in the proof of Corollary 2.49.

Proposition 3.28. *Let (V,π) be an irreducible supercuspidal representation of G, and let ω_π be its central character.*

(1) *Let (U,σ) be a smooth representation of G admitting central character ω_π. For $\widetilde{v}_0 \in \widetilde{V}$ and $u_0 \in U$, the linear map*

$$\begin{aligned} \phi_{\widetilde{v}_0,u_0} : V &\longrightarrow U \\ v &\longmapsto \int_{G/Z} \langle \pi(g^{-1})v, \widetilde{v}_0\rangle \sigma(g) u_0 \, d\dot{g} \end{aligned}$$

is G-equivariant. In particular, it vanishes identically if (U,σ) is irreducible but not isomorphic to (V,π). For $(U,\sigma) = (V,\pi)$, $\phi_{\widetilde{v}_0,u_0} = d_\pi^{-1}\langle u_0, \widetilde{v}_0\rangle \mathrm{Id}_V$.

(2) *For $v_0 \in V$ and $\widetilde{v}_0 \in \widetilde{V}$ let $f_{v_0,\widetilde{v}_0} \in \mathcal{H}(G,\omega_\pi^{-1})$ be the matrix coefficient for $\widetilde{V}$*

$$g \mapsto \langle \pi(g^{-1})v_0, \widetilde{v}_0\rangle = \langle v_0, \widetilde{\pi}(g)\widetilde{v}_0\rangle.$$

Then for any irreducible smooth representation (U,σ) of G having central character ω_π,

$$\operatorname{tr}\sigma(f_{v_0,\widetilde{v}_0}) = \begin{cases} 0 & \text{if } \sigma \not\simeq \pi, \\ d_\pi^{-1}\langle v_0, \widetilde{v}_0\rangle & \text{if } \sigma \simeq \pi. \end{cases}$$

Proof. The first point was proved in the proof of Corollary 2.49. Let (U,σ) be an irreducible smooth representation of G having central character ω_π. For $u \in U$ we have $\sigma(f_{v_0,\widetilde{v}_0})u = \phi_{\widetilde{v}_0,u}(v_0)$. The first point shows that $\sigma(f_{v_0,\widetilde{v}_0}) = 0$ if $\sigma \not\simeq \pi$. The first point also shows that for $v \in V$ we have $\pi(f_{v_0,\widetilde{v}_0})v = \phi_{\widetilde{v}_0,v}(v_0) = d_\pi^{-1}\langle v, \widetilde{v}_0\rangle v_0$ and so $\operatorname{tr}\pi(f_{v_0,\widetilde{v}_0}) = \operatorname{tr}\pi(f_{v_0,\widetilde{v}_0} \mid \mathbb{C}v_0) = d_\pi^{-1}\langle v_0, \widetilde{v}_0\rangle$. $\square$

In particular if we take $v_0 \in V$ and $\widetilde{v}_0 \in \widetilde{V}_0$ such that $\langle v_0, \widetilde{v}_0\rangle = d_\pi$ then we have produced $f \in \mathcal{H}(G,\omega_\pi^{-1})$ distinguishing π among all irreducible smooth representations of G having same central character. Note that for finite (or compact) groups there is a natural choice for such a function, namely the trace of the contragredient of π, but for G this is not a smooth compactly supported function! We would like to have similar functions also for irreducible non-supercuspidal representations of G. It turns out that this is not possible for an irreducible $\mathrm{Ind}_B^G \mu$, but it is almost possible for the Steinberg representation.

Recall that $\widetilde{w} = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} \in G$ normalizes I. Denote $\widetilde{I} = IZ/Z \sqcup \widetilde{w}IZ/Z$, a compact open subgroup of G/Z. Let $\mathrm{sign} : \widetilde{I} \to \{\pm 1\}$ be the character which is trivial on IZ/Z and maps $\widetilde{w}$ to -1. Define $f_{\mathrm{EP}} \in \mathcal{H}(G/Z)$ as $e_{K_0Z/Z} - e_{\widetilde{I},\mathrm{sign}}$ where $e_{\widetilde{I},\mathrm{sign}}$ is $\mathrm{vol}(\widetilde{I})^{-1}\mathrm{sign}$ (extended by zero outside of $\widetilde{I}$).

Proposition 3.29. *For any smooth irreducible representation (V, π) of G having trivial central character, we have*

$$\operatorname{tr}\pi(f_{\mathrm{EP}}) = \begin{cases} -1 & \text{if } \pi \simeq \mathrm{St}, \\ 1 & \text{if } \pi \text{ is trivial}, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The function f_{EP} is bi-IZ/Z-invariant and so $\pi(f)V \subset V^I$, in particular the trace vanishes if $V^I = 0$. We classified the representations of G such that $V^I \neq 0$ in Proposition 2.59 and before Proposition 2.57. The ones having trivial central character are the characters $\chi \circ \det$ for $\chi : \mathbb{Q}_p^\times \to \{\pm 1\}$ unramified (there are two such characters), $(\chi \circ \det) \otimes \mathrm{St}$ for the same χ's, and the irreducible $\operatorname{Ind}_B^G \mu$ with $\mu|_Z = 1$ (i.e. $\mu_1\mu_2 = 1$) and μ unramified. Note that $\operatorname{tr}\pi(e_{K_0 Z/Z}) = \dim V^{K_0}$ and $\operatorname{tr}\pi(e_{\widetilde{I},\mathrm{sign}}) = \dim\ker([I\widetilde{w}I] + 1 \,|\, V^I)$. Recall that we computed the matrix $[I\widetilde{w}I]$ in a basis of $(\operatorname{Ind}_B^G \mu)^I$ and found

$$\begin{pmatrix} 0 & \mu_2(p)p^{1/2} \\ \mu_1(p)p^{-1/2} & \end{pmatrix}.$$

If $\mu|_Z = 1$ this matrix has trace 0 and determinant 1 so its eigenvalues are ± 1. So for $V = \operatorname{Ind}_B^G \mu$ with $\mu|_Z = 1$ we have

$$\dim V^{K_0} = \dim\ker([I\widetilde{w}I] + 1 \,|\, V^I) = 1$$

and the trace of f_{EP} on $\operatorname{Ind}_B^G \mu$ vanishes. If $\chi : \mathbb{Q}_p^\times \to \{\pm 1\}$ is the unramified character of order 2 then for the one-dimensional representation $\chi \circ \det$ of G the two dimensions above are also equal to 1. For the trivial representation V of G we have $\dim V^{K_0} = 1$ and $\ker([I\widetilde{w}I] + 1 \,|\, V^I) = 0$. For $(\chi \circ \det) \otimes \mathrm{St}$ we have $V^{K_0} = 0$ and $\ker([I\widetilde{w}I] + 1 \,|\, V^I)$ has dimension one (resp. zero) if χ is trivial (resp. non-trivial). □

So apart from the trivial representation, $-f_{\mathrm{EP}}$ plays the same role for the Steinberg representation as the matrix coefficient for a supercuspidal representation. We call $-f_{\mathrm{EP}}$ a pseudo-coefficient for St.

Inspired by the supercuspidal case, we ask if the orbital integrals of f_{EP} are related to the Harish-Chandra character of the Steinberg representation.

Theorem 3.30. *Let $\gamma \in G_{\mathrm{rs}}$. Then*

$$O_\gamma(f_{\mathrm{EP}}) = \begin{cases} 0 & \text{if } \gamma \text{ is hyperbolic}, \\ \operatorname{vol}(G_\gamma/Z)^{-1} & \text{if } \gamma \text{ is elliptic.} \end{cases}$$

Exercise: prove the first case using Proposition 3.29 and Proposition 3.23.

For the proof we introduce a geometric tool. Recall that the discrete set $G/K_0 Z$ parametrizes lattices in $\mathbb{Q}_p^2$ up to rescaling, and that G/IZ parametrizes pairs (L, D) where L is a lattice in $\mathbb{Q}_p^2$ and $D \subset L/pL$ is an $\mathbb{F}_p$-line, again up to rescaling. In particular to such a pair (L, D) we can associate another lattice: the preimage of D in L. This motivates the following definition.

Definition 3.31. *Denote $\mathcal{V} = G/K_0Z$. Two lattices-up-to-rescaling $[L_1],[L_2] \subset \mathcal{V}$ are* neighbours *if L_1 and L_2 can be chosen so that $L_2 \subset L_1$ and $|L_1/L_2| = p$.*

This relation is symmetric ($pL_1 \subset L_2$ and $|L_2/pL_1| = p^2/p = p$) and $[L]$ is never its own neighbour. The neighbours of $[L]$ are naturally in bijection with the set of lines in L/pL, in particular $[L]$ has $p+1$ neighbours. Let $\mathcal{E} \subset \mathcal{P}(\mathcal{V})$ be the set of $\{[L_1],[L_2]\}$ which are neighbours. This new notation suggests that $\mathcal{V}$ is a set of vertices and $\mathcal{E}$ is a set of edges, i.e. $(\mathcal{V},\mathcal{E})$ is a graph (in a combinatorial sense). Let $\mathcal{A}$ be the associated topological space:

- For each edge $\{v_1, v_2\}$ in $\mathcal{E}$ choose an ordering (v_1, v_2). In other words choose functions $s, t : \mathcal{E} \to \mathcal{V}$ ("source" and "target") such that for any $e \in \mathcal{E}$ we have $e = \{s(e), t(e)\}$.
- Let $\mathcal{A}$ be the quotient of
$$\mathcal{V} \sqcup (\mathcal{E} \times [0,1])$$
by the equivalence relation generated by the relations $(e,0) \sim s(e)$ and $(e,1) \sim t(e)$ for any $e \in \mathcal{E}$.

The space $\mathcal{A}$ is Hausdorff and locally compact (exercise). The continuous map
$$\mathcal{E} \times [0,1] \longrightarrow \mathcal{A}$$
is surjective because any $v \in \mathcal{V}$ appears in at least one edge (in fact, $p+1$ edges).

Proposition 3.32. *The graph $(\mathcal{V},\mathcal{E})$ is a tree, i.e. it is connected (for any $v, v' \in \mathcal{V}$, there exist $k \geq 2$ and $(v_1,\dots,v_k) \in \mathcal{V}^k$ such that $v_1 = v$, $v_k = v'$ and $\{v_i, v_{i+1}\} \in \mathcal{E}$ for any i, i.e. a path between v and v') and does not contain any cycle (that is, a non-trivial path from v to v such that $v_1,\dots,v_{k-1}$ are pairwise distinct).*

Proof. Recall that G acts transitively on $\mathcal{V}$ and $\mathcal{E}$. In fact the Cartan decomposition says that for any $[L_1]$ and $[L_2]$ in $\mathcal{V}$, there is a basis (e,f) of L_1 and integers $a \geq b$ such that $(p^a e, p^b f)$ is a basis of L_2. From uniqueness in the Cartan decomposition we get that $a - b \in \mathbb{Z}_{\geq 0}$ is uniquely determined by the orbit of $([L_1],[L_2])$ under G. Denote $d([L_1],[L_2]) = a - b$, then $d([L_2],[L_1]) = d([L_1],[L_2])$ and $[L_1]$ and $[L_2]$ are neighbours if and only if $d([L_1],[L_2]) = 1$. Up to rescaling one or both lattices we may assume that $b = 0$. Then
$$[L_1] = [\mathbb{Z}_p e \oplus \mathbb{Z}_p f] \leftrightarrow [\mathbb{Z}_p pe \oplus \mathbb{Z}_p f] \leftrightarrow \cdots \leftrightarrow [\mathbb{Z}_p p^a e \oplus \mathbb{Z}_p f] = [L_2]$$
is a path joining $[L_1]$ and $[L_2]$.

Now assume that $d([L_1],[L_2]) > 0$ (i.e. $[L_2] \neq [L_1]$) and that $[L_3]$ is a neighbour of $[L_2]$ distinct from $[\mathbb{Z}_p p^{a-1} e \oplus \mathbb{Z}_p f]$. This means that we can take L_3 to be the preimage of an $\mathbb{F}_p$-line D in L_2/pL_2 distinct from $\mathbb{Z}_p p^a e \oplus \mathbb{Z}_p pf/L_2$. This means that L_3/pL_2 is generated by $f + \lambda p^a e$ for some $\lambda \in \mathbb{Z}_p$. Up to replacing f by $f + \lambda p^a e$ (note that this does not change the above path from $[L_1]$ to $[L_2]$: $\mathbb{Z}_p p^i e \oplus \mathbb{Z}_p(f + \lambda p^a e) = \mathbb{Z}_p p^i e \oplus \mathbb{Z}_p f$ for $0 \leq i \leq a$), we can assume that $L_3 = \mathbb{Z}_p p^{a+1} e \oplus \mathbb{Z}_p f$. This shows $d([L_1],[L_3]) = d([L_1],[L_2]) + 1$, in particular $[L_3] \neq [L_1]$. ☐

We can extend the function $d(\cdot,\cdot)$ to $\mathcal{A}^2$ as follows:

(1) For $e \in \mathcal{E}$ and $x, y \in [0,1]$ define
$$d((e,x),(e,y)) = |x-y|.$$

(2) For $x, y \in \mathcal{A}$ define
$$d(x,y) = \inf \left\{ \sum_{i=1}^{k-1} d(x_i, x_{i+1}) \,\middle|\, x_1 = x,\, x_k = y,\, \forall i \exists e \in \mathcal{E}, x_i, x_{i+1} \in \{e\} \times [0,1] \right\}$$

Exercise: check that this is well-defined and that this infimum is a minimum, that $d(\cdot,\cdot)$ is a metric on $\mathcal{A}$ (a general fact for any connected graph), that G acts by isometries, and that for any $x, y \in \mathcal{A}$ there is a unique geodesic in $\mathcal{A}$ from x to y, denoted $[x,y]$ (existence is true in any connected graph, uniqueness is true in any tree). Recall that a geodesic is (in this context) a continuous map $f : [0, d(x,y)] \to \mathcal{A}$ such that $f(0) = x$, $f(d(x,y)) = y$ and for any $t_1, t_2 \in [0, d(x,y)]$, $d(f(t_1), f(t_2)) = |t_1 - t_2|$.

The Bruhat-Tits tree (for more general groups, the Bruhat-Tits building, see [BT72] and [BT84]) is a p-adic analogue of symmetric spaces in the theory of real Lie groups. The Cartan fixed point theorem gives a geometric proof of conjugacy of maximal compact subgroups in a connected semisimple Lie group. The following lemma is the analogue in the present context (see [BT72, §3.2] for the general case).

Lemma 3.33. *Let K be a compact subgroup of G/Z. Then $\mathcal{A}^K \neq \emptyset$. In particular, if γ is an elliptic element of G (i.e. if $\gamma \in Z \cup G_{\mathrm{rs}}^{\mathrm{ell}}$) then $\mathcal{A}^\gamma \neq \emptyset$.*

Proof. Choose $v_0 \in \mathcal{V}$ arbitrarily. Then $Kv_0 \subset \mathcal{V}$ is finite. Since any closed ball in $\mathcal{A}$ is compact, there exists $x \in \mathcal{A}$ minimizing $\max\{d(x,y) \,|\, y \in Kv_0\}$. Let us show that x is unique. Let $x' \in \mathcal{A}$ be a different minimizer, and let $x'' \in [x, x']$ distinct from x and x'. For any $y \in \mathcal{A}$, we have $d(x'', y) < \max(d(x,y), d(x',y))$. (Quick and dirty argument: $\mathcal{A} \smallsetminus \{x''\}$ has finitely many connected components, and x and x' lie in different components, so y is not in the same component as x or x', say x. Then the geodesic $[x,y]$ goes through x''.) This gives a contradiction. So x is unique. For any $k \in K$, kx has the same minimizing property, so x is fixed by K.

If γ is elliptic then the closure of the subgroup of G/Z generated by γ is compact. □

Proof of Theorem 3.30. Let $v_0 = [\mathbb{Z}_p^2] \in \mathcal{V}$. For $g \in G$, $g^{-1}\gamma g \in K_0 Z/Z$ if and only if γ fixes $gv_0 \in \mathcal{V}$. Note that $\widetilde{I}$ is the stabilizer of $e_0 = \{[\mathbb{Z}_p^2], [p\mathbb{Z}_p \times \mathbb{Z}_p]\} \in \mathcal{E}$ (observe that $\widetilde{w}$ swaps the two endpoints). Therefore $g^{-1}\gamma g \in \widetilde{I}$ if and only if γ fixes ge_0. Using these facts, we get
$$O_\gamma(f_{\mathrm{EP}}) = \sum_{v \in G_\gamma \backslash \mathcal{V}^\gamma} \mathrm{vol}(\mathrm{Stab}_{G_\gamma/Z}(v))^{-1} - \sum_{e \in G_\gamma \backslash \mathcal{E}^\gamma} \mathrm{vol}(\mathrm{Stab}_{G_\gamma/Z}(e))^{-1} \mathrm{sign}(\gamma, e)$$
where $\mathrm{sign}(\gamma, e) = +1$ if γ fixes e pointwise (i.e. if it fixes the endpoints of e) and $\mathrm{sign}(\gamma, e) = -1$ if it swaps the endpoints.

Let us show that $\mathcal{A}^\gamma$ connected. Let $x, y \in \mathcal{A}^\gamma$, then $\gamma([x,y])$ is a geodesic from $\gamma x = x$ to $\gamma y = y$, so it equals $[x,y]$ and since γ is an isometry every point of $[x,y]$ is fixed by γ.

First we consider the case where γ is elliptic. We have just proved that $\mathcal{A}^\gamma \neq \emptyset$. Now G_γ/Z is compact so by Lemma 3.5 the set $\mathcal{A}^\gamma$ is compact, and we can expand the above expression to get

$$O_\gamma(f_{\mathrm{EP}}) = \sum_{v \in \mathcal{V}^\gamma} \mathrm{vol}(G_\gamma/Z)^{-1} - \sum_{e \in \mathcal{E}^\gamma} \mathrm{vol}(G_\gamma/Z)^{-1} \mathrm{sign}(\gamma, e).$$

Although it is somewhat artificial, we distinguish two cases:

- If there exists $e \in \mathcal{E}^\gamma$ such that $\mathrm{sign}(\gamma, e) = -1$, then denoting by x the middle point of e, x is fixed by γ and γ swaps the two connected components of $\mathcal{A} \smallsetminus \{x\}$, so $\mathcal{V}^\gamma = \emptyset$ and $\mathcal{E}^\gamma = \{e\}$. It follows that $O_\gamma(f_{\mathrm{EP}}) = 1$.

- Otherwise $\mathcal{A}^\gamma$ is a subgraph of $\mathcal{A}$, and we are left to compute the difference between the number of its edges and the number of its vertices (i.e. its Euler characteristic!). Since $\mathcal{A}^\gamma$ is non-empty and connected, it is also a tree and one can give a simple elementary argument, by induction on the number of vertices (remove a vertex from the boundary, as well as the unique edge containing it; repeat until there is only one vertex left).

We now consider the hyperbolic case. If $\mathcal{A}^\gamma = \emptyset$ then the result is obvious, so we might as well assume that it is non-empty. The centralizer $G_\gamma/Z \simeq \mathbb{Q}_p^\times$ is not compact and it acts on $\mathcal{A}^\gamma$ with compact stabilizers, so $\mathcal{A}^\gamma$ is not compact either. If there exists $e \in \mathcal{E}^\gamma$ such that $\mathrm{sign}(\gamma, e) = -1$ then as above $\mathcal{A}^\gamma$ is a point, but it has an action of the non-compact group G_γ/Z and the stabilizer of any point of $\mathcal{A}$ is a compact subgroup of G/Z. So for any $e \in \mathcal{E}^\gamma$ all points of the image of $\{e\} \times [0,1]$ in $\mathcal{A}$ are fixed by γ. In particular $\mathcal{V}^\gamma$ is not empty, i.e. γ is conjugated to an element of $K_0 Z$, and so $v(\det \gamma)$ is even. Up to conjugating, we may assume that $\gamma \in T_{G-\mathrm{reg}}$. Considering the valuation of its determinant we have $\gamma \in Z T_0$. The topological realization $\mathcal{X}$ of the connected subgraph of $(\mathcal{V}, \mathcal{E})$ with vertices $\{[p^a \mathbb{Z}_p \times \mathbb{Z}_p] \,|\, a \in \mathbb{Z}\}$ (an *apartment* in the terminology of Bruhat and Tits, here it is an infinite geodesic), that we encountered in the proof of Proposition 3.32, is included in $\mathcal{A}^\gamma$. The element $t = \mathrm{diag}(p, 1)$ of $G_\gamma = T$ acts simply transitively on the set of vertices of $\mathcal{X}$. For $y \in \mathcal{A} \smallsetminus \mathcal{X}$, there is a unique vertex x of $\mathcal{X}$ such that $d(x', y) > d(x, y)$ for any $x' \in \mathcal{X} \smallsetminus \{x\}$. We call this x the projection of y on $\mathcal{X}$, denote $\mathrm{pr}_{\mathcal{X}}(y)$. The fibres of $\mathrm{pr}_{\mathcal{X}}$ give a partition of $\mathcal{A}^\gamma \smallsetminus \mathcal{X}$, and $t^{\mathbb{Z}}$ acts simply transitively on this partition. Since $\mathcal{A}^\gamma$ is connected, for any $x \in \mathcal{E} \cap \mathcal{X}$ the subset $(\mathrm{pr}_{\mathcal{X}}^{-1}(\{x\}) \cap \mathcal{A}^\gamma) \cup \{x\}$ of $\mathcal{A}^\gamma$ is a finite subtree and x is one of its endpoints. The quotient group $G_\gamma/Z t^{\mathbb{Z}}$ is compact, so the quotient $t^{\mathbb{Z}} \backslash \mathcal{A}^\gamma$ is finite, and arguing as in the elliptic case we see that $O_\gamma(f_{\mathrm{EP}})$ is proportional to the Euler characteristic of the graph $t^{\mathbb{Z}} \backslash \mathcal{A}^\gamma$. Now this graph is very simple: $t^{\mathbb{Z}} \backslash \mathcal{X}$ has one vertex, with one edge from this vertex to itself (a loop), and so $t^{\mathbb{Z}} \backslash \mathcal{A}^\gamma$ is simply obtained by attaching a finite tree to this vertex. The Euler characteristic of this graph is zero: by the same induction as in the elliptic case, we are reduced to the case of a loop, which has one vertex and one edge. ∎

Remark 3.34. (1) *This beautiful geometric argument generalizes to an algebro-topologic one for a general connected reductive group, see* [Ser71] *and* [Kot88].

(2) *The computation of orbital integrals of e_{K_0} (preceding Proposition 3.10), even of other elements of $\mathcal{H}(G, K_0)$, can also be done geometrically using $\mathcal{A}$, see* [Kot05]. *Like the "lattice-theoretic" computation, this is particular to GL_2.*

To conclude, we have constructed a pseudo-coefficient f_π for any essentially square-integrable representation π, whose orbital integrals are given by $\Theta_{\widetilde{\pi}}$ on $G_{\mathrm{rs}}^{\mathrm{ell}}$ and vanishing on $G_{\mathrm{rs}}^{\mathrm{hyp}}$ (in fact any pseudo-coefficient satisfies this, but to prove this we would need the very natural fact that orbital integrals vanish if all traces vanish, and this is not obvious ...).

3.6. Elliptic orthogonality.

Theorem 3.35. *If π_1 and π_2 are irreducible smooth essentially square-integrable representations of G with $\omega_{\pi_1} = \omega_{\pi_2}$, we have*

$$\sum_{T' \in \mathcal{T}_{\mathrm{ell}}} \frac{1}{2} \int_{T'_{G-\mathrm{reg}}/Z} |D(t)| \Theta_{\pi_1}(t) \Theta_{\widetilde{\pi_2}}(t) \operatorname{vol}(T'/Z)^{-1} \, dt = \begin{cases} 1 & \text{if } \pi_1 \simeq \pi_2 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Recall that we have a pseudo-coefficient $f_{\pi_2} \in C_c^\infty(G, \omega_{\pi_2}^{-1})$.

$$\begin{aligned} \operatorname{tr} \pi_1(f_{\pi_2}) &= \int_{G/Z} f_{\pi_2}(g) \Theta_{\pi_1}(g) \, d\dot{g} \\ &= \sum_{T' \in \mathcal{T}} \frac{1}{2} \int_{T'_{G-\mathrm{reg}}/Z} |D(t)| \Theta_{\pi_1}(t) O_t(f_{\pi_2})(t) \, dt \\ &= \sum_{T' \in \mathcal{T}_{\mathrm{ell}}} \frac{1}{2} \int_{T'_{G-\mathrm{reg}}/Z} |D(t)| \Theta_{\pi_1}(t) \Theta_{\widetilde{\pi_2}}(t) \operatorname{vol}(T'/Z)^{-1} \, dt. \end{aligned}$$

□

Remark 3.36. *If ω_{π_i} is unitary (which can always be arranged after twisting), then both π_i's are unitary and $\widetilde{\pi_2} \simeq \overline{\pi_2}$ so that (exercise) $\Theta_{\widetilde{\pi_2}} = \overline{\Theta_{\pi_2}}$ and we recover the more familiar "orthogonality of characters" formulation.*

3.7. Existence of supercuspidal representations.

Theorem 3.37. *Let $\omega : Z \to \mathbb{C}^\times$ be a smooth character. There exists an irreducible supercuspidal representation of G having central character ω.*

Proof. Maybe later. □

4. Trace formulas

We now change notations: G will denote linear algebraic groups etc.

4.1. **Quaternion algebras and inner forms of** GL_2**.** We refer to Appendix D for general results about quaternion algebras. Let us recall the classification results over local and global fields that we shall need.

Theorem 4.1. (1) *Up to isomorphism there are two quaternion algebras over* $\mathbb{R}$*: the split one and the "usual" quaternion algebra* $\mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k$ *with* $i^2 = j^2 = -1$ *and* $ij = -ji = k$.

(2) *Up to isomorphism there are two quaternion algebras over* $\mathbb{Q}_p$*: the split one and* $E \oplus E\varpi$ *where* $E/\mathbb{Q}_p$ *is an unramified quadratic extension,* $\varpi^2 = p$ *and conjugation by* ϖ *on* E *induces the non-trivial element of* $\mathrm{Gal}(E/\mathbb{Q}_p)$.

(3) *The isomorphism class of a quaternion algebra* D *over* $\mathbb{Q}$ *is determined by the finite set* S *of places of* $\mathbb{Q}$ *such that* $D_v := \mathbb{Q}_v \otimes_{\mathbb{Q}} D$ *is not isomorphic to* $M_2(\mathbb{Q}_v)$*, and this set has even cardinality. Conversely any finite set* S *of places of* $\mathbb{Q}$ *having even cardinality is associated to a quaternion algebra over* $\mathbb{Q}$.

The first point is well-known via the relation with quadratic spaces in dimension 3 (see Appendix D), in fact the classification of quadratic spaces over $\mathbb{R}$ by their signature is well-known in any dimension. The second point is contained in Corollary E.5, except for the explicit construction which is left as an exercise. The third point (a special case of the theorem of Hasse-Minkowski, itself a special case of several theorems, many due to Kneser) is harder. See [Ser77, Ch. IV] for an elementary proof (over $\mathbb{Q}$).

If K is a field of characteristic zero (this will be $\mathbb{Q}$ or one of its completions) and D is a quaternion algebra we denote by G the associated algebraic group over K defined as the functor

$$\begin{aligned} K-\mathrm{Alg} &\longrightarrow \mathrm{Groups} \\ R &\longmapsto (R \otimes_K D)^\times \end{aligned}$$

where $K-\mathrm{Alg}$ is the category of commutative K-algebras. In particular the base change of G to some finite extension K' of K is isomorphic to GL_2. The group G can also be described explicitly using the 1-cocycle c introduced in the proof of Proposition D.1. Using the same notation as in this proof, ψ induces a natural isomorphism between the functor

$$R \mapsto \{g \in \mathrm{GL}_2(K' \otimes_K R) \,|\, \forall \sigma \in \mathrm{Gal}(K'/K),\ \mathrm{Ad}(c(\sigma))(\sigma(g)) = g\} \tag{4.1}$$

and G. We call G the *inner form* of GL_2 associated to D (because PGL_2 is the group of inner automorphisms of GL_2).

If D is not split we define a maximal torus of G to be the centralizer (as an algebraic subgroup of G) of an element of $G(K) \smallsetminus K^\times = D \smallsetminus K$. By Lemma D.3 such an element becomes semi-simple regular after extension of scalars to a finite extension K' of K splitting D, so any maximal torus T of G is commutative, satisfies $K' \times_K T \simeq \mathrm{GL}_{1,K}^2$, and is the centralizer of any element of $T(K) \smallsetminus K^\times$.

Lemma 4.2. *Let* K *be a field of characteristic zero. Let* D *be a non-split quaternion algebra over* K*. Let* G *be the associated inner form of* GL_2.

(1) *Two elements of D are conjugated by $G(K)$ if and only if they have the same characteristic polynomial (defined in Lemma D.3).*

(2) *For any maximal torus T of G we have $N_{G(K)}(T(K))/T(K) = \mathbb{Z}/2\mathbb{Z}$.*

Proof. Let $x, y \in D$ have the same characteristic polynomial. If one of them belongs to K then the result is clear using Lemma D.3. Otherwise they are conjugated in $K' \otimes_K D$ for some finite extension K'/K, i.e. there exists $g \in G(K')$ such that $gxg^{-1} = y$. We can assume that K'/K is Galois. Let T be the maximal torus of G which is the centralizer of x. Denote $E = K[x]$, a quadratic extension of K, so that $T \simeq \mathrm{Res}_{E/K}(\mathrm{GL}_1)$. For any $\sigma \in \mathrm{Gal}(K'/K)$ we have $\sigma(g)x\sigma(g)^{-1} = y$ and so $\sigma(g)^{-1}g \in T(K')$, and this defines a 1-cocycle $\mathrm{Gal}(K'/K) \to T(K')$. By Shapiro's lemma we have $H^1(K,T) \simeq H^1(E, \mathrm{GL}_1) = \{1\}$ (by Hilbert 90), so up to replacing K' by a quadratic extension we can find $t \in T(K')$ such that $gt \in G(K)$.

Take $x \in T(K) \smallsetminus K^\times$ and consider $x^{-1} \det x$. It has the same characteristic polynomial as x but is not equal to x, otherwise we would have $x^2 = \det x \in K^\times$ but in this case we have $x^2 = -\det x$. Thus x and $x^{-1}\det x$ are conjugated in $G(K)$ by an element of $N_{G(K)}(T(K)) \smallsetminus T(K)$. It remains to check that $N_{G(K)}(T(K))/T(K)$ has at most two elements. Letting $E = K[x] \subset D$ be the quadratic extension corresponding to T, we have an isomorphism

$$E \otimes_K D \simeq M_2(E)$$

mapping $E \otimes_K E$ to the sub-K-algebra of diagonal matrices, and it is easy to see that we get an embedding of $N_{G(K)}(T(K))/T(K)$ into the Weyl group of the diagonal torus in $\mathrm{GL}_2(E)$, which has two elements. □

Proposition 4.3. *Let D be a non-split quaternion algebra over $\mathbb{Q}_p$. The center of D is $\mathbb{Q}_p$ and the non-central conjugacy classes in $D^\times$ are parametrized by characteristic polynomials: for every non-split $X^2 - tX + d \in \mathbb{Q}_p[X]$ the set of $x \in D^\times$ satisfying $\mathrm{tr}\, x = t$ and $\det x = d$ is a (non-empty!) conjugacy class in $D^\times$.*

Proof. The fact that non-central conjugacy classes are parametrized by characteristic polynomials is not particular to $\mathbb{Q}_p$: see Lemma 4.2. The fact that each non-split polynomial arises follows from Corollary E.5. □

We now recall two theorems proved in Gabriel Dospinescu's course, using a slightly different formulation.

First we mention that for any affine scheme $X = \mathrm{Spec} R$ of finite type over $\mathrm{Spec}(\mathbb{Q})$, the set $X(\mathbb{A})$ of $\mathbb{A}$-points has a natural topology: if we choose $x_1, \dots, x_n$ generating the $\mathbb{Q}$-algebra R then we have a corresponding embedding $X(\mathbb{A}) \hookrightarrow \mathbb{A}^n$, and we can endow $X(\mathbb{A})$ with the induced topology. Since $X(\mathbb{A})$, being defined by polynomial equations, is closed in $\mathbb{A}^n$, it inherits the property of being Hausdorff and locally compact. The problem is to show that this topology does not depend on the choice of $x_1, \dots, x_n$. Exercise: prove that this topology coincides with the topology induced by the embedding $X(\mathbb{A}) \hookrightarrow \mathbb{A}^R$ (here $\mathbb{A}^R$, the set of maps $R \to A$, is endowed with the product topology).

In particular for D a quaternion algebra over $\mathbb{Q}$ and G the associated inner form of GL_2, the group $G(\mathbb{A})$ has a natural topology, making it a locally compact topological group. As in the case of GL_2 this topological group is can also be described as a restricted product over all places of $\mathbb{Q}$, namely the restricted product of the groups $G(\mathbb{Q}_v)$ with respect to the compact open subgroups $G(\mathbb{Z}_p)$. Note that this makes sense: since it is affine of finite type over $\mathbb{Q}$, we can find equations for the group scheme G over $\mathbb{Z}[1/m]$ for some integer $m > 0$ (a *model* of G, i.e. a scheme $\underline{G}$ over $\mathbb{Z}[1/m]$ together with an isomorphism $\mathbb{Q} \times_{\mathbb{Z}[1/m]} \underline{G} \simeq G$); then $G(\mathbb{Z}_p)$ is well-defined for all p not dividing m. If we consider another model then the two possible definitions of $G(\mathbb{Z}_p)$ coincide for almost all p (i.e. all but finitely many). Concretely, a basis of open neighbourhoods of $1 \in G(\mathbb{A})$ consists of $\prod_{v\in S} U_v \times \prod_{p\notin S} G(\mathbb{Z}_p)$ where S is a large enough finite set of places of $\mathbb{Q}$ and $U_v \subset G(\mathbb{Q}_v)$ is an open neighbourhood of $1 \in G(\mathbb{Q}_v)$.

These considerations were unnecessary for GL_2 because this group is naturally defined over $\mathbb{Z}$. There are two somewhat related ways to construct isomorphisms $G(\mathbb{Z}_p) \simeq \mathrm{GL}_2(\mathbb{Z}_p)$ for almost all p.

Proposition 4.4. *Let D be a quaternion algebra over $\mathbb{Q}$ and G the associated inner form of GL_2. For any model $\underline{G}$ of G over $\mathbb{Z}[1/m]$ (for some integer $m \geq 1$) there exists a finite S of prime numbers containing all prime divisors of m and such that for any $p \notin S$ the topological group $\underline{G}(\mathbb{Z}_p)$ is isomorphic to $\mathrm{GL}_2(\mathbb{Z}_p)$.*

Proof. We sketch two different proofs.

(1) An order in D is defined to be a finitely generated sub-$\mathbb{Z}$-module A of D of rank 4, containing $\mathbb{Z}$ and stable under multiplication. It is easy to see that orders exist: if L is any lattice in D then for $n \geq 1$ sufficiently divisible the lattice $\mathbb{Z} + nL$ is an order. An order gives a model $\underline{G}$ of G over $\mathbb{Z}$, defined as a functor on commutative rings by $\underline{G}(R) = (R \otimes_{\mathbb{Z}} A)^\times$. For any prime number p the $\mathbb{Z}_p$-submodule $A_p := \mathbb{Z}_p \otimes_{\mathbb{Z}} A$ of $D_p := \mathbb{Q}_p \otimes_{\mathbb{Q}} D$ is an order. If D_p is split, i.e. if there exists an isomorphism $\psi_p : D_p \simeq M_2(\mathbb{Q}_p)$, then by the same argument as in Lemma 2.1 there exists $g_p \in \mathrm{GL}_2(\mathbb{Q}_p)$ such that $g_p \psi(A_p) g_p^{-1} \subset M_2(\mathbb{Z}_p)$. (In particular we have $\operatorname{tr} x \in \mathbb{Z}_p$ for any $x \in A_p$.) Now the bilinear form

$$\begin{aligned} D \times D &\longrightarrow \mathbb{Q} \\ (x, y) &\longmapsto \operatorname{tr} xy \end{aligned}$$

is non-degenerate, so $A^\sharp := \{x \in D \,|\, \forall y \in A, \operatorname{tr} xy \in \mathbb{Z}\}$ is also lattice in D. For almost all p we have $\mathbb{Z}_p \otimes_{\mathbb{Z}} A^\sharp = A_p$ inside D_p (a general fact about lattices in a $\mathbb{Q}$-vector space) and so with the notation above we have $g_p \psi_p(A_p) g_p^{-1} = M_2(\mathbb{Z}_p)$, and so $\underline{G}(\mathbb{Z}_p)$ is naturally isomorphic to $\mathrm{GL}_2(\mathbb{Z}_p)$. It is easy to deduce from the Cartan decomposition that the normalizer of $\mathrm{GL}_2(\mathbb{Z}_p)$ in $\mathrm{GL}_2(\mathbb{Q}_p)$ is $\mathbb{Q}_p^\times \mathrm{GL}_2(\mathbb{Z}_p)$, so the isomorphism $\underline{G}(\mathbb{Z}_p)$ is uniquely determined up to composition with conjugation by an element of $\mathrm{PGL}_2(\mathbb{Z}_p)$.

This construction can be refined: one can show that there exist maximal orders in D, which become isomorphic to $M_2(\mathbb{Z}_p)$ at any prime p where D splits. Using a maximal order gives a "better" model of G over $\mathbb{Z}$.

(2) Using a cocycle $c \in Z^1(\mathbb{Q}, \mathrm{PGL}_2)$ introduced above, we can give a "concrete" model of G as follows. For some finite Galois extension $K/\mathbb{Q}$, the

cocycle c is inflated from an element of $Z^1(\mathrm{Gal}(K/\mathbb{Q}), \mathrm{PGL}_2(K))$, that we abusively still denote c. Let S be a finite set of primes, large enough so that every prime which ramifies in $K/\mathbb{Q}$ is in S and c takes values in $\mathrm{PGL}_2(\mathcal{O}_{K,S})$, where $\mathcal{O}_{K,S} = \mathcal{O}_K[1/m]$ with $\mathcal{O}_K$ the ring of integers of K and $m = \prod_{p \in S} p$. For simplicity we also assume that $2 \in S$. Then the functor (4.1) makes sense for $\mathbb{Z}[1/m]$-algebras R, giving us a model $\underline{G}$ of G over $\mathbb{Z}[1/m]$. For $p \notin S$, choose a place $\mathfrak{p}$ of K over p, and let $c_\mathfrak{p} \in Z^1(\mathrm{Gal}(K_\mathfrak{p}/\mathbb{Q}_p), \mathrm{PGL}_2(\mathcal{O}_{K,\mathfrak{p}}))$ (here $\mathcal{O}_{K,\mathfrak{p}}$ is the ring of integers of the completion $K_\mathfrak{p}$) be the 1-cocycle obtained by restricting c to $\mathrm{Gal}(K_\mathfrak{p}/\mathbb{Q}_p)$ and using the projection $\mathbb{Z}_p \otimes_{\mathbb{Z}[1/m]} \mathcal{O}_{K,S} \to \mathcal{O}_{K,\mathfrak{p}}$. Writing Shapiro's lemma explicitly, we see that the group scheme $\mathbb{Z}_p \otimes_{\mathbb{Z}[1/m]} \underline{G}$ is given by the analogue of (4.1) for $c_\mathfrak{p}$. But one can show that $H^1(\mathrm{Gal}(K_\mathfrak{p}/\mathbb{Q}_p), \mathrm{PGL}_2(\mathcal{O}_{K,\mathfrak{p}})) = \{1\}$ (hint: first show that $H^1(k/\mathbb{F}_p, \mathrm{PGL}_2(k)) = \{1\}$ for any finite extension $k/\mathbb{F}_p$ using the interpretation with 3-dimensional quadratic spaces, then use the filtration of $\mathrm{GL}_2(\mathcal{O}_{K,\mathfrak{p}})$ by congruence subgroups and $H^1(k/\mathbb{F}_p, k) = \{0\}$). Thus there is an isomorphism $\mathbb{Z}_p \times_{\mathbb{Z}[1/m]} \underline{G} \simeq \mathbb{Z}_p \times_{\mathbb{Z}} \mathrm{GL}_2$, well-defined (from c) up to composing with conjugation by an element of $\mathrm{PGL}_2(\mathbb{Z}_p)$. An alternative way to produce these isomorphisms is to consider orders in quaternion algebras, a maximal order in $\mathrm{M}_2(\mathbb{Q}_p)$ being conjugated to $M_2(\mathbb{Z}_p)$.

Note that for $p \notin S$, we similarly have have $c_\mathfrak{p} \in Z^1(\mathrm{Gal}(K_\mathfrak{p}/\mathbb{Q}_p), \mathrm{PGL}_2(K_\mathfrak{p}))$, and its cohomology class is trivial if and only if $\mathbb{Q}_p \otimes_\mathbb{Q} D$ is split. Therefore under this assumption we get an isomorphism $\mathbb{Q}_p \times_\mathbb{Q} G \simeq \mathbb{Q}_p \times_\mathbb{Z} \mathrm{GL}_2$, well-defined up to composition with conjugation by an element of $\mathrm{GL}_2(\mathbb{Q}_p)$.

$\square$

Recall the following special case of a theorem of Mostow and Tamagawa, proved in Gabriel Dospinescu's course in the non-adélic setting.

Theorem 4.5. (1) *Let D be a non-split quaternion algebra over $\mathbb{Q}$, and G the corresponding inner form of* GL_2*. Then $G(\mathbb{Q}) \backslash G(\mathbb{A}) / \mathbb{A}^\times$ is compact.*

(2) *Let E be a quadratic extension of $\mathbb{Q}$. Then $(\mathbb{A} \otimes_\mathbb{Q} E)^\times / \mathbb{A}^\times$ is compact.*

There are useful variants of this formulation, for example $G(\mathbb{Q}) \backslash G(\mathbb{A}) / \mathbb{R}_{>0}$ is also compact, since the map $G(\mathbb{Q}) \backslash G(\mathbb{A}) / \mathbb{R}_{>0} \to G(\mathbb{Q}) \backslash G(\mathbb{A}) / \mathbb{A}^\times$ is proper: the fibers are isomorphic to $\mathbb{Q}^\times \backslash \mathbb{A}^\times / \mathbb{R}_{>0} \simeq \prod_p \mathbb{Z}_p^\times$.

Let D and G be as in the previous theorem. Let $\omega : \mathbb{Q}^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ be a continuous unitary character. Let $L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}), \omega)$ be the space of measurable functions $\phi : G(\mathbb{Q}) \backslash G(\mathbb{A}) \to \mathbb{C}$ satisfying:

- $\phi(zg) = \omega(z)\phi(g)$ for any $z \in \mathbb{A}^\times$ and $g \in G(\mathbb{A})$, and
- $$\int_{\mathbb{A}^\times G(\mathbb{Q}) \backslash G(\mathbb{A})} |\phi(g)|^2 \, d\dot{g} < \infty,$$

quotiented by the subspace of functions vanishing almost everywhere as usual. It is naturally a unitary representation of $G(\mathbb{A})$ for the action defined by $(g \cdot \phi)(x) = \phi(xg)$, admitting central character ω.

Theorem 4.6. *Let D be a non-split quaternion algebra over $\mathbb{Q}$. Let G be the associated inner form of GL_2. Let $\omega : \mathbb{Q}^\times \backslash \mathbb{A}^\times \to \mathbb{C}^\times$ be a continuous unitary character. Let K_f be a compact open subgroup of $G(\mathbb{A}_f)$. The unitary representation $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)^{K_f}$ of $G(\mathbb{R})$ decomposes discretely.*

Remark 4.7. *There are variants of this formulation, for example the unitary representation $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/R_{>0})$ of $G(\mathbb{A})/\mathbb{R}_{>0}$ also decomposes discretely. This statement is more elegant because it does not isolate the Archimedean place of $\mathbb{Q}$ among all places, but the equivalence between the two statements is not trivial and it will be easier for us to work with levels K_f.*

Recall that the proof relies on a general theorem of Gelfand, Graev and Piatetski-Shapiro: it is enough to show that for any $f \in C_c^0(G(\mathbb{R}), \omega_\infty^{-1})$, the operator

$$\begin{aligned} \rho(f) : L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)^{K_f} &\longrightarrow L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)^{K_f} \\ \phi &\longmapsto \left(g \mapsto \int_{Z(\mathbb{R})\backslash G(\mathbb{R})} \phi(gx) f(x)\, dx \right) \end{aligned}$$

is compact. In fact it is even Hilbert-Schmidt (definition recalled below, as we will also need this notion).

4.2. **Compact, Hilbert-Schmidt and trace-class operators.** Let V be a separable Hilbert space. Our convention is that Hermitian inner products are linear in the first variable. Recall that a continuous operator $T : V \to V$ is said to be compact if the image of any ball is relatively compact. Also recall that compact operators form a closed subspace of the space $\mathcal{B}(V)$ of continuous operators on V (for the strong topology). The spectrum $\sigma(T)$ of a compact operator T is such that for any $\epsilon > 0$, $\{\lambda \in \sigma(T) \,|\, |\lambda| > \epsilon\}$ is finite. We will use the spectral theory of compact operators only in the normal (even self-adjoint semi-positive definite) case. If T is compact and normal then for $\lambda \in \sigma(T) \smallsetminus \{0\}$ the eigenspace $\ker(T - \lambda \mathrm{Id}_V)$ is finite-dimensional, and we have an orthogonal decomposition $V = \bigoplus_{\lambda \in \sigma(T)} \ker(T - \lambda \mathrm{Id})$. Applying this to T^*T, we get the following "explicit" characterization of compact operators on Hilbert spaces.

Lemma 4.8. *An operator $T : V \to V$ is compact if and only if there exist a set J and orthonormal families $(f_j)_{j\in J}$ and $(g_j)_{j\in J}$ in V and a family $(\lambda_j)_{j\in J}$ such that for any $\epsilon > 0$, $\{j \in J \,|\, |\lambda_j| > \epsilon\}$ is finite, and for any $v \in V$*

$$Tv = \sum_{j\in J} \lambda_j (v, f_j) g_j.$$

Proof. It is easy to check that for families as in the lemma, the sum converges for the operator norm, i.e. T is a limit of finite rank operators. Therefore such a T is compact. Moreover it is easy to compute $T^*v = \sum_{j\in J} \overline{\lambda_j}(v, g_j) f_j$, and we see that $f_j \in \ker(T^*T - |\lambda_i|^2 \mathrm{Id}_V)$.

Conversely and guided by this computation, take $(f_i)_{i\in I}$ an orthonormal basis of V consisting of eigenvectors for T^*T, with eigenvalues $\rho_i \in \mathbb{R}_{\geq 0}$, and let $J = \{i \in I \,|\, \rho_i > 0\}$, $g_j = \rho_j^{-1/2} T f_j$ and $\lambda_j = \rho_j^{1/2}$for $j \in J$. $\square$

The proof shows that if we impose $\lambda_j \in \mathbb{R}_{>0}$ then the families are essentially unique (up to reordering and choosing different bases for the eigenspaces of T^*T). Note that Lemma 4.8 is an analogue of the Cartan decomposition $\mathrm{GL}_n(\mathbb{C}) = U(n)D^+U(n)$ where $U(n)$ is the (compact) unitary group and D^+ is the group of diagonal matrices with real, positive and decreasing coefficients on the diagonal. In fact the proof is identical.

Recall that a continuous operator $T : V \to V$ is said to be Hilbert-Schmidt if for some orthonormal basis $(e_i)_{i\in I}$ of V we have $\sum_{i\in I} \|Te_i\|^2 < \infty$. Also recall that any HS operator is compact.

Lemma 4.9. *Let T be a Hilbert-Schmidt operator on V. Then $\|T\|_{\mathrm{HS}}^2 := \sum_{i\in I} \|Te_i\|^2$ does not depend on the choice of an orthonormal basis $(e_i)_{i\in I}$ of V. Moreover $\|T^*\|_{\mathrm{HS}} = \|T\|_{\mathrm{HS}}$, and $\|\cdot\|_{\mathrm{HS}}^2$ defines a Hermitian inner product on the space $\mathcal{B}(V)_{\mathrm{HS}}$ of Hilbert-Schmidt operators on V, endowing it with a Hilbert space structure.*

Finally, writing a compact operator $T : V \to V$ as in Lemma 4.8, we have that T is Hilbert-Schmidt if and only if $\sum_{j\in J} |\lambda_j|^2 < \infty$.

Proof. Let $(f_j)_{j\in J}$ be another orthonormal basis of V. We have

$$\sum_{i\in I} \|Te_i\|^2 = \sum_{i\in I}\sum_{j\in J} |(Te_i, f_j)|^2 = \sum_{i\in I}\sum_{j\in J} |(e_i, T^*f_j)|^2 = \sum_{j\in J} \|T^*f_j\|^2$$

and this implies both independence of the choice of basis and $\|T^*\|_{\mathrm{HS}} = \|T\|_{\mathrm{HS}}$. The rest is easy (that is, left as an exercise): any orthonormal basis $(e_i)_{i\in I}$ identifies $\mathcal{B}(V)_{\mathrm{HS}}$ with $\ell^2(I, V)$, by $T \mapsto (Te_i)_{i\in I}$. □

Let (X, μ) be a separable measured space. Recall that HS operators on $L^2(X, \mu)$ are identified with elements of $L^2(X \times X, \mu \times \mu)$: a kernel $K \in L^2(X \times X, \mu \times \mu)$ defines a Hilbert-Schmidt operator $T_K : L^2(X, \mu) \to L^2(X, \mu)$ defined by

$$T_K(f)(x) = \int_X f(y)K(x, y)d\mu(y).$$

The expression given in Lemma 4.8 amounts to

$$K(x, y) = \sum_{j\in J} \lambda_j g_j(x)\overline{f_j(y)}$$

which is a sum of pairwise orthogonal elements of $L^2(X \times X, \mu \times \mu)$. Exercise: check that $\|T_K\|_{\mathrm{HS}}^2 = \|K\|^2$.

Definition 4.10. *A continuous operator $T : V \to V$ is* trace class *if it is compact and for any set J and any orthonormal families $(e_i)_{i\in I}$ and $(h_i)_{i\in I}$ in V we have $\sum_{i\in I} |(Te_i, h_i)| < \infty$.*

Remark 4.11. *This is not the standard definition, as we impose compactness, but this one does not require us to define $\sqrt{T^*T}$ for arbitrary $T \in \mathcal{B}(V)$.*

Proposition 4.12. *Let V be a Hilbert space.*

(1) *A linear combination of trace class operators $V \to V$ is trace class.*

(2) *The composition of two Hilbert-Schmidt operators is trace class.*

(3) *A continuous operator $T : V \to V$ is trace class if and only if*

$$\sum_{\rho \in \sigma(T^*T)} \sqrt{\rho} < \infty.$$

(Equivalently, if T is compact and if, writing T as in Lemma 4.8, $\sum_{j\in J} |\lambda_j| < \infty$) In particular, any trace class operator is Hilbert-Schmidt.

(4) *If T is trace class then $\operatorname{tr} T := \sum_{i\in I}(Te_i, e_i)$ does not depend on the choice of an orthonormal basis $(e_i)_{i\in I}$ of V.*

Proof. (1) Easy.

(2) Using Cauchy-Schwarz,

$$\sum_{i\in I} |(T_1T_2e_i, h_i)| = \sum_{i\in I} |(T_2e_i, T_1^*h_i)| \leq \sqrt{\sum_{i\in I} \|T_2e_i\|^2}\sqrt{\sum_{i\in I} \|T_1^*h_i\|^2}.$$

(3) Assume that T is of trace class, then T is compact so we can write T as in Lemma 4.8. Taking $e_j = f_j$ and $h_j = g_j$, we see that $\sum_{j\in J} |\lambda_j| < \infty$.

Conversely, if T can be written as in Lemma 4.8 with $\sum_{j\in J} |\lambda_j| < \infty$ then for any orthonormal families $(e_i)_{i\in I}$ and $(h_i)_{i\in I}$ we have

$$\begin{aligned}\sum_{i\in I} |(Te_i, h_i)| &\leq \sum_{i\in I}\sum_{j\in J} |\lambda_j||(e_i, f_j)||(g_j, h_i)| \\ &\leq \sum_{j\in J} |\lambda_j| \sqrt{\sum_{i\in I} |(e_i, f_j)|^2}\sqrt{\sum_{i\in I} |(g_j, h_i)|^2} \\ &\leq \sum_{j\in J} |\lambda_j|\end{aligned}$$

using the Cauchy-Schwarz inequality.

(4) Writing T as in Lemma 4.8 we have

$$\sum_{i\in I}(Te_i, e_i) = \sum_{i\in I}\sum_{j\in J} \lambda_j(e_i, f_j)(g_j, e_i) = \sum_{j\in J} \lambda_j \sum_{i\in I}(e_i, f_j)(g_j, e_i) = \sum_{j\in J} \lambda_j(g_j, f_j)$$

where the exchange of $\sum$ signs is justified by absolute convergence:

$$\left(\sum_{i\in I} |(e_i, f_j)(g_j, e_i)|\right)^2 \leq \left(\sum_{i\in I} |(e_i, f_j)|^2\right)\left(\sum_{i\in I} |(g_j, e_i)|^2\right) = 1.$$

$\square$

It seems that the only practical way of showing that an operator is trace class is to write it as a sum of products of Hilbert-Schmidt operators: an easy computation shows that for $T_1, T_2 \in \mathcal{B}(V)_{\mathrm{HS}}$ we have $\operatorname{tr} T_1 T_2 = (T_2, T_1^*)_{\mathrm{HS}}$. Let us make the trace more explicit in the case where $V = L^2(X, \mu)$. We have $T_i = T_{K_i}$ for $K_i \in L^2(X \times X, \mu \times \mu)$ and so $T_1 T_2 = T_K$ with $K(x,y) = \int_X K_1(x,z) K_2(z,y)\, d\mu(z)$ (exercise). Note that this makes sense in $L^2(X \times X, \mu \times \mu)$: the Cauchy-Schwarz inequality gives us

$$|K(x,y)|^2 \leq \left(\int_X |K_1(x,z)|^2\, d\mu(z) \right) \left(\int_X |K_2(z,y)|^2\, d\mu(z) \right)$$

and so $\|K\|^2 \leq \|K_1\|^2 \|K_2\|^2$. We also deduce that

$$\begin{aligned} \int_X |K(x,x)|\, d\mu(x) &\leq \int_X \sqrt{\int_X |K_1(x,z)|^2\, d\mu(z)} \sqrt{\int_X |K_2(z,x)|^2\, d\mu(z)}\, d\mu(x) \\ &\leq \|K_1\|\, \|K_2\| \end{aligned}$$

where the last inequality is another application of Cauchy-Schwarz. This shows that the restriction of K to the diagonal in $X \times X$ is well-defined (by K_1 and K_2, not by K directly!) in $L^1(X, \mu)$. Thinking of the case where X is finite, we guess that $\operatorname{tr} T_K = \int_K K(x,x)\, d\mu(x)$. This is easy to check when $T_1 = T_2^*$, and the general case follows using the polarization identity for $(\cdot,\cdot)_{\mathrm{HS}}$.

This formula will be crucial for the trace formula. Unfortunately defining the restriction of K to the diagonal properly requires us to write K as the convolution of K_1 and K_2. While this should be possible in the applications in this course (this is the approach taken in [Art74]), we would prefer to write $\operatorname{tr} T_K = \int_X K(x,x) d\mu(x)$ directly in cases where it makes sense. The following theorem (Theorem 4.10 in [War79]) achieves just that.

Theorem 4.13. *Let X be a locally compact, second-countable topological space and μ a Radon measure on X. Let $T_K : L^2(X,\mu) \to L^2(X,\mu)$ be a Hilbert-Schmidt operator. Assume that T_K is trace class, and that K can be chosen (among measurable functions representing a given class in $L^2(X \times X, \mu \times \mu)$, i.e. up to adding a measurable function which vanishes almost everywhere) so that for almost all $y \in X$ the function $K(\cdot, y)$ is continuous. Fix such a representative K. Then $x \mapsto K(x,x)$ is integrable with respect to μ and we have*

$$\operatorname{tr} T_K = \int_x K(x,x) d\mu(x).$$

Proof. As we saw above we can write

$$K(x,y) = \sum_{j \in J} \lambda_j g_j(x) \overline{f_j(y)}$$

with $(f_j)_{j \in J}$ and $(g_j)_{j \in J}$ orthonormal families in $L^2(X,\mu)$, $\lambda_j > 0$ and $\sum_{j \in J} \lambda_j < \infty$. This equality holds in $L^2(X \times X, \mu \times \mu)$, i.e. away from a set of $\mu \times \mu$-measure zero in $X \times X$. Integrating over X, we see that the series $\sum_{j \in J} \lambda_j |g_j(x)|^2$ and

$\sum_{j\in J}\lambda_j|f_j(y)|^2$ converge almost everywhere. Write X as the increasing union of compact subsets $(C_k)_{k\geq 1}$. By Lusin's theorem, for any $k\geq 1$, $j\in J$ and $\epsilon>0$ there is an open subset U_k of C_k such that $\mu(U_k)<\epsilon$ and g_j and f_j are continuous on $C_k\smallsetminus U_k$. By Egorov's theorem, for any $k\geq 1$ and any $\epsilon>0$ there is an open subset U_k of C_k such that $\mu(U_k)<\epsilon$ and $\sum_{j\in J}\lambda_j|g_j|^2$ and $\sum_{j\in J}\lambda_j|f_j|^2$ converge uniformly on $C_k\smallsetminus U_k$. Putting these two results together, we get (exercise) that there is a sequence $(C'_k)_{k\geq 1}$ of compact subsets of X with $C'_k\subset C_k$ and $C'_k\subset C'_{k+1}$, such that for any $k\geq 1$ we have

- $\mu(C_k\smallsetminus C'_k)<1/k$,
- for any $j\in J$, f_j and g_j are continuous on C'_k,
- $\sum_{j\in J}\lambda_j|f_j|^2$ and $\sum_{j\in J}\lambda_j|g_j|^2$ converge uniformly on C'_k.

Replacing C'_k be its smallest closed subset of full measure (note that second countability is used here), we may also assume that C'_k does not admit any proper closed subset of full measure. It follows from the Cauchy-Schwarz inequality that

$$K'(x,y):=\sum_{j\in J}\lambda_j g_j(x)\overline{f_j(y)}$$

converges uniformly on $C'_k\times C'_k$, and so it defines a continuous function on $C'_k\times C'_k$, which coincides with K away from a negligible set. More precisely, let $S_k=\{y\in C'_k \mid \int_{C'_k}|K(x,y)-K'(x,y)|\,d\mu(x)>0\}$, so that $\mu(S_k)=0$. For any $y\in C'_k\smallsetminus S_k$, the set of x in C'_k where $K(x,y)\neq K'(x,y)$ has measure zero, and since both $K(\cdot,y)$ and $K'(\cdot,y)$ are continuous it is also open and by construction of C'_k it is empty.

Let $X'=(\bigcup_k C'_k)\smallsetminus(\bigcup_k S_k)$. Taking all k into consideration, we get that K coincides with K' on $(X')^2$. We have $\mu(X\smallsetminus X')\leq\limsup_k\mu(C_k\smallsetminus C'_k)=0$. Finally

$$\operatorname{tr}T_K=\sum_{j\in J}\lambda_j\int_X g_j(x)\overline{f_j(x)}\,d\mu(x)=\sum_{j\in J}\lambda_j\int_{X'} g_j(x)\overline{f_j(x)}\,d\mu(x)=\int_{X'}K(x,x)\,d\mu(x)$$

where the last equality is given by the dominated convergence theorem (using $|g_j(x)f_j(x)|\leq(|g_j(x)|^2+|f_j(x)|^2)/2$ and $\sum_{j\in J}\lambda_j<\infty$) and also shows that $x\mapsto K(x,x)$ is integrable. $\square$

4.3. **The trace formula for anisotropic groups.** Let D be a non-split quaternion algebra over $\mathbb{Q}$ and denote by G the corresponding inner form of GL_2.

Recall that a smooth function on $G(\mathbb{A})$ is $f:G(\mathbb{A})\to\mathbb{C}$ such that for any $g\in G(\mathbb{A})$, there exists U_∞ an open neighbourhood of g_∞ in $G(\mathbb{R})$ and U_f a neighbourhood of g_f in $G(\mathbb{A}_f)$, and $\psi:U_\infty\to\mathbb{C}$ a smooth function, such that $f(x)=\psi(x_\infty)$ for any $x\in U_\infty\times U_f$. Similarly, for any $k\geq 1$ we define functions of class C^k on $G(\mathbb{A})$ (note that these are "smooth", i.e. locally constant, on the finite adélic factor $G(\mathbb{A}_f)$). Exercise: show that any $f\in C_c^k(G(\mathbb{A}))$ is a linear combination of functions of the form $\prod_v f_v$ where $f_\infty\in C_c^k(G(\mathbb{R})$, for any prime number p $f_p\in C_c^\infty(G(\mathbb{Q}_p))$ and for almost all prime numbers p, f_p is the characteristic function of $G(\mathbb{Z}_p)$. Similarly, for $\omega:\mathbb{A}^\times\to\mathbb{C}^\times$ a continuous character, any function in $C_c^k(G(\mathbb{A}),\omega^{-1})$ is a linear combination of functions of the form $\prod_v f_v$ where for almost all prime numbers p, f_p is supported on $G(\mathbb{Z}_p)Z(\mathbb{Q}_p)$ and for any $k \in G(\mathbb{Z}_p)$ we have $f_p(k) = 1$.

Also recall that any Haar measure on $G(\mathbb{A})$ is given by a collection of Haar measures on $G(\mathbb{Q}_v)$ such that for almost all prime numbers p, $\mathrm{vol}(G(\mathbb{Z}_p)) = 1$.

We will consider orbital integrals of functions on $G(\mathbb{Q}_v)$, v any place of $\mathbb{Q}$, and $G(\mathbb{A})$. For the case where v is non-Archimedean and $\mathbb{Q}_v \otimes_{\mathbb{Q}} D$ is split we defined and studied orbital integrals in Section 3.2. For any place v such that $\mathbb{Q}_v \otimes_{\mathbb{Q}} D$ is not split, for any $\gamma \in G(\mathbb{Q}_v)$ the quotient $G_\gamma(\mathbb{Q}_v)\backslash G(\mathbb{Q}_v)$ is compact and so the theory is easy (of course, explicit computations are not so easy ...). For $\mathrm{GL}_2(\mathbb{R})$, the (formal) definition of $O_\gamma(f)$ in Definition 3.1, the computation for γ semi-simple regular hyperbolic (3.2), and Lemma 3.5 (showing that for any $f \in C_c^0(G(\mathbb{R}))$, $O_\gamma(f)$ is the integral of a continuous compactly supported function) all adapt. Of course in the real case orbital integrals are almost never finite sums as in (3.1). As in the p-adic case we have similar results when $f \in C_c^0(G(\mathbb{R}), \omega^{-1})$. The analogous property in the adélic setting is the following fact.

Lemma 4.14. *Let $\omega : \mathbb{A}^\times \to \mathbb{C}^\times$ a continuous character. Fix Haar measures on $G_{\mathrm{ad}}(\mathbb{Q}_v)$ such that $\mathrm{vol}(G_{\mathrm{ad}}(\mathbb{Z}_p)) = 1$ for almost all prime numbers p. Let $f = \prod_v f_v \in C_c^k(G(\mathbb{A}), \omega^{-1})$ (as discussed above). Let $\gamma \in G(\mathbb{Q})$ be semi-simple. Then for almost all p the set of $[g_p] \in G_\gamma(\mathbb{Q}_p)\backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ such that $g_p^{-1}\gamma g_p \in G(\mathbb{Z}_p)Z(\mathbb{Q}_p)$ is simply $\{[1]\}$, and so $\mathrm{vol}(G_\gamma(\mathbb{Q}_p)/Z(\mathbb{Q}_p))\, O_\gamma(f_p) = 1$ for almost all p. In particular the function $g \mapsto f(g^{-1}\gamma g)$ in $C^k(G_\gamma(\mathbb{A})\backslash G(\mathbb{A}))$ is compactly supported, and $O_\gamma(f) = \prod_v O_\gamma(f_v)$.*

Proof. Left as an exercise, using formula (3.2) and the argument around Lemma 3.8 (for almost all p we have $\mathbb{Z}_p[\gamma] = \mathcal{O}_E$). □

Recall the following theorem which was stated in Gabriel Dospinescu's course, that we will not prove either.

Theorem 4.15 ([DM78]). *For any $f \in C_c^\infty(G(\mathbb{R}))$ there exist $k \geq 1$ and f_1, g_1, ..., f_k, g_k in $C_c^\infty(G(\mathbb{R}))$ such that $f = \sum_i f_i * g_i$.*

Remark 4.16. *In the applications in this course a weaker result would be enough, with $g_i \in C_c^k(G(\mathbb{R}))$ for a large enough integer k. This weaker result is easier to prove (although far from trivial: another use of elliptic operators ...): see* [DL71, §I.1.10] *and* [War79, Theorem 4.3 and Lemma 4.5]. *In fact the reader can check that all consequences of trace formulas that we will prove could be proved by only considering functions of the form $\sum_i f_i * g_i$ (without using that any smooth function can be written in this manner). In other words, these results are not strictly necessary for the purpose of these notes. However, avoiding them would make the formulation of certain results more complicated, and require more computations.*

Corollary 4.17. *Let $\omega : Z(\mathbb{Q})\backslash Z(\mathbb{A}) \to \mathbb{C}^\times$ be a continuous unitary character. Then for any $f \in C_c^\infty(G(\mathbb{A}), \omega^{-1})$ there exist $k \geq 1$ and $f_1, g_1, \ldots, f_k, g_k \in C_c^\infty(G(\mathbb{A}), \omega^{-1})$ such that $f = \sum_i f_i * g_i$.*

Theorem 4.18. *Fix a Haar measure on $G_{\mathrm{ad}}(\mathbb{A})$ and a continuous unitary character $\omega : Z(\mathbb{Q})\backslash Z(\mathbb{A}) \to \mathbb{C}^\times$. For any $f \in C_c^\infty(G(\mathbb{A}), \omega^{-1})$, the operator $\rho(f)$ on*

$L^2(G(\mathbb{Q})\backslash G(\mathbb{A}),\omega)$ is trace class and

$$\operatorname{tr}\rho(f)=\sum_{[\gamma]}\iota(\gamma)^{-1}\operatorname{vol}(G_\gamma(\mathbb{Q})Z(\mathbb{A})\backslash G(\mathbb{A}))O_\gamma(f) \tag{4.2}$$

where the sum is over conjugacy classes of elements γ in $G_{\mathrm{ad}}(\mathbb{Q})$, $\iota(\gamma)$ is the index of $G_\gamma(\mathbb{Q})/Z(\mathbb{Q})$ in $\mathrm{Cent}(\gamma,G_{\mathrm{ad}}(\mathbb{Q}))$ *(exercise: $\iota(\gamma)\in\{1,2\}$, and $\iota(\gamma)=2$ if and only if* $\operatorname{tr}\gamma=0$*), and only finitely many terms in the sum are non-zero.*

Note that the product $\operatorname{vol}(G_\gamma(\mathbb{Q})Z(\mathbb{A})\backslash G(\mathbb{A}))O_\gamma(f)$ does not depend on the choice of a Haar measure on $G_\gamma(\mathbb{A})/Z(\mathbb{A})$. Observe also that f is bi-K_f-invariant for some compact open subgroup K_f of $G(\mathbb{A}_f)$, and so we could replace $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}),\omega)$ by $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}),\omega)^{K_f}$ (same trace), and find ourselves in the setting of Theorem 4.6.

Proof. We have

$$\begin{aligned}(\rho(f)\phi)(x)&=\int_{G_{\mathrm{ad}}(\mathbb{A})}\phi(xg)f(g)\,d\dot g\\&=\int_{G_{\mathrm{ad}}(\mathbb{A})}\phi(y)f(x^{-1}y)\,d\dot y\\&=\int_{G_{\mathrm{ad}}(\mathbb{Q})\backslash G_{\mathrm{ad}}(\mathbb{A})}\phi(y)\sum_{\gamma\in G_{\mathrm{ad}}(\mathbb{Q})}f(x^{-1}\gamma y)\,d\dot y\\&=\int_{G_{\mathrm{ad}}(\mathbb{Q})\backslash G_{\mathrm{ad}}(\mathbb{A})}\phi(y)K_f(x,y)\,d\dot y\end{aligned}$$

with $K_f(x,y)=\sum_{\gamma\in G_{\mathrm{ad}}(\mathbb{Q})}f(x^{-1}\gamma y)$. For x and y in a compact subset C of $G(\mathbb{A})$, there is a finite subset $F(C,\operatorname{supp}(f))$ of $G_{\mathrm{ad}}(\mathbb{Q})$ such that for $x,y\in C$ and $\gamma\in G_{\mathrm{ad}}(\mathbb{Q})\smallsetminus F(C,\operatorname{supp}(f))$ we have $f(x^{-1}\gamma y)=0$, since $G_{\mathrm{ad}}(\mathbb{Q})$ is discrete in $G_{\mathrm{ad}}(\mathbb{A})$. In particular the function K_f on $(G(\mathbb{Q})\backslash G(\mathbb{A}))^2$ is continuous. Moreover $K_f(z_1x,z_2y)=\omega(z_1z_2^{-1})K_f(x,y)$ for $z_1,z_2\in Z(\mathbb{A})$, so $|K_f|$ induces a bounded function on the compact topological space $(G_{\mathrm{ad}}(\mathbb{Q})\backslash G_{\mathrm{ad}}(\mathbb{A}))^2$, in particular $|K_f|\in L^2((G_{\mathrm{ad}}(\mathbb{Q})\backslash G_{\mathrm{ad}}(\mathbb{A}))^2)$. This shows that $\rho(f)$ is Hilbert-Schmidt. To show that it is of trace class, use the Dixmier-Malliavin theorem which expresses $\rho(f)$ as $\sum_i\rho(f_i)\rho(g_i)$ and apply Proposition 4.12. Finally Theorem 4.13 (or the Dixmier-Malliavin expression, see the discussion before Theorem 4.13) shows that

$$\operatorname{tr}\rho(f)=\int_{G_{\mathrm{ad}}(\mathbb{Q})\backslash G_{\mathrm{ad}}(\mathbb{A})}K_f(x,x)\,d\dot x.$$

Note that integrability of K_f (and of $K_{|f|}$, defined analogously even though $|f|$ may not be differentiable ...) can also be checked directly, without using Theorem 4.13: $x\mapsto K_{|f|}(x,x)$ is a continous function on the compact topological space

$G_{\mathrm{ad}}(\mathbb{Q})\backslash G_{\mathrm{ad}}(\mathbb{A})$. This justifies the inversion of integral signs in the following:

$$\begin{aligned}\int_{G_{\mathrm{ad}}(\mathbb{Q})\backslash G_{\mathrm{ad}}(\mathbb{A})} K_f(x,x)\,d\dot{x} &= \sum_{[\gamma]} \int_{G_{\mathrm{ad}}(\mathbb{Q})\backslash G_{\mathrm{ad}}(\mathbb{A})} \sum_{\delta\in[\gamma]} f(x^{-1}\delta x)\,d\dot{x} \\ &= \sum_{[\gamma]} \int_{G_{\mathrm{ad}}(\mathbb{Q})\backslash G_{\mathrm{ad}}(\mathbb{A})} \sum_{\alpha\in \mathrm{Cent}(\gamma, G_{\mathrm{ad}}(\mathbb{Q}))\backslash G_{\mathrm{ad}}(\mathbb{Q})} f(x^{-1}\alpha^{-1}\gamma\alpha x)\,d\dot{x} \\ &= \sum_{[\gamma]} \int_{G_{\mathrm{ad}}(\mathbb{Q})\backslash G_{\mathrm{ad}}(\mathbb{A})} \iota(\gamma)^{-1} \sum_{\alpha\in G_\gamma(\mathbb{Q})\backslash G(\mathbb{Q})} f(x^{-1}\alpha^{-1}\gamma\alpha x)\,d\dot{x}\end{aligned}$$

Thus

$$\begin{aligned}\operatorname{tr}\rho(f) &= \sum_{[\gamma]} \iota(\gamma)^{-1} \int_{(G_\gamma(\mathbb{Q})/Z(\mathbb{Q}))\backslash G_{\mathrm{ad}}(\mathbb{A})} f(x^{-1}\gamma x)\,d\dot{x} \\ &= \sum_{[\gamma]} \iota(\gamma)^{-1} \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} \int_{(G_\gamma(\mathbb{Q})Z(\mathbb{A}))\backslash G_\gamma(\mathbb{A})} f(x^{-1}y^{-1}\gamma y x)\,d\dot{y}\,d\dot{x} \\ &= \sum_{[\gamma]} \iota(\gamma)^{-1} \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} \mathrm{vol}((G_\gamma(\mathbb{Q})Z(\mathbb{A}))\backslash G_\gamma(\mathbb{A})) f(x^{-1}\gamma x)\,d\dot{x} \\ &= \sum_{[\gamma]} \iota(\gamma)^{-1}\,\mathrm{vol}((G_\gamma(\mathbb{Q})Z(\mathbb{A}))\backslash G_\gamma(\mathbb{A}))O_\gamma(f).\end{aligned}$$

Finally we must prove that only finitely many conjugacy classes $[\gamma]$ in $G_{\mathrm{ad}}(\mathbb{Q})$ satisfy $O_\gamma(f) \neq 0$. In fact this follows from the proof of continuity of K_f above: choose a compact subset $C \subset G_{\mathrm{ad}}(\mathbb{A})$ which surjects onto $G_{\mathrm{ad}}(\mathbb{Q})\backslash G_{\mathrm{ad}}(\mathbb{A})$, then on $C \times C$ only finitely many elements on $G_{\mathrm{ad}}(\mathbb{Q})$ contribute to the sum defining K_f. Although it is not absolutely necessary, let us give a direct argument which does not use compactness of $G_{\mathrm{ad}}(\mathbb{Q})\backslash G_{\mathrm{ad}}(\mathbb{A})$. Recall that for any field of characteristic zero F, conjugacy classes in $G(F)$ are parametrized by trace and determinant. This implies that the map $\nu = \mathrm{tr}^2/\det : G_{\mathrm{ad}}(F) \to F$ is an invariant of conjugacy classes in $G_{\mathrm{ad}}(F)$. Consider the compact subset $\mathrm{supp}(f)$ of $G_{\mathrm{ad}}(\mathbb{A})$, and its image $\nu(\mathrm{supp}(f))$ in $\mathbb{A}$. Since $\mathbb{Q}$ is discrete in $\mathbb{A}$, the subset $F \cap \nu(\mathrm{supp}(f))$ of F is finite. Unfortunately the invariant ν does not completely characterize conjugacy, although counter-examples are somewhat rare. More precisely, one can check that if $\nu(\gamma_1) = \nu(\gamma_2)$ do not vanish then γ_1 and γ_2 are conjugated in $G_{\mathrm{ad}}(F)$; but $\nu^{-1}(\{0\})$ is a union of several conjugacy classes in general (if D was split and $G = \mathrm{GL}_2$ a simple example would be $\mathrm{diag}(-1,1)$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$). To conclude we also consider arithmetic invariants of conjugacy classes in $G_{\mathrm{ad}}(\mathbb{Q})$. If $\overline{g} \in G_{\mathrm{ad}}(F)$ then for a lift $g \in G(F)$ of $\overline{g}$, the image $\zeta(g)$ of $\det g$ in $F^\times/F^{\times,2}$ does not depend on the choice of the lift g, and is clearly invariant by conjugation. (Identifying G_{ad} with a special orthogonal group as explained in Section 4.1, $\zeta(g)$ is the spinor norm of g.) There exists a finite set S' of places of $\mathbb{Q}$, containing the Archimedean place and all finite places where ω is ramified (i.e. non-trivial on $\mathbb{Z}_p^\times$) such that $f = f_S f^S$ where $f_S \in C_c^\infty(\prod_{v\in S} G(\mathbb{Q}_v), \omega_S^{-1})$ and $f^S \in \prod_{p\notin S} C^\infty(G(\mathbb{Z}_p)Z(\mathbb{Q}_p), \omega_p^{-1})$. Since $G(\mathbb{Z}_p)$ is a compact subgroup of $G(\mathbb{Q}_p)$ we have $\det G(\mathbb{Z}_p) \subset \mathbb{Q}_p^\times$ and so a necessary condition for the non-vanishing of $O_\gamma(f)$ is that $v_p(\zeta(\gamma)) \in 2\mathbb{Z}$ (note that this parity is well-defined!) for all p not in S. Since S is finite this only leaves finitely many possible values for $\zeta(\gamma)$ in $\mathbb{Q}^\times/\mathbb{Q}^{\times,2}$, and since we have already seen that $(\operatorname{tr}\gamma)^2/\det\gamma$ can only take finitely many values, this implies that up to the action of $Z(\mathbb{Q})$, the pair $(\operatorname{tr}\widetilde{\gamma}, \det\widetilde{\gamma})$, where $\widetilde{\gamma} \in G(\mathbb{Q})$ lifts γ, can only take finitely many values. □

Of course this formula is useful in combination with Theorem 4.6. Recall that this theorem gives a canonical orthogonal decomposition

$$L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)^{K_f} = \widehat{\bigoplus_{\pi_\infty \in \widehat{G(\mathbb{R})}}} \operatorname{Hom}_{G(\mathbb{R})}(\pi_\infty, L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)^{K_f}) \otimes \pi_\infty$$

where each $\operatorname{Hom}_{G(\mathbb{R})}(\pi_\infty, L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)^{K_f})$ is finite-dimensional and has an action of $\mathcal{H}(G(\mathbb{A}_f), K_f, \omega_f^{-1})$. In particular for any irreducible unitary representation π_∞ of $G(\mathbb{R})$ having central character ω_∞,

$$\varinjlim_{K_f} \operatorname{Hom}_{G(\mathbb{R})}(\pi_\infty, L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)^{K_f})$$

is an admissible representation of $G(\mathbb{A}_f)$ having central character ω_f. It is endowed with a natural $G(\mathbb{A}_f)$-invariant Hermitian inner product (canonical up to $\mathbb{R}_{>0}$), and so it is semi-simple. (Exercise: formulate unitarity of the $G(\mathbb{A}_f)$-action in terms of the Hecke algebra action. Which formulation is clearer?)

Thus we have

$$\operatorname{tr}\rho(f) = \sum_\pi m^G(\pi) \operatorname{tr}\pi(f) \tag{4.3}$$

where $m^G(\pi) \in \mathbb{Z}_{\geq 0}$, the sum is over all isomorphism classes of tensor products $\pi = \pi_\infty \otimes \pi_f$ with π_∞ a unitary irreducible representation of $G(\mathbb{R})$ with central character ω_∞ and π_f a smooth admissible unitary irreducible representation of $G(\mathbb{A}_f)$ with central character ω_f, and all but countably many $m^G(\pi)$ vanish. Note that the sum is absolutely convergent (by definition of trace class operators), but has infinitely many terms in general. Also note that each π_f decomposes as a restricted tensor product $\bigotimes'_p \pi_p$ of irreducible smooth representations of $G(\mathbb{Q}_p)$, almost all of which are unramified (and endowed with a non-zero invariant under $G(\mathbb{Z}_p)$ ...). We will say that π is an automorphic representation if $m^G(\pi) > 0$.

Remark 4.19. *Recall from Gabriel Dospinescu's course that if we fix a maximal compact subgroup K_∞ of $G(\mathbb{R})$ then we also have a decomposition of the space of square-integrable automorphic forms*

$$\mathcal{A}^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)^{K_f} \simeq \bigoplus_{\pi = \pi_\infty \otimes \pi_f} \left(\operatorname{HC}(\pi_\infty) \otimes \pi_f^{K_f}\right)^{\oplus m^G(\pi)}$$

where $\operatorname{HC}(\pi_\infty)$ *is the* $(\mathfrak{g}, K_\infty)$*-module (*$\mathfrak{g} := \mathbb{C} \otimes_\mathbb{R} \operatorname{Lie} G(\mathbb{R})$*) consisting of smooth* K_∞*-finite vectors in* π_∞*. This decomposition contains the same information as the decomposition of* L^2 *above since any unitary irreducible representation* π_∞ *is determined by the* $(\mathfrak{g}, K_\infty)$*-module* $\operatorname{HC}(\pi_\infty)$*. Recall that there is also a decomposition of* *the unitary representation $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$ of $G(\mathbb{A})$ (note that this is not a smooth representation of $G(\mathbb{A}_f)$), which also contains the same information, although the relation is non-trivial. Note that in this course we have not studied topological representations of p-adic groups.*

In these notes we will only have to consider topological representations (even unitary on Hilbert spaces) of real groups, and smooth representations of p-adic groups.

Our first application of the trace formula is the existence of automorphic representations whose components at finitely many places are given (Theorem 4.22 below). An obvious necessary condition for existence is that the centrals characters are restrictions to $\mathbb{Q}_v^\times$ of a continuous character $\mathbb{Q}^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$. This condition is made transparent by the following lemma.

Lemma 4.20. *Let S' be a finite set of prime numbers. Let $(\eta_p)_{p\in S'}$ be a family of (automatically unitary) continuous characters $\eta_p : \mathbb{Z}_p^\times \to \mathbb{C}^\times$. Then there exists a continuous unitary character $\omega : \mathbb{Q}^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ such that for any $p \in S'$ we have $\omega|_{\mathbb{Z}_p^\times} = \eta_p$.*

Proof. Use once again $\mathbb{A}^\times = \mathbb{Q}^\times \mathbb{R}_{>0} \widehat{\mathbb{Z}}^\times$. □

Remark 4.21. *The proof shows that we can even find ω which is unramified at all primes not in S' and equal to a given unitary character on $\mathbb{R}_{>0}$. The case of an arbitrary number field instead of $\mathbb{Q}$ is more subtle, the statement is not as simple but it essentially reduces to Dirichlet's unit theorem and [Che51].*

Theorem 4.22. *Let S be the finite set of places where D is not split. Let $\omega : \mathbb{Q}^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ be a continuous unitary character. Let S' be a finite set of prime numbers, and $(\sigma_p)_{p\in S}$ a collection of smooth irreducible representations of $G(\mathbb{Q}_p)$ having central character $\omega_p := \omega|_{\mathbb{Q}_p^\times}$. Assume that for any $p \in S' \smallsetminus S$ the representation σ_p is square-integrable. There exists an irreducible representation $\pi = \bigotimes_v' \pi_v$ in $\varinjlim_{K_f} L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)^{K_f}$ such that $\pi_p \simeq \sigma_p$ for all $p \in S'$.*

Proof. Up to adding to S' a prime number which is not in S, and taking for σ_p a supercuspidal representation of $G(\mathbb{Q}_p)$ having central character ω_p (such a representation exists by Theorem 3.37), we can assume that there exists $p \in S' \smallsetminus S$ such that σ_p is supercuspidal. Let ℓ be a prime number which does not belong to S'. We will apply the trace formula to a function $f \in C_c^\infty(G(\mathbb{A}), \omega^{-1})$ which can be written as a product $\prod_v f_v$.

- For v a place of $\mathbb{Q}$ which does not belong to $S' \cup \{\ell\}$, pick $f_v \in \mathcal{H}(G(\mathbb{Q}_v), \omega_v^{-1})$ (for v the Archimedean place this means $C_c^\infty(G(\mathbb{Q}_v), \omega_v^{-1})$) such that $f_v(1) \neq 0$, and f_p is the characteristic function of $G(\mathbb{Z}_p)$ for almost all primes numbers p.
- For each $p \in S'$, choose a pseudo-coefficient $f_p \in \mathcal{H}(G(\mathbb{Q}_p), \omega_p^{-1})$ for the representation σ_p. Recall that such pseudo-coefficients were constructed in Propositions 3.28 and 3.29 for $p \notin S$, and are easy to construct using finite group representation theory for $p \in S$ (for example $f_p = (\dim \sigma_p)^{-1} \overline{\operatorname{tr} \sigma_p}$).

- Finally, for K_ℓ a compact open subgroup of $\ker(\det : G(\mathbb{Q}_\ell) \to \mathbb{Q}_\ell^\times)$, small enough so that $K_\ell \cap Z(\mathbb{Q}_\ell) = \{1\}$, take $f_\ell \in \mathcal{H}(G(\mathbb{Q}_\ell), \omega_\ell^{-1})$ to be the function with support in $K_\ell Z(\mathbb{Q}_\ell)$ and such that $f(k) = \mathrm{vol}(K_\ell Z(\mathbb{Q}_\ell)/Z(\mathbb{Q}_\ell))^{-1}$ for $k \in K_\ell$.

Now we claim that if K_ℓ is chosen sufficiently small, the only non-vanishing summand on the geometric side of the trace formula (4.2) is for $\gamma = 1$. Start with an arbitrary K_ℓ. The set $X(\mathrm{supp}(f))$ of conjugacy classes $[\gamma]$ in $G_{\mathrm{ad}}(\mathbb{Q})$ having a non-zero contribution in the trace formula is finite. For any non-central γ we have $\nu(\gamma) \neq 4$ (ν as in the proof of Theorem 4.18). By continuity of ν there exists an open subgroup K'_ℓ of K_ℓ such that $\nu(K'_\ell) \cap \nu(X(\mathrm{supp}(f)) \smallsetminus \{[1]\}) = \emptyset$. Thus up to replacing K_ℓ by K'_ℓ, the claim holds true.

So the geometric side of the trace formula is simply $\mathrm{vol}(Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}))f(1)$. Since $f(1) \neq 0$, it does not vanish, and so the spectral side (4.3) does not vanish either. In particular there exists π such that $m^G(\pi) > 0$ and $\mathrm{tr}\,\pi(f) \neq 0$. We have $\mathrm{tr}\,\pi(f) = \prod_v \mathrm{tr}\,\pi_v(f_v)$, so the property of pseudo-coefficients implies that for any $p \in S'$, either $\pi_p \simeq \sigma_p$ or $p \notin S$ and π_p is one-dimensional. Corollary 4.24 below shows that the second possibility contradicts the fact that there exists $p' \in S' \smallsetminus S$ such that $\pi_{p'}$ is supercuspidal. $\square$

Theorem 4.23 (Strong approximation)**.** *Let G' be the algebraic subgroup of G which is the kernel of the determinant morphism. Let v be a place of $\mathbb{Q}$ which is not in S, i.e. $\mathbb{Q}_v \otimes_{\mathbb{Q}} D \simeq M_2(\mathbb{Q}_v)$. Then $G'(\mathbb{Q})G'(\mathbb{Q}_v)$ is dense in $G'(\mathbb{A})$.*

Proof. See [Kne65, §3]. $\square$

Corollary 4.24. *Let $\pi = \pi_\infty \otimes \bigotimes'_p \pi_p$ be an automorphic representation of $G(\mathbb{A})$ having central character ω. Assume that there exists a place v of $\mathbb{Q}$ which is not in S and such that π_v is one-dimensional. Then π is one-dimensional, i.e. for every place w of $\mathbb{Q}$ the representation π_w is one-dimensional.*

Proof. Fix a maximal compact subgroup K_∞ of $G(\mathbb{R})$. Fix $v_0 \in \mathrm{HC}(\pi_\infty) \otimes \bigotimes'_p \pi_p \smallsetminus \{0\}$. For simplicity, assume that v_0 is a pure tensor. There exists a compact open subgroup K_f of $G(\mathbb{A}_f)$ fixing v_0. Let $\varphi : \pi_\infty \otimes \pi_f^{K_f} \to L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)^{K_f}$ be a non-zero continuous $G(\mathbb{R})$-equivariant linear map which is also equivariant for the action of the Hecke algebra $\mathcal{H}(G(\mathbb{A}_f), K_f)$. (Note that this last property is equivalent to requiring that φ extends to a $G(\mathbb{A}_f)$-equivariant map $\pi \to \varinjlim_{K'_f} L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)^{K'_f}$.) Let $f = \varphi(v_0)$, then f is an automorphic form (this non-trivial fact was proved in Gabriel Dospinescu's course), in particular it is continuous. The group $G'(\mathbb{Q}_v) \simeq \mathrm{SL}_2(\mathbb{Q}_v)$ is perfect, so $G'(\mathbb{Q}_v) \subset \ker \pi_v$, and f is right $G'(\mathbb{Q}_v)$-invariant. Let $x \in G(\mathbb{A})$ and $g \in G'(\mathbb{A})$. There are sequences $(\gamma_n)_n$ and $(y_n)_n$ of elements of $G'(\mathbb{Q})$ and $G'(\mathbb{Q}_v)$ such that $(\gamma_n y_n)_n$ converges to xgx^{-1}, so $f(xg) = \lim_{n \to +\infty} f(y_n x) = f(x)$. The representation π is irreducible and so φ is injective, so we deduce that v_0 is fixed by $G'(\mathbb{A})$, and so for every place w of $\mathbb{Q}$ there is a non-zero vector in π_w fixed by $G'(\mathbb{Q}_w)$. Since $G'(\mathbb{Q}_w)$ is distinguished in $G(\mathbb{Q}_w)$ this implies that $G'(\mathbb{Q}_w) \subset \ker \pi_w$ (for $w = \infty$ we use the fact that π_∞ is topologically irreducible, whereas for finite w we use the fact that π_w is simply irreducible). $\square$

Remark 4.25. (1) *Theorem 4.23 and Corollary 4.24 (for* discrete *automorphic representations) are still valid if $S = \emptyset$, i.e. if $G = \mathrm{GL}_2$, with the same proof.*

(2) *If we knew more about the classification of representations of $\mathrm{GL}_2(\mathbb{R})$ and harmonic analysis for this group, including the existence of pseudo-coefficients for square-integrable representations, we could also include the Archimedean place in the set S' in Theorem 4.22. For the case of arbitrary reductive groups over number fields see* [Clo86].

4.4. **The simple trace formula for GL_2.** We would like to prove an analogous formula for GL_2. It turns out that this is much harder, due to the continuous part of the automorphic spectrum on the spectral side, and the contributions of non-elliptic elements on the geometric side (note that $\mathrm{vol}(G_\gamma(\mathbb{Q})Z(\mathbb{A})\backslash G_\gamma(\mathbb{A})) = +\infty$ for γ semi-simple regular hyperbolic). Under a simplifying assumption on the test function, we will get a reasonably simple trace formula for GL_2.

For the algebraic group GL_2 over $\mathbb{Q}$ change the notation used in the first chapters for $\mathrm{GL}_2(\mathbb{Q}_p)$: the letters G, B, T, N will be used to denote the corresponding algebraic groups over $\mathbb{Q}$.

We first recall the fundamental results on the cuspidal automorphic spectrum proved in Gabriel Dospinescu's course. We first introduce cusp forms in the L^2 setting. Let ω be a continuous unitary character of $Z(\mathbb{Q})\backslash Z(\mathbb{A})$.

Lemma 4.26. *Let $\phi \in L^2(G(\mathbb{Q})Z(\mathbb{A})\backslash G(\mathbb{A}), \omega)$. Then for almost all $g \in G(\mathbb{A})$, the integral on the RHS of*
$$\phi_B(g) := \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \phi(ng)\, d\dot{g}$$
converges absolutely. Moreover if $\mathrm{vol}\Big(Z(\mathbb{A})\backslash\{x \in G(\mathbb{Q})\backslash G(\mathbb{A}) \,|\, \phi(x) \neq 0\}\Big) = 0$ then $\phi_B(g) = 0$ for almost all $g \in G(\mathbb{A})$.

Proof. Let $g_0 \in G(\mathbb{A})$. There exists a continuous compactly supported function $T_0 : Z(\mathbb{A})\backslash G(\mathbb{A}) \to \mathbb{R}_{\geq 0}$ such that $T_0(g_0) > 0$. Let $T : Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{R}_{\geq 0}$ be defined by $T(g) = \sum_{\gamma \in Z(\mathbb{Q})\backslash G(\mathbb{Q})} T(\gamma g)$. This function is clearly continuous and compactly supported, so it is bounded and
$$\int_{Z(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})} |\phi(g)|^2 T(g)\, d\dot{g} < \infty.$$
But this equals
$$\begin{aligned}\int_{Z(\mathbb{A})\backslash G(\mathbb{A})} |\phi(g)|^2 T_0(g)\, dg &= \int_{Z(\mathbb{A})N(\mathbb{A})\backslash G(\mathbb{A})} \int_{N(\mathbb{A})} |\phi(ng)|^2 T_0(ng)\, dn\, d\dot{g} \\ &= \int_{Z(\mathbb{A})N(\mathbb{A})\backslash G(\mathbb{A})} \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} |\phi(ng)|^2 \sum_{\gamma \in N(\mathbb{Q})} T_0(\gamma n g)\, d\dot{n}\, d\dot{g}.\end{aligned}$$
The fact that this last integral converges implies both statements in the lemma, using Cauchy-Schwarz (note that $N(\mathbb{Q})\backslash N(\mathbb{A})$ is compact, so that the constant function 1 on it is square-integrable). ∎

Let $L^2_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}),\omega)$ be the subspace of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}),\omega)$ consisting of all ϕ such that for almost all $g \in G(\mathbb{A})$ we have

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \phi(ng)\, d\dot{n} = 0.$$

Theorem 4.27. *Let K_f be a compact open subgroup of $G(\mathbb{A}_f)$. Then the unitary representation $L^2_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}),\omega)^{K_f}$ of $G(\mathbb{R})$ decomposes discretely.*

Recall that this theorem is also proved using the theorem of Gelfand, Graev and Piatetski-Shapiro, that is by proving that for any $f \in C^0_c(K_f\backslash G(\mathbb{A})/K_f,\omega^{-1})$, the operator $\rho_{\mathrm{cusp}}(f)$, which is the restriction of $\rho(f)$ to $L^2_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}),\omega)^{K_f}$, is compact, even Hilbert-Schmidt. Note that the proof did not exhibit an explicit kernel for this Hilbert-Schmidt operator. Nevertheless, we shall see that in the case of a cuspidal test function f, essentially the same arguments, applied to the kernel instead of automorphic forms, do give an explicit kernel.

Before we can achieve this in Lemma 4.34 below, we need to recall two essential tools: reduction theory and the Poisson summation formula.

Let K_0 be the maximal compact subgroup $O_2(\mathbb{R}) \times G(\widehat{\mathbb{Z}})$ of $G(\mathbb{A})$. Define

$$B(\mathbb{A})^1 = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in B(\mathbb{A}) \,\middle|\, |a|_{\mathrm{ad}} = |c|_{\mathrm{ad}} = 1 \right\}.$$

Note that this subgroup of $B(\mathbb{A})$ contains $T(\mathbb{Q})N(\mathbb{A})$, in particular it contains $B(\mathbb{Q})$. For $\eta > 0$ define

$$S(\eta) = \{\mathrm{diag}(x,y) \in G(\mathbb{R}) \,|\, x,y > 0 \text{ and } x/y \geq \eta\}.$$

We also introduce the function $H : G(\mathbb{A}) \to \mathbb{R}$ defined using the Iwasawa decomposition by $H(bk) = \log|x/y|$ if $b = \begin{pmatrix} x & * \\ 0 & y \end{pmatrix} \in B(\mathbb{A})$ and $k \in K_0$. Clearly H is left $B(\mathbb{A})^1$ and right K_0-invariant. We have a "product formula" (actually a sum because of the logarithm ...) $H(g) = \sum_v H_v(g_v)$. Note that we have

$$B(\mathbb{A})^1 S(\eta) = S(\eta) B(\mathbb{A})^1 = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in B(\mathbb{A}) \,\middle|\, |a|/|c| \geq \eta \right\}.$$

$$\text{and } B(\mathbb{A})^1 S(\eta) K_0 = H^{-1}([\log\eta, +\infty[).$$

Theorem 4.28. *There exists a compact subset Ω of $B(\mathbb{A})^1$ and $\eta > 0$ such that $G(\mathbb{A}) = G(\mathbb{Q})\Omega S(\eta) K_0$.*

Remark 4.29. *This is a coarser version of the classical fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on the Poincaré upper-half plane; explicitly we may take $\eta = \sqrt{3}/2$,*

$$\Omega = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \,\middle|\, |x| \leq 1/2 \right\} \subset N(\mathbb{R}).$$

See [Ser77, Ch. VII]*. For arbitrary reductive groups over number fields the first part, together with the compactness criterion for arithmetic quotients, are theorems due to Borel, Harish-Chandra, Mostow, Tamagawa, Godement, Weil (see* [God95] *and* [Spr94]*; the latter also covers reductive groups of positive characteristic).*

The first point in the following lemma gives "coordinates near the cusps" on $G(\mathbb{Q})\backslash G(\mathbb{A})$. The second point is a slight generalization that will be useful later.

Lemma 4.30. (1) *For any place v of $\mathbb{Q}$, $g \in G(\mathbb{Q}_v)$ and $n \in N(\mathbb{Q}_v)$ we have $H_v(nwg) \leq -H_v(g)$. In particular for $g \in G(\mathbb{A})$ and $n \in N(\mathbb{A})$ we have $H(nwg) \leq -H(g)$. In particular for $\kappa > 1$ we have an embedding*

$$B(\mathbb{Q})\backslash B(\mathbb{A})^1 S(\kappa) K_0 \hookrightarrow G(\mathbb{Q})\backslash G(\mathbb{A})$$

(and similarly if we take quotients by $Z(\mathbb{A})$).

(2) *Let $\eta > 0$. Let C be a compact subset of $G_{\mathrm{ad}}(\mathbb{A})$. There exists $\kappa > 0$ (depending on η and C) such that for any $x \in B(\mathbb{A})^1 S(\kappa) K_0$, $\gamma \in G(\mathbb{Q})$ and $y \in B(\mathbb{A})^1 S(\eta) K_0$ satisfying $x^{-1}\gamma y \in C$, we have $\gamma \in B(\mathbb{Q})$.*

Proof. (1) We consider the Archimedean and non-Archimedean cases separately. In any case we can assume that $g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, and that $n = 1$ since H_v is left $Z(\mathbb{Q}_v)N(\mathbb{Q}_v)$-invariant. We have $wg = \begin{pmatrix} 0 & -1 \\ a & -b \end{pmatrix}$.

In the real case we compute $wg\,{}^t(wg) = \begin{pmatrix} 1 & b \\ b & a^2+b^2 \end{pmatrix}$ and solve for $x\,{}^t x = wg\,{}^t(wg)$ with $x \in B(\mathbb{Q}_v)$. We find $x \in Z(\mathbb{Q}_v) \begin{pmatrix} a/(a^2+b^2) & b/(a^2+b^2) \\ 0 & 1 \end{pmatrix}$, and $\log(a/(a^2+b^2)) \leq \log(a/a^2) = -\log a$.

For a prime number p, doing column operations on wg we find that if $b/a \in \mathbb{Z}_p$ then $H_p(wg) = -H_p(g)$ whereas if $b/a \in \mathbb{Q}_p \smallsetminus \mathbb{Z}_p$ then $H_p(wg) = -H_p(g) - 2\log(|b/a|)$.

The last assertion follows from the Bruhat decomposition for $\mathrm{GL}_2(\mathbb{Q})$: if $H(g) > 0$ and $\gamma \in \mathrm{GL}_2(\mathbb{Q}) \smallsetminus B(\mathbb{Q})$ then $H(\gamma g) \leq -\inf_{n \in N(\mathbb{Q})} H(wng) \leq 0$.

(2) Up to replacing C by $K_0 C K_0$ we may assume that C is bi-K_0-invariant. Then $B(\mathbb{A})^1 S(\kappa) K_0 C = B(\mathbb{A})^1 S(\kappa) C$. There exists $\epsilon > 0$ such that $C \subset B(\mathbb{A})^1 S(\epsilon) K_0$, so that $B(\mathbb{A})^1 S(\kappa) K_0 C \subset B(\mathbb{A})^1 S(\epsilon\kappa) K_0$. Assume that κ is large enough so that $\kappa\epsilon\eta > 1$. We will show that for any $\gamma \in G(\mathbb{Q})$, if $\gamma B(\mathbb{A})^1 S(\eta) K_0 \cap B(\mathbb{A})^1 S(\kappa) K_0 C \neq \emptyset$ then $\gamma \in B(\mathbb{Q})$. Since $B(\mathbb{Q})B(\mathbb{A})^1 = B(\mathbb{A})^1$ we may assume that $\gamma = 1$ or $\gamma \in wN(\mathbb{Q})$ (Bruhat decomposition). The previous point shows that $H(wN(\mathbb{Q})B(\mathbb{A})^1 S(\eta) K_0) \subset]-\infty, -\log(\eta)]$, whereas $H(B(\mathbb{A})^1 S(\kappa) K_0 C) \subset [\log(\kappa\epsilon), +\infty[$.

□

I do not know of a reference for a generalization of the second part to arbitrary reductive groups, but there is no doubt that such a generalization exists ...

We now recall the Poisson summation formula. First, the classical form, which is well-known. The assumptions we put are far from optimal.

Proposition 4.31. *For $f \in C_c^1(\mathbb{R})$ we have $\sum_{u \in \mathbb{Z}} f(u) = \sum_{v \in \mathbb{Z}} \widehat{f}(v)$, where $\widehat{f}(x) = \int_{\mathbb{R}} f(t) e^{-2i\pi xt}\, dt$ and the right-hand side is absolutely convergent.*

Proof. The function $t \mapsto \sum_{u\in\mathbb{Z}} f(t+u)$ on $\mathbb{R}$ is $\mathbb{Z}$-periodic and C^1 so it is the sum of its Fourier series, convergent for the sup norm. □

Recall that $\mathbb{A} = \mathbb{Q} + \mathbb{R} + \widehat{\mathbb{Z}}$, and that the kernel of the surjection $\mathbb{R} \times \widehat{\mathbb{Z}} \to \mathbb{Q}\backslash\mathbb{A}$ is $\mathbb{Z}$. It follows that there is a unique continuous (automatically unitary) character $\psi_0 : \mathbb{Q}\backslash\mathbb{A}/\widehat{\mathbb{Z}} \to \mathbb{C}^\times$ whose restriction to $\mathbb{R}$ is $t \mapsto \exp(2i\pi t)$.

Exercise 4.32. *Show that $\mathbb{Q} \to \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Q}\backslash\mathbb{A}, \mathbb{C}^\times)$, $\lambda \mapsto \psi_0(\lambda\cdot)$ is an isomorphism of topological groups. (This fact is not strictly necessary for what follows but it explains the adélic Poisson summation formula in the general framework of Pontryagin duality.)*

For $f \in L^1(\mathbb{A})$ define $\widehat{f} : \mathbb{A} \to \mathbb{C}$ by $\widehat{f}(x) = \int_{\mathbb{A}} f(t)\psi_0(-tx)\,|dt|$. Note that the Haar measure $|dt|$ on $\mathbb{A}$ is characterized by the fact that $\mathrm{vol}(\mathbb{Q}\backslash\mathbb{A}) = 1$. For $x = (x_v)_v \in \mathbb{A}^\times$ we denote

$$|x|_{\mathrm{ad}} := \prod_v |x_v|_v$$

the norm of x, where $|\cdot|_\infty$ is the usual absolute value and $|x_p|_p := p^{-v_p(x_p)}$, which is equal to 1 for almost all prime numbers p. Note for $x \in \mathbb{Q}^\times$ we have $|x|_{\mathrm{ad}} = 1$.

Proposition 4.33. *Let $f \in C^1_c(\mathbb{A})$.*

(1) *There exists an integer $m \geq 1$ such that f is invariant under $m\widehat{\mathbb{Z}}$.*

(2) *For any $a \in \mathbb{A}^\times$ we have*

$$\sum_{v\in\mathbb{Q}} f(a^{-1}v) = |a|_{\mathrm{ad}} \sum_{v\in\mathbb{Q}} \widehat{f}(av).$$

(3) *There exists a constant $c > 0$ such that if f is of class C^2 (for the real variable) then for any $a \in \mathbb{A}^\times$ we have*

$$\left|\sum_{v\in\mathbb{Q}} f(a^{-1}v) - |a|_{\mathrm{ad}}\widehat{f}(0)\right| \leq c|a|_{\mathrm{ad}}^{-1} m^2 \|f''\|_{L^1}$$

where f'' is the second derivative of f with respect to the real variable.

Proof. The first point follows from the isomorphism $C^1_c(\mathbb{A}) \simeq C^1_c(\mathbb{R}) \otimes_{\mathbb{C}} C^1_c(\mathbb{A}_f)$ and the corresponding statement for $C^1_c(\mathbb{A}_f)$. For any compact subset C of $\mathbb{A}$ there exists a finite subset S of $\mathbb{Q}$ such that for $x \in C$ and $v \in \mathbb{Q} \smallsetminus S$ we have $f(x+v) = 0$. Define $F \in C^1(\mathbb{Q}\backslash\mathbb{A})$ by $F(x) = \sum_{v\in\mathbb{Q}} f(x+v)$. This defines a function on $\mathbb{Q}\backslash\mathbb{A}/m\widehat{\mathbb{Z}}$. By density of $\mathbb{Q}+\mathbb{R}$ in $\mathbb{A}$ (i.e. strong approximation for the additive group) the natural morphism $\mathbb{R} \to \mathbb{Q}\backslash\mathbb{A}/m\widehat{\mathbb{Z}}$ is surjective. It has kernel $m\mathbb{Z}$ and $\mathbb{R}/m\mathbb{Z}$ is compact, so we have an isomorphism of topological groups

$$\mathbb{R}/m\mathbb{Z} \simeq \mathbb{Q}\backslash\mathbb{A}/m\widehat{\mathbb{Z}}.$$

Define $G \in C^1(\mathbb{R}/\mathbb{Z})$ by $G(x) = F(mx)$. We have

$$G(0) = \sum_{n\in\mathbb{Z}} \widehat{G}(n).$$

We compute

$$\begin{aligned}\widehat{G}(n) &= \int_{\mathbb{R}/\mathbb{Z}} F(mt)\exp(-2i\pi nt)\,dt \\ &= \int_{\mathbb{R}/m\mathbb{Z}} F(u)\exp\left(-2i\pi\frac{n}{m}u\right)\frac{du}{m} \\ &= \int_{\mathbb{Q}\backslash\mathbb{A}} F(t)\psi_0\left(-\frac{n}{m}t\right)|dt| \\ &= \int_{\mathbb{A}} f(t)\psi_0\left(-\frac{n}{m}t\right)|dt| \\ &= \widehat{f}(n/m).\end{aligned}$$

We obtain the formula

$$\sum_{v\in\mathbb{Q}} f(v) = \sum_{n\in\mathbb{Z}} \widehat{f}(n/m).$$

For $v \in \mathbb{Q} \smallsetminus m^{-1}\mathbb{Z}$ there exists $x \in m\widehat{\mathbb{Z}}$ such that vx belongs to $\mathbb{A}_f \smallsetminus \widehat{\mathbb{Z}}$ and so $\psi_0(vx)$ is not equal to 1, and we have

$$\begin{aligned}\widehat{f}(v) &= \int_{\mathbb{A}} f(t)\psi_0(-tv)\,|dt| \\ &= \int_{\mathbb{A}} f(u+x)\psi_0(-(u+x)v)\,|du| \\ &= \psi_0(vx)^{-1}\int_{\mathbb{A}} f(u)\psi_0(-uv)\,|du| \\ &= \psi_0(vx)^{-1}\widehat{f}(v)\end{aligned}$$

which shows that $\widehat{f}(v)$ vanishes. So the above formula can also be written

$$\sum_{v\in\mathbb{Q}} f(v) = \sum_{v\in\mathbb{Q}} \widehat{f}(v).$$

For $a \in \mathbb{A}^\times$ define $f_a \in C_c^1(\mathbb{A})$ by $f_a(x) = f(a^{-1}x)$. For $x \in \mathbb{A}$ we have

$$\begin{aligned}\widehat{f_a}(x) &= \int_{\mathbb{A}} f(a^{-1}t)\psi_0(-tx)\,|dt| \\ &= \int_{\mathbb{A}} f(u)\psi_0(-uax)|a|\,|du| \\ &= |a|\widehat{f}(ax)\end{aligned}$$

and so applying the above to f_a we obtain

$$\sum_{v\in\mathbb{Q}} f(a^{-1}v) = \sum_{v\in\mathbb{Q}} f_a(v) = \sum_{v\in\mathbb{Q}} \widehat{f_a}(v) = |a|_{\mathrm{ad}}\sum_{v\in\mathbb{Q}} \widehat{f}(av).$$

Note that this formula does not change if we replace a by an element of $a\mathbb{Q}^\times$. Because of the decomposition $\mathbb{A}^\times = \mathbb{Q}^\times\mathbb{R}_{>0}\widehat{\mathbb{Z}}^\times$ this allows us to assume that a

belongs to $\mathbb{R}_{>0}\widehat{\mathbb{Z}}^\times$. In this case the function f_a is also invariant under $m\widehat{\mathbb{Z}}$ and we have $\widehat{f}(av) = 0$ for $v \in \mathbb{Q} \smallsetminus m^{-1}\mathbb{Z}$. If f is of class C^2 the for $x \in \mathbb{A}$ we have, using integration by parts,

$$\widehat{f''}(x) = -4\pi^2 x_\infty^2 \widehat{f}(x).$$

In particular we have

$$\sum_{n\in\mathbb{Z}\smallsetminus\{0\}} \widehat{f}(an/m) = \frac{-m^2}{4\pi^2 a_\infty^2} \sum_{n\in\mathbb{Z}\smallsetminus\{0\}} \frac{\widehat{f''}(an/m)}{n^2}.$$

Note that for $a \in \mathbb{R}_{>0}\widehat{\mathbb{Z}}^\times$ we have $|a_{\mathrm{ad}}| = |a_\infty|$. Using the estimate

$$|\widehat{f''}(x)| \leq \|f''\|_{L^1}$$

we obtain the inequality in the proposition. $\square$

Lemma 4.34. *As above K_f is a compact open subgroup of $G(\mathbb{A}_f)$ and ω is a unitary continuous character of $Z(\mathbb{Q})\backslash Z(\mathbb{A})$. Let $f \in C_c^2(K_f\backslash G(\mathbb{A})/K_f, \omega^{-1})$ be cuspidal, i.e. for any $x, y \in G(\mathbb{A})$ we have $\int_{N(\mathbb{A})} f(xny)\, dn = 0$. Then the operator $\rho(f)$ on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$ has image contained in $L^2_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$ and is Hilbert-Schmidt with kernel $K_f : (G(\mathbb{Q})\backslash G(\mathbb{A}))^2 \to \mathbb{C}$, $(x, y) \mapsto \sum_{\gamma\in Z(\mathbb{Q})\backslash G(\mathbb{Q})} f(x^{-1}\gamma y)$. This kernel is continuous and bounded.*

Proof. Note that bounded implies square-integrable modulo $(Z(\mathbb{Q})\backslash Z(\mathbb{A}))^2$. In fact this is how we will prove that ρ_f is Hilbert-Schmidt. As in the anisotropic case the function K_f is continuous, satisfies $K_f(z_1x, z_2y) = \omega(z_1z_2^{-1})K_f(x, y)$ for $z_1, z_2 \in Z(\mathbb{A})$ and for any $\phi \in L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$ we have

$$(\rho(f)\phi)(x) = \int_{G_{\mathrm{ad}}(\mathbb{Q})\backslash G_{\mathrm{ad}}(\mathbb{A})} \phi(y)K_f(x, y)\, d\dot{y}.$$

If X is a compact subset of $G_{\mathrm{ad}}(\mathbb{A})$, for $x \in X$, for any $\gamma \in G_{\mathrm{ad}}(\mathbb{Q})$ and $y \in G_{\mathrm{ad}}(\mathbb{A})$ such that $x^{-1}\gamma y$ we have that y belongs to the (compact) image of $X\mathrm{supp}(f)$ in $G_{\mathrm{ad}}(\mathbb{Q})\backslash G_{\mathrm{ad}}(\mathbb{A})$. This shows that for $x \in G_{\mathrm{ad}}(\mathbb{Q})\backslash G_{\mathrm{ad}}(\mathbb{Q})X$ the support of $|K_f(x, \cdot)|$ is contained in a compact subset of $G_{\mathrm{ad}}(\mathbb{Q})\backslash G_{\mathrm{ad}}(\mathbb{A})$ which does not depend on x (of course it depends on X and f), in particular it is bounded independently of x. The kernel K_f is also cuspidal in the first variable: for any $x, y \in G(\mathbb{A})$ we have

$$\begin{aligned}
\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} K_f(nx, y)\, d\dot{n} &= \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \sum_{\gamma\in G_{\mathrm{ad}}(\mathbb{Q})} f(x^{-1}n^{-1}\gamma y)\, d\dot{n} \\
&= \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \sum_{\gamma\in N(\mathbb{Q})\backslash G_{\mathrm{ad}}(\mathbb{Q})} \sum_{\alpha\in N(\mathbb{Q})} f(x^{-1}n^{-1}\alpha^{-1}\gamma y)\, d\dot{n} \\
&= \sum_{\gamma\in N(\mathbb{Q})\backslash G_{\mathrm{ad}}(\mathbb{Q})} \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \sum_{\alpha\in N(\mathbb{Q})} f(x^{-1}n^{-1}\alpha^{-1}\gamma y)\, dn \\
&= \sum_{\gamma\in N(\mathbb{Q})\backslash G_{\mathrm{ad}}(\mathbb{Q})} \int_{N(\mathbb{A})} f(x^{-1}n^{-1}\gamma y)\, dn \\
&= 0
\end{aligned}$$

where the third equality is justified by absolute convergence ($K_{|f|}$ is also continuous and $N(\mathbb{Q})\backslash N(\mathbb{A})$ is compact so the first integral is finite) and the last equality follows from cuspidality of f. Now for any $x \in G(\mathbb{A})$ the image of $N(\mathbb{A})x$ in $G_{\mathrm{ad}}(\mathbb{Q})\backslash G_{\mathrm{ad}}(\mathbb{A})$ is compact, so $|K_f|$ is bounded on $N(\mathbb{A})x \times (G_{\mathrm{ad}}(\mathbb{Q})\backslash G_{\mathrm{ad}}(\mathbb{A}))$. Since $|\phi|$ is integrable on $G_{\mathrm{ad}}(\mathbb{Q})\backslash G_{\mathrm{ad}}(\mathbb{A})$ (it is square-integrable and $\mathrm{vol}(G_{\mathrm{ad}}(\mathbb{Q})\backslash G_{\mathrm{ad}}(\mathbb{A}))$ is finite) we have

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \int_{G_{\mathrm{ad}}(\mathbb{Q})\backslash G_{\mathrm{ad}}(\mathbb{A})} |\phi(y) K_f(nx, y)| \, d\dot{y} \, d\dot{n} < \infty$$

so we can swap integral signs and deduce that we have

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} (\rho(f)\phi)(nx) \, d\dot{n} = 0.$$

This shows that the image of $\rho(f)$ is contained in $L^2_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)$.

Let us now show that $|K_f|$ is bounded. Let $C \subset G_{\mathrm{ad}}(\mathbb{A})$ be $K_0 \mathrm{supp}(f) K_0$, and write $C = B_C K_0$ for some compact subset B_C of $B_{\mathrm{ad}}(\mathbb{A})$ which is right $B_{\mathrm{ad}}(\mathbb{A}) \cap (K_0/(K_0 \cap Z(\mathbb{A})))$-invariant. Let $\Omega \subset B(\mathbb{A})^1$ and $\eta > 0$ be as in Theorem 4.28. Let $\kappa > 0$ be as in (2) of Lemma 4.30 (with respect to η and C). Let $x = o_x \mathrm{diag}(a_x, 1) k_x$ with $o_x \in \Omega$, $a_x \in \mathbb{R}_{\geq \eta}$ and $k_x \in K_0$, and similarly $y = o_y \mathrm{diag}(a_y, 1) k_y$. Assume that $a_x > \kappa$. By Lemma 4.30, if $\gamma \in G_{\mathrm{ad}}(\mathbb{Q})$ is such that $x^{-1}\gamma y \in C$ then $\gamma \in B_{\mathrm{ad}}(\mathbb{Q})$. We then also have (in $G_{\mathrm{ad}}(\mathbb{A})$) $k_x x^{-1} \gamma y k_y^{-1} \in K_0 C K_0 = B_C K_0$, and moreover $k_x x^{-1} \gamma y k_y^{-1} \in B_{\mathrm{ad}}(\mathbb{A})$ so writing $\gamma = \begin{pmatrix} a_\gamma & * \\ 0 & 1 \end{pmatrix}$ we obtain $a_x^{-1} a_\gamma a_y \in C'$ where $C' \subset \mathbb{A}^\times$ is a compact subset which depends on C and Ω (explicitly C' is the image of $\Omega B_C \Omega^{-1}$). Using the decomposition $\mathbb{A}^\times = \mathbb{Q}^\times \mathbb{R}_{>0} \widehat{\mathbb{Z}}^\times$ (which is a homeomorphism) we conclude that there exists $\epsilon > 0$ and a finite set $F \subset B_{\mathrm{ad}}(\mathbb{Q})/N(\mathbb{Q})$ (depending on Ω, η and C) such that for $x \in \Omega S(\kappa) K_0$, $y \in \Omega S(\eta) K_0$ and $\gamma \in G_{\mathrm{ad}}(\mathbb{Q})$, if $x^{-1}\gamma y \in C$ then $\gamma \in B_{\mathrm{ad}}(\mathbb{Q})$, $a_x/a_y \in [\epsilon, \epsilon^{-1}]$ and the image of γ in $B_{\mathrm{ad}}(\mathbb{Q})/N(\mathbb{Q})$ lies in F.

This argument is symmetric in x and y, up to replacing C by the larger compact subset of $G_{\mathrm{ad}}(\mathbb{A})$:

$$K_0 \{ g \in G_{\mathrm{ad}}(\mathbb{A}) \,|\, g \in \mathrm{supp}(f) \text{ or } g^{-1} \in \mathrm{supp}(f) \} K_0.$$

Let $\widetilde{F}$ be the preimage of F in $\{\mathrm{diag}(a, 1) \,|\, a \in \mathbb{Q}^\times\}$, naturally in bijection with F. We have shown that for $(x, y) \in (\Omega S(\eta) K_0)^2$, one of them in $\Omega S(\kappa) K_0$, we have

$$K_f(x, y) = \sum_{\gamma \in \widetilde{F}} \sum_{n \in N(\mathbb{Q})} f(x^{-1} \gamma n y).$$

Note that the image of $(\Omega S(\eta) K_0 \smallsetminus \Omega S(\kappa) K_0)^2$ in $(G(\mathbb{Q}) Z(\mathbb{A}) \backslash G(\mathbb{A}))^2$ is relatively compact (essentially because the interval $[\eta, \kappa]$ is compact). In order to bound K_f on $(G(\mathbb{Q})\backslash G(\mathbb{A}))^2$, it is therefore enough to bound the sum over $n \in N(\mathbb{Q})$ when x or y belongs to $\Omega S(\kappa) K_0$. For this we will use the Poisson summation formula. Write $\gamma = \mathrm{diag}(a_\gamma, 1)$ and $n = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ with $u \in \mathbb{Q}$, so that

$$x^{-1} \gamma n y = \underbrace{k_x^{-1} \begin{pmatrix} a_x^{-1} & 0 \\ 0 & 1 \end{pmatrix} o_x^{-1} \begin{pmatrix} a_x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_x^{-1} a_\gamma a_y & 0 \\ 0 & 1 \end{pmatrix}}_{\beta_1(x,y,\gamma)} \begin{pmatrix} 1 & a_y^{-1} u \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} a_y^{-1} & 0 \\ 0 & 1 \end{pmatrix} o_y \begin{pmatrix} a_y & 0 \\ 1 & 0 \end{pmatrix} k_y}_{\beta_2(y)}.$$

Observe that the set $\{\mathrm{diag}(a^{-1},1)o\mathrm{diag}(a,1)\,|\,o\in\Omega,\ a\in\mathbb{R}_{\geq\eta}\}$ is relatively compact in $B(\mathbb{A})^1$. Together with the relation $a_x/a_y\in[\epsilon,\epsilon^{-1}]$ observed above, this implies that the function β_1 (resp. β_2) is bounded on $(\Omega S(\eta)K_0)^2\times\widetilde{F}$ (resp. $\Omega S(\eta)K_0$), in the sense that its image is relatively compact in $G(\mathbb{A})$. Define

$$\begin{aligned}\Xi_{f,x,y,\gamma}:\mathbb{A}&\longrightarrow\mathbb{C}\\ u'&\longmapsto f\left(\beta_1(x,y,\gamma)\begin{pmatrix}1&u'\\0&1\end{pmatrix}\beta_2(y)\right).\end{aligned}$$

The Poisson summation formula (Proposition 4.33) reads

$$\sum_{n\in N(\mathbb{Q})}f(x^{-1}\gamma ny)=\text{constant}\times\sum_{v\in\mathbb{Q}}a_y\,\widehat{\Xi_{f,x,y,\gamma}}(a_yv).$$

We finally use the assumption that f is cuspidal, which implies that $\widehat{\Xi_{f,x,y,\gamma}}(0)=0$. Thanks to the boundedness of β_1 and β_2, the integer m in (3) of Proposition 4.33 may be found independently of x,y as above, and the L^1 norm is bounded uniformly. We obtain that K_f goes to 0 at infinity, i.e. for any $\delta>0$ there exists a compact subset C_δ of $(G_{\mathrm{ad}}(\mathbb{Q})\backslash G_{\mathrm{ad}}(\mathbb{A}))^2$ such that for any $(x,y)\notin C_\delta$ we have $|K_f(x,y)|<\delta$. In particular $|K_f|$ is bounded. $\square$

Theorem 4.35. *Let K_f be a compact open subgroup of $G(\mathbb{A}_f)$ and ω a unitary continuous character of $Z(\mathbb{Q})\backslash Z(\mathbb{A})$. Let $f\in C_c^\infty(K_f\backslash G(\mathbb{A})/K_f,\omega^{-1})$ be cuspidal. Assume that for any $x\in G(\mathbb{A})$ and $\gamma\in G_{\mathrm{ad}}(\mathbb{Q})$ such that $f(x^{-1}\gamma x)\neq0$, γ is semi-simple regular elliptic (over $\mathbb{Q}$). Then*

$$\mathrm{tr}\,\rho(f)=\mathrm{tr}\,\rho_{\mathrm{cusp}}(f)=\sum_{[\gamma]}\iota(\gamma)^{-1}\,\mathrm{vol}(G_\gamma(\mathbb{Q})Z(\mathbb{A})\backslash G(\mathbb{A}))O_\gamma(f)$$

where the sum is over conjugacy classes of semi-simple regular elliptic elements γ in $G(\mathbb{Q})$, and only finitely many terms in the sum are non-zero.

Proof. Recall from Gabriel Dospinescu's course that the action $\rho_{\mathrm{cusp}}(f)$ of any element f of $C_c^\infty(G(\mathbb{A}),\omega^{-1})$ on $L^2_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}),\omega)$ is a Hilbert-Schmidt operator (we proved this in the previous lemma for cuspidal f, but it holds for arbitrary f if we restrict to the space of cusp forms). So we can argue as in the proof of Theorem 4.18 using a Dixmier-Malliavin expression for f to conclude that $\rho_{\mathrm{cusp}}(f)$ is trace class. Thanks to the previous lemma, for f cuspidal $\rho(f)$ is also trace class and $\mathrm{tr}\,\rho(f)=\mathrm{tr}\,\rho_{\mathrm{cusp}}(f)$ (note that $L^2_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}),\omega)$ and its orthogonal in $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}),\omega)$ are stable under $\rho(f)$).

The main difference with the anisotropic case is that it is not true that $K_{|f|}$ is also bounded ($|f|$ is not cuspidal ...), so while we still have

$$\mathrm{tr}\,\rho(f)=\int_{G_{\mathrm{ad}}(\mathbb{Q})\backslash G_{\mathrm{ad}}(\mathbb{A})}K_f(x,x)\,d\dot{x}$$

thanks to Theorem 4.13, we cannot blindly insert the definition of K_f and exchange sums and integrals. Nevertheless, the proof of the previous lemma shows that there exists a compact subset $C(f)$ of $G_{\mathrm{ad}}(\mathbb{Q})\backslash G_{\mathrm{ad}}(\mathbb{A})$ such that for $x \in G(\mathbb{A})$ which does not map to $C(f)$, if $\gamma \in G_{\mathrm{ad}}(\mathbb{Q})$ is such that $f(x^{-1}\gamma x) \neq 0$ then γ is conjugated (in $G_{\mathrm{ad}}(\mathbb{Q})$) to an element of $B_{\mathrm{ad}}(\mathbb{Q})$. Together with the assumption in the theorem, this implies that K_f has compact support on the diagonal (even that each term in the sum defining $K_f(x,x)$ vanishes when x is outside a compact subset of $G_{\mathrm{ad}}(\mathbb{Q})\backslash G_{\mathrm{ad}}(\mathbb{A})$, and we can conclude as in the proof of Theorem 4.18 (including the last finiteness assertion). □

Theorem 4.36. *Let $\omega : \mathbb{Q}^\times\backslash\mathbb{A}^\times \to \mathbb{C}^\times$ be a continuous unitary character. Let S' be a finite set of prime numbers, and $(\sigma_p)_{p\in S}$ a collection of smooth irreducible square-integrable representations of $G(\mathbb{Q}_p)$ having central character $\omega_p := \omega|_{\mathbb{Q}_p^\times}$. There exists an irreducible representation $\pi = \bigotimes'_v \pi_v$ in $\varinjlim_{K_f} L^2_{\mathrm{cusp}}(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)^{K_f}$ such that $\pi_p \simeq \sigma_p$ for all $p \in S'$.*

Proof. Of course the idea is the same as in Theorem 4.22, but now our simple trace formula does not allow us to use functions f satisfying $f(1) \neq 0$. Adding one prime to S' if necessary and thanks to Theorem 3.37 we can assume that at least one σ_p is supercuspidal.

First we fix, for each $p \in S$, a pseudo-coefficient f_p for σ_p. Note that the fact that there exists $p \in S'$ such that σ_p is cuspidal implies that f_p is cuspidal, and so will be any product $\prod_f f_v$. Thanks to the elliptic orthogonality formula (Theorem 3.35) applied to (σ_p, σ_p) we know that there exists a semi-simple regular elliptic conjugacy class $[\gamma_p]$ in $G(\mathbb{Q}_p)$ such that $O_{\gamma_p}(f_p) \neq 0$. Moreover we may assume that $\operatorname{tr}\gamma_p \neq 0$ since the set of elements of vanishing trace in $\mathrm{PGL}_2(\mathbb{Q}_p)$ has measure zero. It follows from Krasner's lemma and smoothness of orbital integrals (Lemma 3.5) that there exists $\epsilon > 0$ such that for any $p \in S'$, $a \in \mathbb{Q}_p$ and $b \in \mathbb{Q}_p^\times$ satisfying $|a - \operatorname{tr}\gamma_p|_p < \epsilon$ and $|b - \det\gamma_p|_p < \epsilon$, the conjugacy class in $G(\mathbb{Q}_p)$ defined by the characteristic polynomial $X^2 - aX + b$ contains an element δ_p in the anisotropic maximal torus $\mathbb{Q}_p[\gamma_p]^\times$ of $G(\mathbb{Q}_p)$ which is regular and sufficiently close to γ_p so that $O_{\delta_p}(f_p) = O_{\gamma_p}(f_p)$. We can find $a \in \mathbb{Q}$ and $b \in \mathbb{Q}^\times$ in these p-adic balls for all $p \in S'$: for a this is essentially the Chinese remainder theorem (we can even assume that a is integral at finite places not in S'), for b it follows from Dirichlet's theorem on primes in arithmetic progressions. (These two existence results are known as *weak approximation* for the additive and multiplicative groups. In fact the additive group even has strong approximation.) Note that for ϵ small enough the above inequalities imply $a \neq 0$ because $\operatorname{tr}\gamma_p \neq 0$. Let γ be an element of $G(\mathbb{Q})$ having characteristic polynomial $X^2 - aX + b$. Note that γ is semi-simple regular, and elliptic over $\mathbb{Q}$ since it is elliptic over $\mathbb{Q}_p$ for some p.

As in the proof of Theorem 4.22, fix ℓ a prime number which does not belong to S'. Fix $f^{S'\cup\{\ell\}} = \prod_{v\notin S} f_v$ with $f_v \in C_c^\infty(G(\mathbb{Q}_v), \omega_v^{-1})$ almost all trivial, such that for any $v \notin S' \cup \{\ell\}$ we have $O_\gamma(f_v) \neq 0$. (Exercise: such a function exists.) Finally, take K_ℓ an open compact subgroup of $\mathrm{SL}_2(\mathbb{Q}_\ell)$ such that $K_\ell \cap Z(\mathbb{Q}_\ell) = \{1\}$ and define $f_\ell \in C_c^\infty(G(\mathbb{Q}_\ell), \omega_\ell^{-1})$ supported in $\gamma K_\ell Z(\mathbb{Q}_\ell)$, right K_ℓ-invariant and such that $f_\ell(\gamma) = 1$. By essentially the same argument as in the proof of Theorem 4.22 we see that if K_ℓ is small enough then the assumption of Theorem 4.35 is satisfied and the only non-vanishing term in the sum on the geometric side is $\iota(\gamma)^{-1}\operatorname{vol}(G_\gamma(\mathbb{Q})Z(\mathbb{A})\backslash G(\mathbb{A}))O_\gamma(f)$, which does not vanish. Note that we arranged that $\nu(\gamma) \neq 0$ and so ν distinguishes γ from other conjugacy classes in $\mathrm{PGL}_2(\mathbb{Q})$ (this fact is crucial for the argument to work). We conclude as in the proof of Theorem 4.22, using Corollary 4.24. $\square$

5. Comparison of trace formulas

5.1. Separation of representations. To compare trace formulas, we start with a simplification lemma, to get rid of the infinite sums on the spectral side of trace formulas.

Recall the notation $f^*(g) := \overline{f(g^{-1})}$.

Lemma 5.1. *Let $\omega : \mathbb{R}^\times \to \mathbb{C}^\times$ be a continuous unitary character. Let $(V_i, \pi)_{i \in I}$ be a family of irreducible unitary representations of $\mathrm{GL}_2(\mathbb{R})$ having central character ω and pairwise non-isomorphic. Let $(\lambda_i)_{i \in I}$ be a family of complex numbers such that for any $f \in C_c^\infty(\mathrm{GL}_2(\mathbb{R}), \omega^{-1})$, the operator $\bigoplus_{i \in I} \lambda_i \pi_i(f^* * f)$ on $\widehat{\bigoplus_{i \in I}^{\perp} V_i}$ is trace class, and $\sum_{i \in I} \lambda_i \operatorname{tr} \pi_i(f^* * f) = 0$. Then all $\lambda_i = 0$.*

Proof. Assume that there exists $i_0 \in I$ such that $\lambda_{i_0} \neq 0$. Up to multiplying all λ_i's by $-\lambda_{i_0}^{-1}$, we can assume that $\lambda_{i_0} = -1$, so that $\operatorname{tr} \pi_{i_0}(f^* * f) = \sum_{i \in I'} \lambda_i \operatorname{tr} \pi_i(f^* * f)$ with $I' = \{i \in I \mid i \neq i_0 \text{ and } \lambda_i \neq 0\}$. For each $i \in I'$, fix an orthonormal basis $(e_{i,j})_{j \in J_i}$ of V_i. The trace class assumption implies that for any $f \in C_c^\infty(\mathrm{GL}_2(\mathbb{R}), \omega^{-1})$ we have $\sum_{i \in I'} \sum_{j \in J_i} |\lambda_i| \|\pi_i(f) e_{i,j}\|_{V_i}^2 < \infty$. Let V be the completion of $\bigoplus_{i \in I'} V_i^{J_i}$ for the Hermitian inner product

$$\|((v_{i,j})_{j \in J_i})_{i \in I'}\|_V^2 = \sum_{i \in I'} |\lambda_i| \sum_{j \in J_i} \|v_{i,j}\|_{V_i}^2.$$

It is naturally a representation of G, which is clearly continuous and unitary. Consider the subspace

$$W_0 = \left\{ ((\pi_i(f) e_{i,j})_{j \in J_i})_{i \in I'} \,\middle|\, f \in C_c^\infty(\mathrm{GL}_2(\mathbb{R}), \omega^{-1}) \right\}$$

of V, and let W be its closure in V, a subrepresentation of V. Let $v \in V_{i_0}$ be such that $\|v\|_{V_{i_0}} = 1$. Completing this to form an orthonormal basis of V_{i_0} and writing traces in this basis, we obtain that for any $f \in C_c^\infty(\mathrm{GL}_2(\mathbb{R}), \omega^{-1})$

$$\|\pi_{i_0}(f) v\|_{V_{i_0}}^2 \leq \operatorname{tr} \pi_{i_0}(f^* * f) = \sum_{i \in I'} \lambda_i \operatorname{tr} \pi_i(f^* * f) \leq \sum_{i \in I'} |\lambda_i| \sum_{j \in J_i} \|\pi_i(f) e_{i,j}\|_{V_i}^2.$$

This inequality implies the existence and uniqueness of a continuous linear map $\Xi : W \to V_{i_0}$ mapping $((\pi_i(f) e_{i,j})_{j \in J_i})_{i \in I'}$ to $\pi_{i_0}(f) v$. This characterization shows that Ξ is $\mathrm{GL}_2(\mathbb{R})$-equivariant. Moreover we know that there exists f such that $\pi_{i_0}(f) v \neq 0$, thus $\Xi \neq 0$. We can uniquely extend Ξ to a linear map $V \to V_{i_0}$, abusively still denoted Ξ, by imposing that $\Xi|_{W^\perp} = 0$ (here $W^\perp$ is the orthogonal of W in V). This extension is clearly also continuous $\mathrm{GL}_2(\mathbb{R})$-equivariant. But the restriction of Ξ to each factor $((V_i)_{j \in J_i})_{i \in I'}$ is zero since $\pi_i \not\simeq \pi_{i_0}$, so $\Xi = 0$ by definition of V. We have obtained a contradiction, so the assumption that there exists $i_0 \in I$ such that $\lambda_{i_0} \neq 0$ was absurd. $\square$

5.2. **Multiplicity one results.** For the proof of Theorem 1.1 we will need to admit a few important theorems, which rely on theories which were not developed in this course.

Theorem 5.2 (Multiplicity one)**.** *Let $\omega : Z(\mathbb{Q})\backslash Z(\mathbb{A}) \to \mathbb{C}^\times$ be a unitary continuous character. Any cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A})$ occurs with multiplicity one in $\varinjlim_{K_f} L^2(G(\mathbb{Q})\backslash G(\mathbb{A}), \omega)^{K_f}$, i.e. $m_{\mathrm{cusp}}^{\mathrm{GL}_2}(\pi) = 1$.*

This was proved in [JL70] (over arbitrary global fields), and generalized to GL_n in [Sha74]. The proof uses Whittaker models, in particular their local uniqueness (this generalizes to all quasi-split reductive groups) and the fact that a cusp form can be reconstructed from Whittaker functionals (this is particular to general linear groups).

Theorem 5.3 (Strong multiplicity one)**.** *Let π and π' be cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A})$. Assume that there exists a finite set S of prime numbers such that for all $p \notin S$ we have $\pi_p \simeq \pi'_p$. Then $\pi \simeq \pi'$.*

See [PS79] for a proof using Kirillov models (related to the Whittaker models). Using Rankin-Selberg L-functions (again, relying on Whittaker models), a much more general result is proved in [JS81]. Morally, Čebotarev density theorem and linear independence of characters.

For inner forms, these methods do not adapt, essentially because there is no Whittaker model (at all non-split places, and thus globally). Ultimately one can show strong multiplicity one results, but using the trace formula and after proving the local Jacquet-Langlands correspondence.

Nevertheless, Godement-Jacquet L-functions and ϵ factors [GJ72] (this theory generalizes the abelian case of Tate's thesis and does not use Whittaker models) can be used to prove the following weaker result, which will be crucial for the proof of the local Jacquet-Langlands correspondence. As usual we specialize to the cases relevant to this course.

Theorem 5.4. *Let D be a quaternion algebra over $\mathbb{Q}$, G the associated inner form of GL_2. Let S be a finite set of prime numbers, and $(\sigma_v)_{v \notin S}$ a collection of smooth irreducible representations of $G(\mathbb{Q}_v)$. Then $\sum_\pi m^G(\pi) < \infty$ where the sum is over automorphic representations π of $G(\mathbb{A})$ such that $\pi_v \simeq \sigma_v$ for all places $v \notin S$.*

See [DKV84, Lemme B.1.e p. 80].

5.3. **Easy transfer.** To compare trace formulas we have to produce *matching functions* on different groups, i.e. functions which have the same orbital integrals (note that this requires an identification of conjugacy classes and of centralizers in the two groups).

Lemma 5.5. *Let p be a prime number. Let $\omega : Z(\mathrm{GL}_2(\mathbb{Q}_p)) \to \mathbb{C}^\times$ be a smooth character. Recall that $\mathcal{T}$ denotes a set of representatives for the (finitely many) conjugacy classes of maximal tori in $\mathrm{GL}_2(\mathbb{Q}_p)$. Fix Haar measures on $\mathrm{PGL}_2(\mathbb{Q}_p)$ and on each $T'/Z(\mathrm{GL}_2(\mathbb{Q}_p))$ for $T' \in \mathcal{T}$. Let $(F_{T'})_{T'\mathcal{T}}$ be a family of smooth functions $T'_{G-\mathrm{reg}} \to \mathbb{C}$ such that $F_{T'}$ is ω^{-1}-equivariant, $N_{\mathrm{GL}_2(\mathbb{Q}_p)}(T')$-invariant and compactly supported modulo $Z(\mathrm{GL}_2(\mathbb{Q}_p))$. Then there exists $f \in C_c^\infty(\mathrm{GL}_2(\mathbb{Q}_p), \omega^{-1})$ whose* *support is contained in the set of regular semisimple elements in* $\mathrm{GL}_2(\mathbb{Q}_p)$ *such that for any* $T' \in \mathcal{T}$ *and any* $t \in T'_{G-\mathrm{reg}}$ *we have* $O_t(f) = F(t)$*, and for any* $T' \in \mathcal{T}$ *such that* $F_{T'}$ *vanishes identically* f *also vanishes on all elements conjugate to elements of* T'*.*

Note that the assumption is that the support of $F_{T'}$ is a compact subset of $T'_{G-\mathrm{reg}}/Z(\mathrm{GL}_2(\mathbb{Q}_p))$, not just a relatively compact subset of $T'/Z(\mathrm{GL}_2(\mathbb{Q}_p))$.

Proof. We use the functions $\phi_{T'}$ defined in Section 3.3. For $T' \in \mathcal{T}$ let $U_{T'}$ be a non-empty compact open subset of $T'\backslash\mathrm{GL}_2(\mathbb{Q}_p)$ such that $w'U_{T'} \cap U_{T'} = \emptyset$, where w' is the non-trivial element of $N_{\mathrm{GL}_2(\mathbb{Q}_p)}(T')$. Let $f(\phi_{T'}(t, \dot{g})) = \mathrm{vol}(U_{T'})^{-1}F(t)/2$ if $\dot{g} \in U$, zero otherwise. Then

$$O_t(f) = \int_{T'\backslash\mathrm{GL}_2(\mathbb{Q}_p)} f(g^{-1}tg)\, d\dot{g} = \int_{U_{T'}\sqcup w'U_{T'}} f(g^{-1}tg)\, d\dot{g} = \frac{F(t) + F(t^{w'})}{2} = F(t).$$

□

5.4. **Proof of the local Jacquet-Langlands correspondence.** We can finally prove Theorem 1.1. In this section D will denote a non-split quaternion algebra over $\mathbb{Q}_p$. The first step is to associate to an essentially square-integrable representation τ_p of $\mathrm{GL}_2(\mathbb{Q}_p)$ an irreducible representation of $D^\times$, satisfying the relation between Harish-Chandra characters. If $\tau_p \simeq (\chi_p \circ \det) \otimes \mathrm{St}$, we have already seen (Corollary 3.25) that the representation $\chi \circ \det$ of $D^\times$ corresponds to τ_p.

Thus we can assume that τ_p is supercuspidal. As in the Steinberg case, it is enough to prove the result with τ_p replaced by $(\chi_p \circ \det) \otimes \tau_p$ for some smooth character $\chi_p : \mathbb{Q}_p^\times \to \mathbb{C}^\times$. Let $\ell_1 \neq \ell_2$ be prime numbers distinct from p. Let $\omega : Z(\mathbb{Q})\backslash Z(\mathbb{A}) \to \mathbb{C}^\times$ be a continuous unitary character such that $\omega|_{\mathbb{Z}_p^\times} = \omega_{\tau_p}$, $\omega|_{\mathbb{Z}_{\ell_1}^\times} = 1$. Let χ_p be one of the two unramified characters of $\mathbb{Q}_p^\times$ satisfying $(\chi \circ \det|_{Z(\mathbb{Q}_p)})\omega_{\tau_p} = \omega_p$. Let $\chi_{\ell_1} : \mathbb{Q}_{\ell_1}^\times \to \mathbb{C}^\times$ be one of the two unramified characters such that $\chi_{\ell_1} \circ \det|_{Z(\mathbb{Q}_{\ell_1})} = \omega_{\ell_1}$. Choose an irreducible supercuspidal representation τ_{ℓ_2} of $\mathrm{GL}_2(\mathbb{Q}_{\ell_2})$ having central character ω_{ℓ_2} (Theorem 3.37). Let D_{glob} be a quaternion algebra corresponding to $S = \{p, \ell_1\}$, and let G be the associated inner form of GL_2. In particular we have an isomorphism $G(\mathbb{Q}_p) \simeq D^\times$, well-defined up to inner composing with an inner automorphism. Fix Haar measures on $G_{\mathrm{ad}}(\mathbb{Q}_v)$ (for all places v, so that $G_{\mathrm{ad}}(\mathbb{Z}_p)$ has volume 1 for almost all p) and endow $G_{\mathrm{ad}}(\mathbb{A})$ with the product of Haar measures. We will apply the trace formula with functions $f_v \in C_c^\infty(G(\mathbb{Q}_v), \omega_v)$ as follows:

- f_{ℓ_1} is a coefficient for the representation $\chi_{\ell_1} \circ \det$ of $G(\mathbb{Q}_{\ell_1})$. To be explicit, we can take $f_{\ell_1}(g) = \mathrm{vol}(G_{\mathrm{ad}}(\mathbb{Q}_p))^{-1}\chi_{\ell_1}(\det g)^{-1}$.
- f_{ℓ_2} is a coefficient for τ_{ℓ_2}.
- f_p is any smooth function which vanishes on $Z(\mathbb{Q}_p)$.
- for $v \notin \{p, \ell_1, \ell_2\}$, f_v is arbitrary, we only impose that f_p is the unit in $\mathcal{H}(G(\mathbb{Q}_p), G(\mathbb{Z}_p), \omega_p^{-1})$ for almost all prime numbers p.

Recall that there is a family $(\psi_v)_{v\notin\{p,\ell_1\}}$ of isomorphisms $\psi_v : G\times_{\mathbb{Q}}\mathbb{Q}_v \simeq \mathrm{GL}_{2,\mathbb{Q}_v}$, well-defined up to composition on the right with $\mathrm{Ad}(g)$ for some $g\in \mathrm{GL}_2(\mathbb{A}^{(p,\ell_1)})$ where $\mathbb{A}^{(p,\ell_1)} = \prod'_{v\notin\{p,\ell_1\}}\mathbb{Q}_v = \mathbb{A}/\mathbb{Q}_p\mathbb{Q}_{\ell_1}$. Fix Haar measures on $\mathrm{PGL}_2(\mathbb{Q}_v)$ for all places v, so that for any For $v\notin\{p,\ell_1\}$, endow $\mathrm{PGL}_2(\mathbb{Q}_v)$ with the Haar measure transported from that on $G_{\mathrm{ad}}(\mathbb{Q}_v)$ via ψ_v. Endow $\mathrm{PGL}_2(\mathbb{Q}_p)$ and $\mathrm{PGL}_2(\mathbb{Q}_{\ell_1})$ with arbitrary Haar measures.

Now choose corresponding functions $f_v^{\mathrm{GL}}\in C_c^\infty(\mathrm{GL}_2(\mathbb{Q}_v),\omega_v^{-1})$ as follows.

- $f_{\ell_1}^{\mathrm{GL}}$ is the pseudo-coefficient for the representation $(\chi_{\ell_1}\circ\det)\otimes\mathrm{St}$ constructed in Proposition 3.29. Note that by Theorem 3.30 this implies that the orbital integrals of $f_{\ell_1}^{\mathrm{GL}}$ are opposite to that of f_{ℓ_1}. Note that there is an isomorphism between the centralizers $(\mathrm{GL}_2)_t(\mathbb{Q}_{\ell_1})$ and $G_t(\mathbb{Q}_{\ell_1})$, well-defined up to normalizers, so we can transport Haar measures between $(\mathrm{GL}_2)_t(\mathbb{Q}_{\ell_1})/Z(\mathbb{Q}_{\ell_1})$ and $G_t(\mathbb{Q}_{\ell_1})/Z(\mathbb{Q}_{\ell_1})$, and comparing orbital integrals makes sense.

- f_p^{GL} is supported on the set of semisimple regular elliptic elements in $\mathrm{GL}_2(\mathbb{Q})$ and such that for any semisimple regular $t\in\mathrm{GL}_2(\mathbb{Q}_p)$ we have

$$O_t(f_p^{\mathrm{GL}}) = \begin{cases} -O_{t'}(f_p) & \text{if } t'\in G(\mathbb{Q}_p) \text{ has same characteristic polynomial as } t\\ 0 & \text{if } t \text{ is hyperbolic.}\end{cases}$$

  The existence of such $f_p\in C_c^\infty(\mathrm{GL}_2(\mathbb{Q}_p),\omega_p^{-1})$ follows from Lemma 5.5. The same remark as at the place ℓ_1 applies for the comparison of Haar measures of centralizers.

- for any $v\notin\{\ell_1,p\}$, there is an isomorphism $\psi_v : \mathrm{GL}_{2,\mathbb{Q}_v}\simeq G\times_{\mathbb{Q}}\mathbb{Q}_v$, and we let $f_v^{\mathrm{GL}} = f_v\circ\psi_v^{-1}$. In particular f_v is trivial for almost all v, and the orbital integrals of f_v do not depend on the choice of ψ_v.

These choices were made so that the geometric sides of the traces formulas for GL_2 (Theorem 4.35) and G (Theorem 4.18) are equal (again, the centralizers G_γ and $(\mathrm{GL}_2)_\gamma$ can be identified up to conjugation by the Weyl group, and the signs at ℓ_1 and p cancel each other so the global orbital integrals are equal). Therefore the spectral sides are equal:

$$\sum_\pi m_{\mathrm{cusp}}^{\mathrm{GL}_2}(\pi)\,\mathrm{tr}\,\pi(f^{\mathrm{GL}}) = \sum_{\pi'} m^G(\pi')\,\mathrm{tr}\,\pi'(f).$$

Let σ be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ with central character ω such that

- $\sigma_p\simeq(\chi_p\circ\det)\otimes\tau_p$,
- $\sigma_{\ell_1}\simeq(\chi_{\ell_1}\circ\det)\otimes\mathrm{St}$, and
- $\sigma_{\ell_2}\simeq\tau_{\ell_2}$.

The existence of such a σ follows from Theorem 4.36.

Applying Lemma 5.1 (using ψ_∞ to identify $\mathrm{GL}_2(\mathbb{R})$ and $G(\mathbb{R})$), we obtain

$$\sum_{(\pi_q)_q} m_{\mathrm{cusp}}^{\mathrm{GL}_2}\Big(\sigma_\infty \otimes {\bigotimes_q}' \pi_q\Big) \prod_q \operatorname{tr} \pi_q(f_q) = \sum_{(\pi'_q)_q} m^G\Big(\sigma_\infty \otimes {\bigotimes}' \pi'_q\Big) \prod_q \operatorname{tr} \pi'_q(f_q)$$

where the products are over all prime numbers q. Note that both sides are traces in an admissible representation. As recalled at the beginning of Section 3.4, the theory of finite-dimensional representation of associative $\mathbb{C}$-algebras (for the Hecke algebra $\mathcal{H}(\mathrm{GL}_2(\mathbb{A}^{(\infty,p,\ell_1,\ell_2)}), (\omega^{(\infty,p,\ell_1,\ell_2)})^{-1})$) tells us that this implies

$$\sum_{(\pi_p,\pi_{\ell_1},\pi_{\ell_2})} m_{\mathrm{cusp}}^{\mathrm{GL}_2}\Big(\pi_p \otimes \pi_{\ell_1} \otimes \pi_{\ell_2} \otimes {\bigotimes_{v\notin\{p,\ell_1,\ell_2\}}}' \sigma_v\Big) \operatorname{tr} \pi_p(f_p^{\mathrm{GL}}) \operatorname{tr} \pi_{\ell_1}(f_{\ell_1}^{\mathrm{GL}}) \operatorname{tr} \pi_{\ell_2}(f_{\ell_2}^{\mathrm{GL}})$$
$$= \sum_{(\pi'_p,\pi'_{\ell_1},\pi'_{\ell_2})} m^G\Big(\pi'_p \otimes \pi'_{\ell_1} \otimes \pi'_{\ell_2} \otimes {\bigotimes_{v\notin\{p,\ell_1,\ell_2\}}}' \sigma_v\Big) \operatorname{tr} \pi'_p(f_p) \operatorname{tr} \pi'_{\ell_1}(f_{\ell_1}) \operatorname{tr} \pi_{\ell_2}(f_{\ell_2}^{\mathrm{GL}})$$

Again this uses $(\psi_v)_{v\notin\{p,\ell_1,\ell_2,\infty\}}$. Since f_{ℓ_2} and $f_{\ell_2}^{\mathrm{GL}}$ are coefficients for the supercuspidal representation $\tau_{\ell_2} \simeq \sigma_{\ell_2}$, this implies

$$\sum_{(\pi_p,\pi_{\ell_1})} m_{\mathrm{cusp}}^{\mathrm{GL}_2}\Big(\pi_p \otimes \pi_{\ell_1} \otimes {\bigotimes_{v\notin\{p,\ell_1\}}}' \sigma_v\Big) \operatorname{tr} \pi_p(f_p^{\mathrm{GL}}) \operatorname{tr} \pi_{\ell_1}(f_{\ell_1}^{\mathrm{GL}})$$
$$= \sum_{(\pi'_p,\pi'_{\ell_1})} m^G\Big(\pi'_p \otimes \pi'_{\ell_1} \otimes {\bigotimes_{v\notin\{p,\ell_1\}}}' \sigma_v\Big) \operatorname{tr} \pi'_p(f_p) \operatorname{tr} \pi'_{\ell_1}(f_{\ell_1})$$

By the same argument as in Theorem 4.36, the analogue of Corollary 4.24 for GL_2 implies that for any non-vanishing term on the left-hand side, $\pi_{\ell_1} \simeq \sigma_{\ell_1}$ (that is, π_{ℓ_1} is not isomorphic to $\chi_{\ell_1} \circ \det$), and so $\operatorname{tr} \pi_{\ell_1}(f_{\ell_1}^{\mathrm{GL}}) = 1$. On the right-hand side, any non-vanishing term has $\pi'_{\ell_1} \simeq \chi_{\ell_1} \circ \det$, and so $\operatorname{tr} \pi'_{\ell_1}(f_{\ell_1}) = 1$. Therefore

$$\sum_{\pi_p} m_{\mathrm{cusp}}^{\mathrm{GL}_2}\Big(\pi_p \otimes {\bigotimes_{v\neq p}}' \sigma_v\Big) \operatorname{tr} \pi_p(f_p^{\mathrm{GL}}) = \sum_{\pi'_p} m^G\Big(\pi'_p \otimes (\chi_{\ell_1} \circ \det) {\bigotimes_{v\notin\{p,\ell_1\}}}' \sigma_v\Big) \operatorname{tr} \pi'_p(f_p)$$

Thanks to the strong multiplicity one theorem (Theorem 5.3), we know that the left-hand side is simply $\operatorname{tr} \sigma_p(f_p^{\mathrm{GL}})$. The right-hand side is the trace of f_p on a semisimple admissible smooth representation of $G(\mathbb{Q}_p)$. Thanks to Theorem 5.4 we know that it is in fact a finite length (i.e. finite-dimensional) representation, that we denote $\mathrm{JL}(\sigma_p)$. It is indeed determined by σ_p up to isomorphism because its trace is (there is a restriction on f_p, but $G(\mathbb{Q}_p) \smallsetminus Z(\mathbb{Q}_p)$ is an open and dense subset of $G(\mathbb{Q}_p)$). Namely, we have $\Theta_{\sigma_p}(t) = -\Theta_{\mathrm{JL}(\sigma_p)}(t')$ for any semisimple regular $t \in \mathrm{GL}_2(\mathbb{Q}_p)$ and $t' \in D^\times$ having the same characteristic polynomial (note the minus sign which comes from the definition of f_p^{GL}). We need to show that $\mathrm{JL}(\sigma_p)$ is irreducible. This follows from elliptic orthogonality (Theorem 3.35 and its easier analogue for $D^\times$, which follows from the analogous Weyl integration formula): if $\mathrm{JL}(\sigma_p) \simeq \oplus_i \rho_i^{\oplus m_i}$ with distinct irreducible ρ_i's, comparing the two elliptic orthogonality relations we obtain $\sum_i m_i^2 = 1$. Elliptic orthogonality also implies that the map JL is injective on isomorphism classes (an irreducible essentially square-integrable representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ is determined by the restriction of its Harish-Chandra character to the semisimple regular elliptic locus).

Finally we need to show that for any irreducible smooth representation ρ of $D^\times$, there exists an irreducible essentially square-integrable representation τ_p of $\mathrm{GL}_2(\mathbb{Q}_p)$ such that $\mathrm{JL}(\tau_p) \simeq \rho$. The argument is almost the same as above, except that we start with an automorphic representation σ of $G(\mathbb{A})$ such that σ_p is a twist of ρ, $\sigma_{\ell_1} \simeq \chi_{\ell_1} \circ \det$ and σ_{ℓ_2} is supercuspidal (such a σ exists thanks to Theorem 4.22). We obtain

$$\sum_{\pi_p} m_{\mathrm{cusp}}^{\mathrm{GL}_2}\Big(\pi_p \otimes \big((\chi_{\ell_1} \circ \det) \otimes \mathrm{St}\big) \otimes {\bigotimes_{v \notin \{p,\ell_1\}}}' \sigma_v\Big) \operatorname{tr} \pi_p(f_p^{\mathrm{GL}}) = \sum_{\pi'_p} m^G\Big(\pi'_p \otimes {\bigotimes_{v \neq p}}' \sigma_v\Big) \operatorname{tr} \pi'_p(f_p)$$

By the strong multiplicity one theorem, the left-hand side has at most one non-zero term, for which the multiplicity is 1. The right-hand side is the trace of f_p on a non-zero representation of $G(\mathbb{Q}_p)$ of finite length, in particular there exists f_p such that the right-hand side does not vanish. Thus there exists a unique π_p contributing to the left-hand side, and going back to the previous argument we have $\mathrm{JL}(\pi_p) = \rho$.

Appendix A. Summary of integration theory on groups and homogeneous spaces

For X a topological space, $C_c(X)$ denotes the space of continuous and compactly supported functions $X \to \mathbb{C}$.

Theorem A.1 (Riesz-Markov-Kakutani Representation Theorem). *Let X be a locally compact Hausdorff topological space. The map*

$$\mu \longmapsto \left(f \mapsto \int_X f(x) d\mu(x) \right)$$

is a bijection between

- *the set of Radon measures on X, i.e. measures μ on the σ-algebra of Borel sets on X satisfying*
  - *for any compact subset K of X we have $\mu(K) < \infty$,*
  - *for any Borel subset S of X we have*

    $$\mu(S) = \inf\{\mu(U) \,|\, U \text{ open subset of } X \text{ containing } S\},$$

  - *for any Borel subset S of X which is either open in X or satisfying $\mu(S) < \infty$, we have*

    $$\mu(S) = \sup\{\mu(K) \,|\, K \text{ compact subset of } X \text{ contained in } S\}.$$

    (note that if X is σ-compact, as will always be the case in these notes, this property then holds for any Borel subset of X),

  (resp. complex Radon measures on X, i.e. complex linear combinations of Radon measures),
- *the set of linear maps $I : C_c(X) \to \mathbb{C}$ which are positive, i.e. for any $f \in C_c(X)$ satisfying $f(X) \subset \mathbb{R}_{\geq 0}$ we have $I(f) \in \mathbb{R}_{\geq 0}$ (resp. continuous for the topology on $C_c(X)$ obtained by realizing $C_c(X)$ as the direct limit of the Banach spaces $C(K)$ where K ranges over all compact subsets of X).*

Proof. See [Rud87, Theorem 2.14]. □

Because of this equivalence, when using integration theory on locally compact Hausdorff topological spaces it is more convenient to deal with linear forms on $C_c(X)$ rather than measures.

For details and proofs for all that follows see [Bou63, Ch. 7].

Theorem A.2. *Let H be a locally compact Hausdorff topological group.*

- *There exists a non-zero Radon measure μ_0 on H which is left H-invariant, i.e. for any $f \in C_c(H)$ and any $g \in H$ we have*

  $$\int_H f(gh) d\mu_0(h) = \int_H f(h) d\mu_0(h).$$

  (equivalently, for any Borel subset S of X and any $g \in H$ we have $\mu_0(gS) = \mu_0(S)$).

- *Fix μ_0 as in the previous point. Let μ be a complex Radon measure on H which is also left H-invariant. There exists a unique $c \in \mathbb{C}$ such that we have $\mu = c\mu_0$.*

A non-zero left H-invariant Radon measure on H is called a left Haar measure. By essentially the same proof, or applying the result to the opposite group, the same result holds with "left" replaced by "right".

For any $x \in H$ there is a unique $\Delta_H(x) \in \mathbb{R}_{>0}$ such that for any left H-invariant complex Radon measure μ on H we have

$$\int_H f(hx) d\mu(h) = \Delta_H(x)^{-1} \int_H f(h) d\mu(h).$$

The map

$$\Delta_H : H \longrightarrow \mathbb{R}_{>0}$$

is easily seen to be a continuous character. For any left Haar measure μ on H the Radon measure $\Delta_H^{-1}\mu$ is a right Haar measure. For $g \in H$ and $f \in C_c(H)$ we have

$$\int_H f(gh)\Delta_H^{-1}(h) d\mu(h) = \int_H f(h)\Delta_H^{-1}(g^{-1}h) d\mu(h) = \Delta_H(g) \int_H f(h)\Delta_H^{-1}(h) d\mu(h). \tag{A.1}$$

The topological group H is called *unimodular* if the character Δ_H is trivial. This is the case if H is commutative, or discrete (the counting measure is a left and right Haar measure), or compact ($\Delta_H(H)$ is a compact subgroup of $\mathbb{R}_{>0}$, so it is trivial).

Let μ be a left Haar measure on H. For any $f \in L^1(H, \mu)$ the function $h \mapsto f(h^{-1})\Delta_H(h)^{-1}$ on H is also integrable with respect to μ and we have

$$\int_H f(h^{-1})\Delta_H(h)^{-1} d\mu(h) = \int_H f(h) d\mu(h).$$

Let X be a locally compact Hausdorff topological space endowed with a left action of H which is continuous and proper. Let $\chi : H \to \mathbb{C}^\times$ be a continuous character. For $f \in C_c(X)$ define $f^\chi : X \to \mathbb{C}$ by

$$f^\chi(x) = \int_H f(h \cdot x)\Delta_H^{-1}(h)\chi(h)^{-1} d\mu(h)$$

Let $C_c(X, H, \chi)$ be the space of continuous functions $f : X \to \mathbb{C}$ satisfying:

- for any $g \in H$ and $x \in X$ we have $f(g \cdot x) = \chi(g) f(x)$,
- the support of f is compact modulo H, i.e. its image in $H\backslash X$ is compact (equivalently, the set of $H \cdot x$ in $H\backslash X$ such that $f(x) \neq 0$ is relatively compact in $H\backslash X$).

In order to endow $C_c(X, H, \chi)$ with a topology, realize it as $\varinjlim_K C(H \cdot K, H, \chi)$ where the direct limit is over all compact subsets K of X. For any such K the space $C(H \cdot K, H, \chi)$ is a Banach space for the norm $f \mapsto \sup_K |f|$. Endow $C_c(X, H, \chi)$ with the final topology. If χ is trivial then $C_c(X, H, \chi)$ is identified with $C_c(H\backslash X)$ and this topology coincides with the final topology on $\varinjlim_{K'} C(K')$, where the inductive limit is taken over all compact subsets K' of $H\backslash X$.

Let $f \in C_c(X)$. For $g \in H$ we have

$$\begin{aligned} f^\chi(g\cdot x) &= \int_H f(hg\cdot x)\Delta_H^{-1}(h)\chi(h)^{-1}d\mu(h) \\ &= \Delta_H(g)\chi(g)\int_H f(hg\cdot x)\Delta_H^{-1}(hg)\chi(hg)^{-1}d\mu(h) \\ &= \chi(g)\int_H f(h\cdot x)\Delta_H^{-1}(h)\chi(h)^{-1}d\mu(h) \\ &= \chi(g)f^\chi(x). \end{aligned}$$

Moreover f^χ vanishes away from the preimage of the image of $\operatorname{supp} f$ in $H\backslash X$, so we have $f^\chi \in C_c(X,H,\chi)$.

Proposition A.3. *The map*

$$\begin{aligned} A_\chi : C_c(X) &\longrightarrow C_c(X,H,\chi) \\ f &\longmapsto f^\chi \end{aligned}$$

is surjective, and identifies the topological complex vector space $C_c(X,H,\chi)$ with the quotient of $C_c(X)$ by the kernel of A_χ. If χ takes values in $\mathbb{R}_{>0}$ then A_χ maps the subset of real non-negative functions in $C_c(X)$ onto the subset of real non-negative functions in $C_c(X,H,\chi)$.

Proof. Surjectivity is proved in [Bou63, Ch. 7, §2, Proposition 2], translating between left and right actions. (This assumes that χ takes values in $\mathbb{R}_{>0}$ but it is clear that the proof does not use this assumption.) The assertion in the case where χ takes values in $\mathbb{R}_{>0}$ is also proved loc. cit.

To prove that the topology on $C_c(X,H,\chi)$ defined above is the quotient topology it is enough to prove that A_χ is continuous and open. Continuity is easy using properness of the action, and openness is easily proved using [Bou63, Ch. 7, §2, Lemme 1] (which is used to prove Proposition 2 loc. cit.). $\square$

We wish to understand under which assumption a complex Radon measure on X, seen as a continuous linear map $C_c(X) \to \mathbb{C}$, factors through A_χ. Note that by the previous proposition, such a factorization is unique. For $f \in C_c(X)$ and $g \in H$ consider the function $f' : X \to \mathbb{C}$ defined by $f'(x) = f(g\cdot x)$. We have

$$\begin{aligned} (f')^\chi(x) &= \int_H f'(h\cdot x)\Delta_H^{-1}(h)\chi(h)^{-1}d\mu(h) \\ &= \int_H f(gh\cdot x)\Delta_H^{-1}(h)\chi(h)^{-1}d\mu(h) \\ &= \Delta_H(g)\chi(g)\int_H f(gh\cdot x)\Delta_H^{-1}(gh)\chi(gh)^{-1}d\mu(h) \\ &= \Delta_H(g)\chi(g)\int_H f(h\cdot x)\Delta_H^{-1}(h)\chi(h)^{-1}d\mu(h) \\ &= \Delta_H(g)\chi(g)f^\chi(x). \end{aligned}$$

This suggests the following proposition.

Proposition A.4. *Let ν be a complex Radon measure on X. Assume that ν is left H-equivariant for the character $\Delta_H \chi$, i.e. for any $f \in C_c(X)$ and $g \in H$ we have*

$$\int_X f(g \cdot x) d\nu(x) = \Delta_H(g)\chi(g) \int_X f(x) d\nu(x).$$

There exists a unique linear map $I_{\nu,\mu,\chi} : C_c(X, H, \chi) \to \mathbb{C}$ such that for any $f \in C_c(X)$ we have

$$\int_X f(x) d\nu(x) = I_{\nu,\mu,\chi}(f^\chi).$$

The map $I_{\nu,\mu,\chi}$ is continuous. If χ takes values in $\mathbb{R}_{>0}$ and ν is a Radon measure then $I_{\nu,\mu,\chi}$ is positive, i.e. it maps non-negative functions to $\mathbb{R}_{\geq 0}$.

Proof. This is one of the implications in [Bou63, Ch. 7, §2, Proposition 3], translating between left and right actions, at least under the assumption that χ takes values in $\mathbb{R}_{>0}$.

For the convenience of the reader we translate the proof. Thanks to the previous proposition it is enough to prove that the kernel of A_χ is contained in the kernel of the integration map on X with respect to ν. For any $f_1, f_2 \in C_c(X)$ we have

$$\begin{aligned}
\int_X f_1^\chi(x) f_2(x) d\nu(x) &= \int_X \int_H f_1(h \cdot x) f_2(x) \Delta_H^{-1}(h) \chi(h)^{-1} d\mu(h) d\nu(x) \\
&= \int_H \Delta_H^{-1}(h)\chi(h)^{-1} \int_X f_1(h \cdot x) f_2(x) d\nu(x) d\mu(h) \\
&= \int_H \int_X f_1(y) f_2(h^{-1} \cdot y) d\nu(y) d\mu(h) \\
&= \int_X f_1(y) \int_H f_2(h^{-1} \cdot y) \Delta_H(h^{-1})^{-1} \Delta_H(h)^{-1} d\mu(h) d\nu(y) \\
&= \int_X f_1(y) \int_H f_2(h \cdot y) \Delta_H(h)^{-1} d\mu(h) d\nu(y) \\
&= \int_X f_1(y) f_2^1(y) d\nu(y).
\end{aligned}$$

Assume that $f_1 \in C_c(X)$ is such that f_1^χ vanishes identically. By the previous proposition and Urysohn's lemma there exists $f_2 \in C_c(X)$ such that f_2^1 is constant equal to 1 on the support of f_1. The above calculation shows that we have

$$\int_X f_1(x) d\nu(x) = 0.$$

This shows the existence of $I_{\nu,\mu,\chi}$. $\square$

Example A.5. *Assume that X is a locally compact topological group and that H is a closed subgroup, with the obvious left action by multiplication. Let ν be a right Haar measure on X. For $g \in H$ we have ((A.1) above)*

$$\int_X f(gx) d\nu(x) = \Delta_X(g) \int_X f(x) d\nu(x).$$

Therefore the assumption of Proposition A.4 holds for $\chi = \Delta_X|_H \Delta_H^{-1}$ (and only for this character). Let K be a compact open subgroup of X and $x \in X$. The function $(1_{xK})^\chi$ clearly vanishes away from HxK. For $y = xk \in xK$ we have

$$\begin{aligned}(1_{xK})^\chi(y) &= \int_H 1_{xK}(hxk)\Delta_H^{-1}(h)\chi(h)^{-1}d\mu(h)\\ &= \int_{H\cap xKx^{-1}} \Delta_X^{-1}(h)d\mu(h)\\ &= \mu(H \cap xKx^{-1})\end{aligned}$$

because the character Δ_X takes values in $\mathbb{R}_{>0}$ and so it is trivial on any compact subgroup of X. Let f_{xK} be the unique element of $C_c(X, H, \chi)$ supported on HxK and such that $f_{xK}(xk) = 1$ for any $k \in K$. We thus have

$$I_{\nu,\mu,\chi}(f_{xK}) = \mu(H \cap xKx^{-1})^{-1}\nu(xK) = \Delta_X^{-1}(x)\frac{\nu(K)}{\mu(H \cap xKx^{-1})}. \tag{A.2}$$

If any neighbourhood of 1 in X contains a compact open subgroup of X then the subspace $C_c^\infty(X, H, \chi)$ of $C_c(X, H, \chi)$ consisting of locally constant functions is dense and any element of $C_c^\infty(X, H, \chi)$ is a linear combination of functions f_{xK} as above. In particular Formula (A.2) above computes the "quotient measure" $I_{\nu,\mu,\chi}$.

Appendix B. Summary of p-adic manifolds and integration using differential forms

We sketch the foundations of p-adic manifolds (sometimes called p-adic *analytic* manifolds) in order to state the "change of variables" formula for measures associated to differential forms (introduced in [Wei82]) that is useful for the study of intertwining operators (Proposition 2.31) and harmonic analysis. We emphasize differences with the Archimedean case. For details (and proper foundations) see [Ser06], [Sch11].

Let $n \geq 1$ be an integer. Let U be an open subset of $\mathbb{Q}_p^n$, and $f : U \to \mathbb{Q}_p$ a function. We say that f is locally analytic at $x_0 \in U$ if there is a family $(a_\alpha)_{\alpha \in \mathbb{N}^n}$ of elements of $\mathbb{Q}_p$ and $r > 0$ such that $(|a_\alpha| r^{|\alpha|})_\alpha$ is bounded (notation: $|\alpha| = \sum_i \alpha_i$) and for any $x \in U \cap D(x_0, r)$ (open disk of radius r) we have $f(x) = \sum_\alpha a_\alpha (x - x_0)^\alpha$ (notation: $z^\alpha = \prod_i z_i^{\alpha_i}$). The same proof as in the complex setting shows that f is then continuous on $U \cap D(x_0, r)$ and locally analytic at any point of $U \cap D(x_0, r)$. The main difference from the complex setting is that any U is totally disconnected: for example, for any function $\mathbb{F}_p \to \mathbb{Q}_p$, the composition $U := \mathbb{Z}_p \to \mathbb{F}_p \to \mathbb{Q}_p$ is locally analytic (even locally constant!) at every point of U.

We have the notion of a locally analytic function $U \to \mathbb{Q}_p^m$ (coordinate-wise), and the composition of two locally analytic functions is again locally analytic. A locally analytic function is differentiable (obvious definition . . .) and its differential (taking values in $\mathrm{Hom}_{\mathbb{Q}_p}(\mathbb{Q}_p^n, \mathbb{Q}_p^m) \simeq \mathbb{Q}_p^{mn}$) is again locally analytic.

Theorem B.1 (Inverse function theorem)**.** *Let U be an open subset of $\mathbb{Q}_p^n$ and $x_0 \in U$. Let $f = (f_1, \dots, f_n) : U \to \mathbb{Q}_p^n$ be a locally analytic function. Assume that the differential of f at x_0 is invertible. Then up to replacing U by an open subset containing x_0, $f(U)$ is open in $\mathbb{Q}_p^n$, f is injective and its inverse $f(U) \to U$ is also locally analytic.*

Proof. Using the usual reductions (translations so that $x_0 = 0$ and $f(x_0) = 0$, post-composing f with the inverse of its differential, pre- and post-composing f with homotheties) we may assume that $U = p\mathbb{Z}_p^n$ and each f_i is a power series, i.e. for $x \in U$ we have

$$f_i(x) = x_i + \sum_{\substack{\alpha \\ |\alpha| \geq 2}} a_{i,\alpha} x^\alpha$$

with $a_{i,\alpha} \in \mathbb{Z}_p$. Note that under this condition, f_i converges on $p\mathbb{Z}_p^n$ and maps $p\mathbb{Z}_p^n$ to itself. We look for g satisfying the same conditions: $g_j(y) = y_j + \sum_\beta b_{j,\beta} y^\beta$. Solving the equation of *formal* power series $f \circ g = \mathrm{Id}$, we see that there is a unique solution. More precisely, by induction on $|\beta|$ we see that $b_{j,\beta} = P_{j,\beta}(a_{i,\alpha}, |\alpha| \leq |\beta|)$ where $P_{j,\beta}$ is a polynomial with coefficients in $\mathbb{Z}_p$.

Reversing the role of f and g, we get that they are inverse maps of each other $p\mathbb{Z}_p^n \to p\mathbb{Z}_p^n$. □

This local theory allows to define p-adic manifolds, obtained by gluing open subsets of $\mathbb{Q}_p^n$ (or $\mathbb{Z}_p^n$) using locally analytic maps to change coordinates. More precisely, if X is a topological space:

- a chart on X is an open subset U of X together with a homeomorphism $\phi : U \to \phi(U)$ where $\phi(U)$ is an open subset of $\mathbb{Q}_p^n$ for some n,

- an atlas on X is a family of charts covering X which are pairwise compatible, i.e. such that the transition maps $\phi' \circ \phi^{-1} : \phi(U \cap U') \to \phi'(U \cap U')$ are locally analytic (exchanging U and U' we get that $n = n'$ if $U \cap U' \neq \emptyset$).

We say that two atlases are compatible if their charts are pairwise compatible. This is an equivalence relation, and we get the notion of a p-adic manifold: a topological space X with an equivalence class of atlases on X, or equivalently a maximal atlas. Note that the dimension n is a locally constant function on X. We will only consider p-adic manifolds of constant dimension.

Clearly a p-adic manifold is locally compact (if one uses a definition of "locally compact" that does not include "Hausdorff"). All examples that we will encounter will also be Hausdorff and paracompact (i.e. every open cover has a refinement that is locally finite; this condition holds if X is a countable union of compact subsets).

We have the obvious notion of morphism between p-adic manifolds: continuous maps which are locally analytic in local coordinates given by charts. As in the Archimedean case one can define tangent and cotangent bundles, and tensor, symmetric and exterior powers of these bundles, for example differential k-forms. The differential of a morphism is a morphism between tangent bundles. Fibers of a submersion are also p-adic manifolds (use the inverse function theorem).

Example: for any smooth algebraic variety X over $\mathbb{Q}_p$, $X(\mathbb{Q}_p)$ is naturally endowed with the structure of a p-adic manifold. This is the case for G, B, T, N, and the group structure is compatible, i.e. multiplication and inversion are morphisms of p-adic manifolds, so these are p-adic Lie groups. Of course open subgroups of these, in particular compact open subgroups, are also p-adic Lie groups.

One may also define submanifolds and quotients of manifolds as in the Archimedean case. If an equivalence relation $R \subset X \times X$ is a submanifold and the first projection $\mathrm{pr}_1 : R \to X$ is a submersion then the quotient manifold exists (without assuming that R is a submanifold, there is at most one manifold structure on the quotient such that the projection $X \to X/R$ is a submersion).

It is easy to see that any compact open subset of $\mathbb{Q}_p^n$ is a disjoint union of balls, which are isomorphic to $\mathbb{Z}_p^n$. With this observation and a little argument, one may deduce the non-trivial direction in the following theorem, which is the analogue of the existence of partitions of unity in the Archimedean setting, except much stronger.

Theorem B.2. *Let X be a p-adic manifold. Assume that X is Hausdorff. The following are equivalent.*

(1) *X is paracompact.*

(2) *X is isomorphic to a disjoint union of balls, i.e. p-adic manifolds isomorphic to $\mathbb{Z}_p^n$ for some n.*

Moreover in (2) the balls can be chosen to refine any given cover of X by open subsets.

Proof. See [Ser06, Part II, Chapter III, Appendix 2, Theorem 1]. □

Recall that partitions of unity are the essential technical tool to setup the theory of integration of top degree differential forms on real manifolds. We want to mimic this theory in the non-Archimedean setting, but we are interested in integrating complex-valued functions, whereas differential forms have p-adic coefficients. For this we will consider the "norm" of differential forms, and so use complex-valued partitions of unity. By Theorem B.2 for a Hausdorff paracompact p-adic manifolds we have (complex-valued) partitions of unity adapted to any given open cover, consisting of characteristic functions of compact open subsets.

Give $\mathbb{Q}_p$ the Haar measure such that $\mathrm{vol}(\mathbb{Z}_p) = 1$, and give $\mathbb{Q}_p^n$ the product measure, that we denote $|dx_1| \dots |dx_n|$. If U is an open subset of $\mathbb{Q}_p^n$ and $\omega \in \Omega^n(U)$, which can be written uniquely as $\psi(x) dx_1 \wedge \dots \wedge dx_n$ where ψ is a locally analytic function, then we may consider the Radon measure $|\omega|$: for any continuous compactly supported continuous function $f : U \to \mathbb{C}$,

$$\int_U f|\omega| := \int_U f(x)|\psi(x)| \, |dx_1| \dots |dx_n|.$$

We want to globalize this notion, to define $\int_X f|\omega|$ for any (nice enough) p-adic manifold X of dimension n, $\omega \in \Omega^n(X)$ and $f \in C_c(X)$.

Lemma B.3. *Let $g \in \mathrm{GL}_n(\mathbb{Q}_p)$, then $\mathrm{vol}(g(p^a\mathbb{Z}_p^n)) = |\det g| p^{-an}$.*

Proof. We only sketch the proof. Use the Iwasawa decomposition for $\mathrm{GL}_n(\mathbb{Q}_p)$ to reduce to the case where g is upper triangular, then use Fubini to compute the volume. The invariance by translation of the Haar measure on $\mathbb{Q}_p$ implies that the volume only depends on the diagonal of g. The case $n = 1$ is elementary. □

Lemma B.4. *Let ω be a locally analytic differential form of degree n on $\mathbb{Z}_p^n$. Let $a \in \mathbb{Z}$, and let $(\phi_i : p^a\mathbb{Z}_p^n \to \mathbb{Z}_p^n)_{i \in I}$ be a decomposition of $\mathbb{Z}_p^n$ into balls (i.e. each ϕ_i is injective, locally analytic with everywhere invertible differential, and $\mathbb{Z}_p^n = \bigsqcup_{i \in I} \phi_i(p^a\mathbb{Z}_p^n)$). For any continuous function $f : \mathbb{Z}_p^n \to \mathbb{C}$ we have*

$$\int_{\mathbb{Z}_p^n} f(x)|\omega| = \sum_i \int_{p^a\mathbb{Z}_p^n} (f \circ \phi_i)|\phi_i^* \omega|. \tag{B.1}$$

Proof. Write $\omega = \psi(x) dx_1 \wedge \dots \wedge dx_n$ where ψ is a locally function on $\mathbb{Z}_p^n$. Replacing f by $f|\psi|$, we can assume that we have $\omega = dx_1 \wedge \dots \wedge dx_n$. By density of $C^\infty(\mathbb{Z}_p^n)$ in $C(\mathbb{Z}_p^n)$ we may assume that f is smooth. Note that I is finite because $\mathbb{Z}_p^n$ is compact. For any $i \in I$ and $r \in p^a\mathbb{Z}_p^n$ there exists $b \geq a$ such that we have:

- f is constant on $\phi_i(r + p^b\mathbb{Z}_p^n)$,
- decomposing

$$\begin{aligned} \phi'_{i,r} : p^b\mathbb{Z}_p^n &\longrightarrow \mathbb{Z}_p^n \\ x &\longmapsto \phi_i(r + x) \end{aligned}$$

as $\phi_i(r) + (d\phi_i)_r \circ \phi''_{i,r}$, the function $\phi''_{i,r}$ is given by a power series

$$\phi''_{i,r}(x) = x + \sum_{\substack{\alpha \\ |\alpha| \geq 2}} a_\alpha x^\alpha$$

with $a_\alpha \in p^{(1-b)(|\alpha|-1)}\mathbb{Z}_p^n$. As we saw in the proof of Theorem B.1, this implies that $\phi''_{i,r}$ is an isomorphism $p^b\mathbb{Z}_p^n \to p^b\mathbb{Z}_p^n$. This also easily implies $(d\phi''_{i,r})_x \in 1 + pM_n(\mathbb{Z}_p)$ for any $x \in p^b\mathbb{Z}_p^n$. Moreover these two properties continue to hold at any element of $r + p^b\mathbb{Z}_p^n$.

By a compactness argument, for $b \geq a$ large enough, for any set R of representatives for the quotient of $p^a\mathbb{Z}_p^n$ by $p^b\mathbb{Z}_p^n$, the above properties hold for any $i \in I$ and $r \in R$. By linearity we may replace $(\phi_i)_{i\in I}$ by $(\phi'_{i,r})_{i\in I, r\in R}$. For any $x \in p^b\mathbb{Z}_p^n$ we have $(\phi''_{i,r})^*(dx_1 \wedge \cdots \wedge dx_n)_x = \psi(x) dx_1 \wedge \cdots \wedge dx_n$ with

$$|\psi(x)| = |\det(d\phi''_{i,r})(x)| = 1.$$

From this equality and the previous lemma applied to $(d\phi_i)_r$ we deduce the formula

$$\mathrm{vol}(\phi'_{i,r}(p^b\mathbb{Z}_p^n)) = \int_{p^b\mathbb{Z}_p^n} |(\phi'_{i,r})^*(dx_1 \wedge \cdots \wedge dx_n)|. \tag{B.2}$$

Multiplying by $f(\phi_i(r))$ and summing over all $(i,r) \in I \times R$ yields Formula (B.1). $\square$

Proposition B.5. *Let X be a Hausdorff and paracompact p-adic manifold of constant dimension n. Let ω be a differential form of degree n on X. Let $(\phi_i : \mathbb{Z}_p^n \to X)_i$ and $(\phi'_j : \mathbb{Z}_p^n \to X)_j$ be two decompositions of X into balls. The two Radon measures $C_c(X) \to \mathbb{C}$,*

$$f \longmapsto \sum_i \int_{\mathbb{Z}_p^n} (f \circ \phi_i)|\phi_i^*\omega| \qquad \text{and}$$
$$f \longmapsto \sum_j \int_{\mathbb{Z}_p^n} (f \circ \phi'_j)|(\phi'_j)^*\omega|$$

are equal.

Proof. Decompose each $\phi_i(\mathbb{Z}_p^n) \cap \phi'_j(\mathbb{Z}_p^n)$ into (finitely many) balls $(\phi''_{i,j,k} : \mathbb{Z}_p^n \to \phi_i(\mathbb{Z}_p^n) \cap \phi'_j(\mathbb{Z}_p^n))_k$ and apply the previous lemma to the decompositions $(\phi_i^{-1} \circ \phi''_{i,j,k})_{j,k}$ and $((\phi'_j)^{-1} \circ \phi''_{i,j,k})_{i,k}$. $\square$

Definition B.6. *Let X be a Hausdorff and paracompact p-adic manifold of constant dimension n. Let ω be a differential form of degree n on X. Recall that decompositions of X into balls exist by Theorem B.2. The Radon measure in the previous proposition will be denoted $f \mapsto \int_X f|\omega|$.*

Essentially the same argument as in the proof of Proposition B.5 is used to prove the following "change of variables integration formula".

Theorem B.7. *Let $\phi : X \to Y$ be a morphism between Hausdorff and paracompact p-adic manifolds of constant dimensions such that the differential of ϕ is everywhere invertible (in particular, $\dim X = \dim Y$). Assume that the fibers of ϕ have bounded cardinality, and denote $c_\phi : Y \to \mathbb{Z}_{\geq 0}$, $y \mapsto \mathrm{card}(\phi^{-1}(\{y\})$. Then for any differential form ω on Y and any function $f : Y \to \mathbb{C}$ that is integrable with respect to $|\omega|$, we have*

$$\int_X f \circ \phi \; |\phi^*\omega| = \int_Y f c_\phi \; |\omega|.$$

Proof. Left as an exercise, using the density of $C_c(Y)$ in $L^1(Y, |\omega|)$. $\square$

Example B.8. *On $G = \mathrm{GL}_2(\mathbb{Q}_p)$, denoting $x = (x_{i,j})_{1 \leq i,j \leq 2} \in G$, the differential form $\omega = \det(x)^{-2} \bigwedge_{1 \leq i,j \leq 2} x_{i,j}$ (choose an arbitrary order to take wedges) is both left- and right-invariant (exercise). This gives a "differential" definition of the Haar measure on G, and shows that it is unimodular.*

Appendix C. PGL_2 and SO_3

Let B be a commutative ring with unity, and denote $S = \operatorname{Spec} B$. (The assumption that S is affine is not necessary for what follows, but it simplifies the formulation.) Recall that a scheme over S (also called a scheme over B) is a scheme X together with a morphism of schemes $X \to S$. A scheme X over S has an associated functor

$$\begin{aligned} F_X : B-\mathrm{Alg} &\longrightarrow \mathrm{Sets} \\ R &\longmapsto X(R) := \mathrm{Hom}_{\mathrm{Sch}/B}(\operatorname{Spec} R, X) \end{aligned}$$

where $B - \mathrm{Alg}$ denotes the category of commutative algebras over B (with unity) and Sets is the category of sets.

Lemma C.1. *Let X be a scheme over S. The functor F_X associated to X is a sheaf on the big affine Zariski site of S, i.e. for any commutative B-algebra R and any family $(f_1, \dots, f_n)$ of elements of R generating its unit ideal, the map*

$$X(R) \to \prod_{i=1}^{n} X(R_{f_i})$$

identifies $X(R)$ with the set of $(s_i)_{1 \leq i \leq n}$ such that for any $1 \leq i < j \leq n$, the images of s_i and s_j in $X(R_{f_i f_j})$ coincide.

Proof. Exercise. □

A morphism $\phi : X \to Y$ of schemes over S induces a natural transformation $F_\phi : F_X \to F_Y$.

Lemma C.2. *Let X and Y be schemes over S. The map $\phi \mapsto F_\phi$ is a bijection between the set of morphism of schemes $X \to Y$ and the set of natural transformations $F_X \to F_Y$.*

Proof. If X is affine, say $X \simeq \operatorname{Spec} A$, this is formal using the point in $X(A)$ corresponding to the identity $A \to A$ (same proof as for Yoneda's lemma).

In general there exists an open cover $X = \bigcup_i U_i$ where each U_i is affine. A morphism $\phi : X \to Y$ amounts to a collection of morphisms $(\phi_i : U_i \to Y)_i$ such that for any indices i, j the restrictions $\phi_i|_{U_i \cap U_j}$ and $\phi_j|_{U_i \cap U_j}$ coincide. Using the fact that each $U_i \cap U_j$ can also be covered by affine open subschemes, one can reduce to the previous case. Details are left to the reader. □

These two lemmas tell us that F defines a fully faithful functor from the category of schemes over S to the category of sheaves on the big affine Zariski site of S. This functorial point of view on schemes has one advantage: one gets morphisms of schemes "for free" from the previous lemma. This is especially convenient for algebraic groups. Recall that an algebraic group over S is a scheme G over S together with morphisms of schemes $e_G : S \to G$ (a section of $G \to S$), $m_G : G \times_S G \to G$ and $i_G : G \to G$, such that the usual diagrams are commutative (expressing the facts that e_G is neutral on both sides for m_G, that i_G is "inversion" with respect to m_G, and that m_G is associative). Thanks to the above fully faithful embedding, this amounts to a functor $B-\mathrm{Alg}\to\mathrm{Groups}$ such that the composition with the forgetful functor $\mathrm{Groups}\to\mathrm{Sets}$ is representable by a scheme. For example for $n\geq 1$ the algebraic group GL_n over $\mathbb{Z}$ is defined as the functor

$$\begin{aligned}\mathbb{Z}-\mathrm{Alg} &\longrightarrow \mathrm{Groups}\\ R &\longmapsto \mathrm{GL}_n(R).\end{aligned}$$

This functor is representable by the scheme $\operatorname{Spec} A$, where

$$A=\mathbb{Z}[Y,X_{i,j},1\leq i,j\leq n]/(Y\det((X_{i,j})_{i,j})-1).$$

Our next goal is to define the algebraic group PGL_n over $\mathbb{Z}$, which intuitively should be the quotient of GL_n by its center GL_1 (diagonal matrices). Unfortunately the functor $R\mapsto \mathrm{GL}_n(R)/R^\times$ is not representable, in fact it is not even a sheaf. The $\mathbb{Z}$-algebra A has a natural grading obtained from the grading on $\mathbb{Z}[Y,X_{i,j},1\leq i,j\leq n]$ for which $\deg X_{i,j}=1$ and $\deg Y=-n$. Let A^0 be the degree zero subalgebra, and define $\mathrm{PGL}_n=\operatorname{Spec} A^0$. We could easily deduce the multiplication, neutral and inversion maps for PGL_n from those for GL_n to define a group scheme structure on GL_n (and even write these explicitly), but this will ultimately not be necessary thanks to the functorial point of view. Denote $\zeta:\mathrm{GL}_1\to\mathrm{GL}_n$ the morphism of schemes defined functorially as

$$\begin{aligned}\mathrm{GL}_1(R)=R^\times &\longmapsto \mathrm{GL}_n(R)\\ x &\longmapsto \mathrm{diag}(x,\dots,x).\end{aligned}$$

The morphism (of schemes) ζ is clearly a morphism of group schemes and a closed immersion.

Lemma C.3. *Let R be a commutative ring. Denote $r:\mathrm{GL}_n(R)\to\mathrm{PGL}_n(R)$.*

(1) *If R is local then r is surjective.*

(2) *For any $g\in\mathrm{GL}_n(R)$ we have $r^{-1}(\{r(g)\})=g\zeta(R^\times)$.*

Proof. (1) Let $\phi:A^0\to R$ be a morphism of rings, i.e. an element of $\mathrm{PGL}_n(R)$. We claim that there exists $1\leq i,j\leq n$ such that $\phi(X_{i,j}^nY)$ is invertible. For m a positive integer we write

$$1=\det((X_{i,j})_{i,j})^mY^m \text{ in } A^0.$$

For $m>>0$ each monomial in the expansion of $\det((X_{i,j})_{i,j})^m$ is divisible by $X_{i,j}^n$ for some pair (i,j). This proves the claim because R is local (a sum of non-invertible elements is not invertible). So we fix a pair (i,j) such that $\phi(X_{i,j}^nY)$ is invertible. For $1\leq k,l\leq n$ define

$$x_{k,l}=\frac{\phi(X_{i,j}^{n-1}X_{k,l}Y)}{\phi(X_{i,j}^nY)}.$$

In particular $x_{i,j}=1$. We have

$$\det((x_{k,l})_{k,l})=\frac{\phi(X_{i,j}^{n(n-1)}Y^n\det((X_{k,l})_{k,l}))}{\phi(X_{i,j}^{n^2}Y^n)}=\phi(X_{i,j}^nY)^{-1}\in R^\times$$

and so $(x_{k,l})_{k,l}$ defines an element of $\mathrm{GL}_n(R)$ that we denote $\phi' : A \to R$, which maps $X_{k,l}$ to $x_{k,l}$ and Y to $\phi(X_{i,j}^n Y)$. It remains to check that the restriction of ϕ' to A^0 is equal to ϕ. It is clear that ϕ and ϕ' map Y and the $X_{i,j}^{n-1} X_{k,l} Y$ to the same values in R. Let $f \in A^0$. For $m >> 0$ there exists a polynomial P in n^2 variables with coefficients in $\mathbb{Z}$ such that we have

$$f \times (X_{i,j}^n Y)^m = P((X_{i,j}^{n-1} X_{k,l} Y)_{k,l}).$$

Applying ϕ and ϕ' to this equality, we obtain $\phi(f) = \phi'(f)$.

(2) The inclusion $g\zeta(R^\times) \subset r^{-1}(\{r(g)\})$ is easy and left to the reader.

To prove the other inclusion we first assume that R is local in order to use the previous construction. Let $g_1, g_2 \in \mathrm{GL}_n(R)$. Denote $\phi_1, \phi_2 : A \to R$ the corresponding morphisms of rings. Assume that we have $r(g_1) = r(g_2)$, i.e. that ϕ_1 and ϕ_2 have equal restriction ϕ to A^0. As above we may choose a pair (i,j) such that $\phi(X_{i,j}^n Y)$ is invertible in R, and construct $\phi' : A \to R$ extending ϕ, corresponding to $g \in \mathrm{GL}_n(R)$. It is clear on the definition that for $k \in \{1,2\}$ we have $g_k = \zeta(\phi_k(x_{i,j}))g$ in $\mathrm{GL}_n(R)$. In particular $g_1 g_2^{-1}$ belongs to $\zeta(R^\times)$. This concludes the proof under the assumption that R is local.

For arbitrary R, let $g_1, g_2 \in \mathrm{GL}_n(R)$ mapping to the same element of $\mathrm{PGL}_n(R)$. We know that for any maximal ideal $\mathfrak{m}$ of R, the image of $g_1 g_2^{-1}$ in $\mathrm{GL}_n(R_\mathfrak{m})$ belongs to $\zeta(R_\mathfrak{m}^\times)$. Since ζ is a closed immersion this implies $g_1 g_2^{-1} \in \zeta(R^\times)$. □

Corollary C.4. *The scheme* PGL_n*, seen as a sheaf on the big affine Zariski site of* $\mathbb{Z}$*, is the sheafification of the functor (presheaf)*

$$\begin{aligned} \mathbb{Z} - \mathrm{Alg} &\longrightarrow \mathrm{Sets} \\ R &\longmapsto \mathrm{GL}_n(R)/R^\times \end{aligned}$$

In particular PGL_n *is naturally endowed with a group scheme structure.*

Concretely this means that an element of $\mathrm{PGL}_n(R)$ is given by two families $(f_1, \dots, f_k)$ and $(g_1, \dots, g_k)$, where $f_i \in R$ and $g_i \in \mathrm{GL}_n(R_{f_i})$, satisfying:

- $(f_1, \dots, f_k) = R$,
- for all $1 \le i < j \le k$ the images of g_i and g_j in $\mathrm{GL}_n(R_{f_i f_j})/R_{f_i f_j}^\times$ coincide.

(Of course such a representation of an element of $\mathrm{PGL}_n(R)$ is not unique, and writing down the equivalence relation is left to the reader.)

For any commutative ring R we have an exact sequence

$$1 \to R^\times \to \mathrm{GL}_n(R) \to \mathrm{PGL}_n(R) \to H^1_{\mathrm{Zar}}(R, \mathrm{GL}_1) \to H^1_{\mathrm{Zar}}(R, \mathrm{GL}_n)$$

where H^1_{Zar} is Čech cohomology for the Zariski topology. The group $H^1_{\mathrm{Zar}}(R, \mathrm{GL}_1)$ is naturally isomorphic to the Picard group $\mathrm{Pic}(R)$ of R, i.e. the group of isomorphism classes of line bundles on $\mathrm{Spec}\, R$ (equivalently, finitely generated projective

R-modules of constant rank 1). Similarly the set $H^1_{\mathrm{Zar}}(R, \mathrm{GL}_n)$ parametrizes the isomorphism classes of vector bundles of rank n on $\operatorname{Spec} R$ (idem with rank n).

Now we consider more closely the case where $n = 2$, and relate PGL_2 to the special orthogonal group in 3 variables. Consider the quadratic form on $V := \mathbb{Z}^3$:

$$q : (x, y, z) \longmapsto x^2 + yz.$$

Note that this definition is universal, i.e. it makes sense with $\mathbb{Z}$ replaced by an arbitrary commutative ring. In fact q comes from a unique element of $\mathrm{Sym}^2 \mathrm{Hom}_{\mathbb{Z}}(V, \mathbb{Z})$, where the symmetric product is defined as a quotient of the tensor product. For any commutative ring R we have a natural identification

$$R \otimes_{\mathbb{Z}} \mathrm{Sym}^2 \mathrm{Hom}_{\mathbb{Z}}(V, \mathbb{Z}) \simeq \mathrm{Sym}^2 \mathrm{Hom}_R(R \otimes_{\mathbb{Z}} V, R).$$

Denote (e_1, e_2, e_3) the standard basis of V. Let B_q be the symmetric bilinear form on V associated to q:

$$B_q(v, w) := q(v + w) - q(v) - q(w).$$

We have

$$(B_q(e_i, e_j))_{1 \le i,j \le 3} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Let $C(q)$ be the Clifford algebra associated to q, i.e. the quotient of the tensor algebra of V by the bilateral ideal generated by $v \otimes v - q(v)$ for $v \in V$. The tensor algebra of V is graded (by non-negative integers), inducing a $\mathbb{Z}/2\mathbb{Z}$ grading on $C(q)$. Denote $C(q)^+$ (resp. $C(q)^-$) the even (resp. odd) part. The Clifford algebra, like the exterior algebra, also inherits a filtration from the grading on the tensor algebra. The graded pieces of these filtrations are naturally isomorphic, and so lifting a basis of the tensor algebra gives us a basis of the Clifford algebra. For example

$$\begin{aligned} C(q)^+ &= \mathbb{Z} \oplus \mathbb{Z}e_1e_2 \oplus \mathbb{Z}e_1e_3 \oplus \mathbb{Z}e_2e_3 \quad \text{and} \\ C(q)^- &= \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3 \oplus \mathbb{Z}e_1e_2e_3. \end{aligned}$$

In particular the canonical map $V \to C(q)^-$ realizes V as a factor of $C(q)^-$. We have $e_1^2 = 1$, $e_2^2 = e_3^2 = 0$, $e_2e_1 = -e_1e_2$, $e_3e_1 = -e_1e_3$ and $e_3e_2 = 1 - e_2e_3$. We have an anti-automorphism $x \mapsto x^*$ of $C(q)$, mapping $v_1 \otimes \cdots \otimes v_k$ to $v_k \otimes \cdots \otimes v_1$.

Lemma C.5. (1) *We have an isomorphism of (non-commutative) $\mathbb{Z}$ algebras*

$$\begin{aligned} \alpha : C(q)^+ &\longrightarrow M_2(\mathbb{Z}) \\ e_1e_2 &\longmapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ e_2e_3 &\longmapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ e_1e_3 &\longmapsto \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \end{aligned}$$

For any $x \in C(q)^+$ we have $\alpha(x^*) = \alpha(x)^*$, where for $y \in M_2(\mathbb{Z})$ we also denote $y^* = \operatorname{tr} y - y$.

(2) *The element* $\lambda = 2e_1e_2e_3 - e_1$ *of* $C(q)^-$ *is central in* $C(q)$ *and satisfies* $\lambda^2 = 1$ *and* $\lambda^* = -\lambda$. *In particular we have* $\mathbb{Z}[\lambda] = \mathbb{Z} \oplus \mathbb{Z}\lambda$.

(3) *The multiplication map*

$$\mathbb{Z}[\lambda] \otimes_{\mathbb{Z}} C(q)^+ \longrightarrow C(q)$$

is an isomorphism of $\mathbb{Z}$*-algebras.*

Proof. (1) The existence of α simply follows from the computation of the multiplication table of $C(q)^+$. The fact that α is surjective is clear, and injectivity follows from the equality of ranks as finite free $\mathbb{Z}$-modules.

(2) Computation left to the reader.

(3) The fact that this is surjective is a simple computation:

$$e_1 = \lambda(2e_2e_3 - 1),\ e_2 = \lambda e_1 e_2,\ e_3 = -\lambda e_1 e_3,\ e_1e_2e_3 = \lambda e_2 e_3.$$

The fact that it is an isomorphism follows because it is a morphism between finite free $\mathbb{Z}$-modules of equal ranks.

□

Remark C.6. *This is a special case of a general structure theorem for Clifford algebra, see* [Bou07, §9 n.4] *for the general result over a field (the proof can be adapted to work over an arbitrary ring using the "right" definitions).*

The first point implies that for any commutative ring R the map

$$\begin{aligned} R \otimes_{\mathbb{Z}} C(q)^+ &\longrightarrow R \otimes_{\mathbb{Z}} C(q)^+ \\ x &\longmapsto xx^* \end{aligned}$$

takes values in R because $\alpha(xx^*) = \alpha(x)\alpha(x)^* = \det \alpha(x)$. This also shows that $\alpha^{-1}(\mathrm{GL}_2(R)) = (R \otimes_{\mathbb{Z}} C(q))^\times$ is the group of $x \in R \otimes_{\mathbb{Z}} C(q)^+$ satisfying $xx^* \in R^\times$.

The third point of the lemma elucidates the structure of $C(q)^-$ as a bi-$C(q)^+$-module: we have an isomorphism

$$\begin{aligned} \beta : C(q)^- &\longrightarrow M_2(\mathbb{Z}) \\ \lambda y &\longmapsto \alpha(y) \end{aligned}$$

which satisfies $\beta(x_1 y x_2) = \alpha(x_1)\beta(y)\alpha(x_2)$ for any $x_1, x_2 \in C(q)^+$ and $y \in C(q)^-$. A simple computation shows that we have

$$\beta(V) = \{X \in M_2(\mathbb{Z}) \mid \operatorname{tr} X = 0\}.$$

In particular for any commutative ring R the sub-R-module $R \otimes_{\mathbb{Z}} V$ of $R \otimes_{\mathbb{Z}} C(q)^-$ is stable under conjugation by $(R \otimes_{\mathbb{Z}} C(q)^+)^\times = \alpha^{-1}(\mathrm{GL}_2(R))$. We obtain a morphism

$$(R \otimes_{\mathbb{Z}} C(q)^+)^\times \longrightarrow \mathrm{GL}(R \otimes_{\mathbb{Z}} V)$$

and one can check that it takes values in $\mathrm{SL}(R \otimes_{\mathbb{Z}} V)$ (more generally, it is well-known that the conjugacy action of GL_n on M_n factors through the special linear group). This conjugation action preserves additional structure. For $y \in R \otimes_{\mathbb{Z}} C(q)^-$ we can write $y = \lambda x$ with $x \in C(q)^+$ and we have

$$yy^* = -xx^* = -\det \alpha(x) = -\det \beta(y).$$

In particular for $y \in R \otimes_{\mathbb{Z}} V$ we have

$$q(y) = y^2 = yy^* = -\det \beta(y).$$

Of course the quadratic form $\det \circ \beta$ on $R \otimes_{\mathbb{Z}} V$ (and on $R \otimes C(q)^-$) is preserved by the conjugation action of $(R \otimes_{\mathbb{Z}} C(q)^+)^\times$. Let SO_3 be the algebraic group over $\mathbb{Z}$ defined as the functor

$$R \mapsto \{g \in \mathrm{SL}(R \otimes_{\mathbb{Z}} V) \,|\, q \circ g = q\}.$$

We have just seen that the conjugation action (inside the Clifford algebra) gives us a morphism

$$(R \otimes_{\mathbb{Z}} C(q)^+)^\times \longrightarrow \mathrm{SO}_3(R)$$

which is clearly functorial in R. Composing with $\alpha^{-1}|_{\mathrm{GL}_2}$ gives us a morphism of group schemes $\mathrm{GL}_2 \to \mathrm{SO}_3$ with kernel GL_1, so it factors to give an injective morphism $\pi : \mathrm{PGL}_2 \to \mathrm{SO}_3$.

Lemma C.7. *The morphism $\pi : \mathrm{PGL}_2 \to \mathrm{SO}_3$ is an isomorphism.*

Proof. It remains to check surjectivity, i.e. that the initial morphism $\mathrm{GL}_2 \to \mathrm{SO}_3$ is surjective (as sheaves on the big affine Zariski site of $\mathbb{Z}$). Equivalently, we have to show that for any commutative local ring R the morphism $\mathrm{GL}_2(R) \to \mathrm{SO}_3(R)$ is surjective. Let $\mathfrak{m}$ be the maximal ideal of R. First we check that $\mathrm{GL}_2(R)$ acts transitively on the set of $v \in R \otimes_{\mathbb{Z}} V$ which map to a non-zero vector in $R/\mathfrak{m} \otimes_{\mathbb{Z}} V$ and satisfy $q(v) = 0$ in R, or equivalently $v^2 = 0$ in $R \otimes_{\mathbb{Z}} C(q)$. Write $v = \lambda x$ where $x \in R \otimes_{\mathbb{Z}} C(q)^+$. The matrix $\alpha(x) = \beta(v) \in M_2(R)$ maps to a non-zero matrix $\overline{\beta(v)}$ in $M_2(R/\mathfrak{m})$ and satisfies $\beta(v)^2 = 0$. There exists $\overline{f_2} \in (R/\mathfrak{m})^2$ such that $\overline{\beta(v)}\,\overline{f_2} \neq 0$ (in $(R/\mathfrak{m})^2$). Let $f_2 \in R^2$ be any lift of $\overline{f_2}$ and let $f_1 = \beta(v) f_2$. The family $(\overline{f_1}, \overline{f_2})$ is a basis of $(R/\mathfrak{m})^2$ so by Nakayama's lemma (f_1, f_2) is a basis of R^2. The matrix of this basis gives $g \in \mathrm{GL}_2(R)$ such that

$$\pi(g) v = \beta^{-1}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \lambda e_1 e_2 = e_2 \text{ in } R \otimes_{\mathbb{Z}} V \subset R \otimes_{\mathbb{Z}} C(q)^-.$$

It remains to show that the stabilizer of e_2 in $\mathrm{GL}_2(R)$ surjects onto the stabilizer of e_2 in $\mathrm{SO}_3(R)$. Consider $h \in \mathrm{SO}_3(R)$ fixing e_2. Then h stabilizes the orthogonal of e_2 for B_q, which is $e_2^\perp = Re_1 \oplus Re_2$, and h acts as the identity on $R \otimes_{\mathbb{Z}} V/e_2^\perp$. So in the basis (e_2, e_1, e_3) of $R \otimes_{\mathbb{Z}} V$ the matrix of h takes the form

$$\begin{pmatrix} 1 & a & c \\ 0 & d & b \\ 0 & 0 & 1 \end{pmatrix}.$$

We have $d = 1$ because $\det h = 1$. A simple computation shows that we have $a = -2b$ and $c = -b^2$. Another simple computation shows that we have

$$h = \pi\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right).$$

∎

Of course the use of Clifford algebras is not necessary to state and prove this result: we could have simply considered the conjugation action of GL_2 on the vector bundle

$$\mathfrak{sl}_2 : R \mapsto \{X \in M_2(R) \mid \operatorname{tr} X = 0\},$$

observed that this action factors through the algebraic subgroup $\mathrm{SL}(\mathfrak{sl}_2)$ of $\mathrm{GL}(\mathfrak{sl}_2)$ and preserves the quadratic form given by the determinant on $\mathfrak{sl}_2$, and proved that the resulting morphism $\mathrm{GL}_2 \to \mathrm{SO}_3$ is surjective (as a morphism of sheaves on the big affine Zariski site of $\mathbb{Z}$) and has kernel GL_1. For this it is not even necessary to define PGL_2 first. However, the arguments above generalize in higher dimension, to define (general) spin groups and prove that there is a surjective morphism $\mathrm{GSpin}_{2n+1} \to \mathrm{SO}_{2n+1}$ with kernel GL_1 for any n (although the proof of Lemma C.7 is particular to the $n = 1$ case). More importantly for us, the fact that the constructions above are universal, in the sense that they are multi-linear and compatible with any extension of scalars $\mathbb{Z} \to R$, means that they can be generalized without much effort to *forms* (in the sense of [Ser94, §III.1]) of the algebra M_2 and the quadratic space $(\mathfrak{sl}_2, \det)$.

Appendix D. Quaternion algebras an quadratic forms in dimension 3

For simplicity we work over a field K of characteristic zero. Most of what follows could be generalized over an arbitrary scheme instead of $\operatorname{Spec} K$. A quaternion algebra over K is a (non-commutative) K-algebra D such that there exists a finite extension K'/K and an isomorphism $K' \otimes_K D \simeq M_2(K')$. If D itself is isomorphic to $M_2(K)$ then we say that D is *split*. Recall that the group of automorphisms of the K'-algebra $M_2(K')$ is $\mathrm{PGL}_2(K')$ (via the adjoint action): to prove this, consider the idempotents $e = \mathrm{diag}(1,0)$ and $f = \mathrm{diag}(0,1)$ which satisfy $ef = fe = 0$, and show that any pairs of non-zero idempotents satisfying this relation is conjugated to (e, f).

Proposition D.1. *Fix an algebraic closure $\overline{K}$ of K. The pointed set $H^1(K, \mathrm{PGL}_2) = \varinjlim_{K'} H^1(\mathrm{Gal}(K'/K), \mathrm{PGL}_2(K'))$, where the direct limit is over the finite Galois extensions of K in K_s and the transition maps are given by inflation and are injective, parametrizes quaternion algebras over K up to isomorphism, by associating to $c \in Z^1(\mathrm{Gal}(K'/K), \mathrm{PGL}_2(K'))$ the algebra*

$$D = \{X \in M_2(K') \,|\, \forall \sigma \in \mathrm{Gal}(K'/K),\ \mathrm{Ad}(c(\sigma))(\sigma(X)) = X\}. \tag{D.1}$$

Proof. To check that (D.1) defines a quaternion algebra, it is enough to check that the natural map $K' \otimes_K D \to M_2(K')$ is an isomorphism. This follows from Hilbert's theorem 90 (see [Ser68, Ch. X Proposition 3]), seeing $\mathrm{Ad}(c)$ as an element of $Z^1(\mathrm{Gal}(K'/K), \mathrm{GL}(M_2(K')))$.

Conversely, let D be a quaternion algebra over K. There exists a finite extension K'/K and an isomorphism $\psi : M_2(K') \simeq K' \otimes_K D$. We may assume that K' is a subextension of $\overline{K}$ and that it is Galois. For $\sigma \in \mathrm{Gal}(K'/K)$, σ acts on $K' \otimes_K D$ (in the natural way on K' and trivially on D), and $c(\sigma) := \psi^{-1} \circ \sigma \circ \psi \circ \sigma^{-1}$ is an automorphism of the K'-algebra $M_2(K')$, i.e. an element of $\mathrm{PGL}_2(K')$. We obtain a 1-cocycle $c : \mathrm{Gal}(K'/K) \to \mathrm{PGL}_2(K')$, and it is easy to check that a different choice of ψ amounts to taking another representative in $H^1(\mathrm{Gal}(K'/K), \mathrm{PGL}_2(K'))$. Compatibility with inflation (taking a larger K') is formal. The fact that the two constructions are inverse of each other is left to the reader. $\square$

Remark D.2. *If D is a K-algebra such that there exists a field extension K'/K (not assumed to be finite or even algebraic) for which $K' \otimes_K D$ is isomorphic to $M_2(K')$ then D is a quaternion algebra. See* [Ser94, §III.1 Proposition 2].

Because the trace and determinant maps on $M_2(K')$ are Galois-equivariant and invariant under conjugation, they descend to give trace and determinant maps on D taking values in K and $K^\times$.

Let D be a quaternion algebra over K. Choose a finite Galois extension K'/K and an isomorphism $\psi : M_2(K') \simeq K' \otimes_K D$. Because the trace and determinant maps on $M_2(K')$ are $\mathrm{Gal}(K'/K)$-equivariant and invariant under conjugation,

$$\mathrm{tr} \circ \psi^{-1} : K' \otimes_K D \to K' \text{ and } \det \circ \psi^{-1} : K' \otimes_K D \to K'$$

are also $\mathrm{Gal}(K'/K)$-equivariant. Taking $\mathrm{Gal}(K'/K)$-invariants, they restrict to maps $D \to K$ which are respectively linear and homogeneous polynomial of degree 2. It is easy to check that they do not depend on the choice of K' and ψ.

We still denote these maps tr and det and call them trace and determinant. The trace map gives us the conjugation map $D \mapsto D$, $x \mapsto x^* := \operatorname{tr} x - x$ which is an anti-automorphism of D. For any $x \in D$ we have $xx^* = x^*x = \det x$.

Lemma C.7 gives us an isomorphism

$$H^1(K, \mathrm{PGL}_2) \simeq H^1(K, \mathrm{SO}_3)$$

and an argument similar to the proof of Proposition D.1 shows that $H^1(K, \mathrm{SO}_3)$ parametrizes non-degenerate quadratic vector spaces over K of dimension 3 and discriminant -1. In fact the construction in the previous section extend to forms, and the correspondence between quaternion algebras and quadratic spaces admits a natural description:

- If D is a quaternion algebra over K, the 3-dimensional subspace
$$V = \{X \in D \mid \operatorname{tr} X = 0\}$$
is endowed with the non-degenerate quadratic form $q = -\det$, which has discriminant -1 because in the above interpretation using Galois cohomology, it is obtained from $(\mathfrak{sl}_2, -\det)$ by twisting using a cocycle taking values in SO_3 and not just O_3.
- If (V, q) is a non-degenerate quadratic vector space of dimension 3 and discriminant -1 then we can form the Clifford algebra $C(q) = C(q)^+ \oplus C(q)^-$ and define D as its even part $C(q)^+$. One can check that D is a quaternion algebra (again using the Galois cohomology interpretation allows one to reduce to the split case).

Lemma D.3. *Let D be a non-split quaternion algebra over K.*

(1) *Any non-zero element of D is invertible in D.*

(2) *Let $x \in D \smallsetminus K$. Then its characteristic polynomial, defined as $X^2 - (\operatorname{tr} x)X + \det x$, does not split, i.e. it does not have a root in K.*

(3) *For $x \in D \smallsetminus K$ the subalgebra $K[x]$ of D is a quadratic field extension of K.*

Proof. Let K' be a finite extension of K such that $K' \otimes_K D$ is isomorphic to $M_2(K')$.

(1) If $x \in D \smallsetminus \{0\}$ is not invertible then its image in $M_2(K')$ admits 0 as an eigenvalue.

If this image is semi-simple then $\operatorname{tr} x$ is the other eigenvalue so it does not vanish and we consider $y = 2x/(\operatorname{tr} x) - 1$. We have $\operatorname{tr} y = 0$ and $\det y = -1$, so the quadratic space (V, q) corresponding to D has a vector y satisfying $q(y) = 1$ and so it is split, a contradiction.

Otherwise we have $\operatorname{tr} x = 0$ and $\det x = 0$ so x itself defines a vector in V satisfying $q(x) = 0$ and so (V, q) is split, a contradiction.

(2) By the Cayley-Hamilton theorem applied in $K' \otimes_K D \simeq M_2(K')$, the characteristic polynomial of x kills x. So if this polynomial is equal to $(X-\alpha)(X-\beta)$ with $\alpha, \beta \in K$ then $x - \alpha$ or $x - \beta$ is not invertible in D, and so x is equal to α or β.

(3) This follows from the previous point.

□

Appendix E. Quadratic forms over $\mathbb{Q}_p$ in dimension ≤ 3

For $a, b \in \mathbb{Q}_p^\times$ define the Hilbert symbol

$$(a,b) = \begin{cases} 1 & \text{if there exists } x, y \in \mathbb{Q}_p \text{ such that } ax^2 + by^2 = 1, \\ -1 & \text{otherwise.} \end{cases}$$

It clearly factors through $(\mathbb{Q}_p^\times / \mathbb{Q}_p^{\times,\square})^2$, where $\mathbb{Q}_p^{\times,\square}$ is the subgroup of squares in $\mathbb{Q}_p^\times$. The Hilbert symbol can be computed explicitly (see [Ser77, Ch. III Théorème 1]) and on this explicit formula the following result is evident.

Theorem E.1 ([Ser77, Ch. III Théorème 2])**.** *The Hilbert symbol defines a non-degenerate bilinear form on the $\mathbb{F}_2$-vector space $\mathbb{Q}_p^\times / \mathbb{Q}_p^{\times,\square}$.*

Corollary E.2. *The map*

$$E \longmapsto N_{E/\mathbb{Q}_p}(E^\times)$$

defines a bijection between the set of isomorphism classes of quadratic extensions of $\mathbb{Q}_p$ and the set of index 2 subgroups of $\mathbb{Q}_p^\times$.

Proof. Recall that quadratic extensions of $\mathbb{Q}_p$ are parametrized by the non-trivial elements in $\mathbb{Q}_p^\times / \mathbb{Q}_p^{\times,\square}$, via $b \mapsto \mathbb{Q}_p(\sqrt{b})$. For $b \in \mathbb{Q}_p^\times \smallsetminus \mathbb{Q}_p^{\times,\square}$, $E = \mathbb{Q}_p(\sqrt{b})$ and $a \in \mathbb{Q}_p^\times$ it is easy to see that $a \in N_{E/\mathbb{Q}_p}(E^\times)$ if and only if $(a,b) = 1$. This shows that $N_{E/\mathbb{Q}_p}(E^\times)$ is a subgroup of $\mathbb{Q}_p^\times$ of index 2. Conversely any index 2 subgroup of $\mathbb{Q}_p^\times$ is the orthogonal for the Hilbert symbol of a unique $b \in \mathbb{Q}_p^\times / \mathbb{Q}_p^{\times,\square} \smallsetminus \{1\}$ (observe that any line in an $\mathbb{F}_2$-vector space contains a unique non-zero vector!). $\square$

Theorem E.3. *A non-degenerate two-dimensional quadratic space (V, q) over $\mathbb{Q}_p$ is isomorphic to exactly one of the following:*

- *if it has discriminant -1 (in $\mathbb{Q}_p^\times / \mathbb{Q}_p^{\times,\square}$), it is split, i.e. isomorphic to $\mathbb{Q}_p^2$ endowed with the quadratic form $(x,y) \mapsto xy$,*
- *if it has discriminant $-b$ with $b \in \mathbb{Q}_p^\times \smallsetminus \mathbb{Q}_p^{\times,\square}$, denoting $E = \mathbb{Q}_p(\sqrt{b})$ then (V,q) is either isomorphic to $(E, N_{E/\mathbb{Q}_p})$ or to $(E, \lambda N_{E/\mathbb{Q}_p})$ where $\lambda \in \mathbb{Q}_p^\times \smallsetminus N_{E/\mathbb{Q}_p}(E^\times)$ (the isomorphism class of this quadratic space does not depend on the choice of such a λ).*

In particular the isomorphism class of (V,q) is determined by

$$S := \{q(v) \,|\, v \in V \smallsetminus \{0\}\} :$$

- *if 0 belongs to S then (V,q) is split,*
- *otherwise the subset $SS^{-1} = \{xy^{-1} \,|\, x, y \in S\}$ of $\mathbb{Q}_p^\times$ is an index two subgroup of $\mathbb{Q}_p^\times$, which is equal to $N_{E/\mathbb{Q}_p}(E^\times)$ for a unique (up to isomorphism) quadratic extension E of $\mathbb{Q}_p$, and S is either $N_{E/\mathbb{Q}_p}(E^\times)$ or $\mathbb{Q}_p^\times \smallsetminus N_{E/\mathbb{Q}_p}(E^\times)$.*

Proof. The case of discriminant -1 is well-known and not particular to $\mathbb{Q}_p$. It is also well-known that if $0 \in S$ then (V,q) is split. Otherwise let $-b$ be the discriminant of (V,q) and take $\lambda \in S$, then (V,q) is isomorphic to

$$\left(\mathbb{Q}_p^2, (x,y) \mapsto \lambda x^2 - \lambda b y^2\right),$$

i.e. to $(E, \lambda N_{E/\mathbb{Q}_p})$. For any $\mu \in E^\times$ we have $(E, \lambda N_{E/\mathbb{Q}_p}) \simeq (E, \lambda N_{E/\mathbb{Q}_p}(\mu) N_{E/\mathbb{Q}_p})$. Details are left to the reader. $\square$

Theorem E.4. *There are up to isomorphism exactly two non-degenerate three-dimensional quadratic spaces (V, q) over $\mathbb{Q}_p$ of discriminant -1, distinguished by*

$$S = \{q(v) \,|\, v \in V \smallsetminus \{0\}\} :$$

- *The space $\mathbb{Q}_p^3$ with quadratic form*

$$(x, y, z) \longmapsto x^2 + yz.$$

  In this case we have $S = \mathbb{Q}_p$.

- *For a quadratic extension $E = \mathbb{Q}_p(\sqrt{b})$ of $\mathbb{Q}_p$, the space $\mathbb{Q}_p \oplus E$ with quadratic form $(x, t) \mapsto bx^2 + \lambda N_{E/\mathbb{Q}_p}(t)$ where $\lambda \in \mathbb{Q}_p^\times \smallsetminus N_{E/\mathbb{Q}_p}(E^\times)$. Up to isomorphism this three-dimensional quadratic space does not depend on the choice of E, and we have $S = \mathbb{Q}_p^\times \smallsetminus \mathbb{Q}_p^{\times,\square}$.*

Proof. Let (V, q) be a non-degenerate three-dimensional quadratic space over $\mathbb{Q}_p$, and S the corresponding set of values. If $1 \in S$ then (V, q) is isomorphic to the direct sum of $(\mathbb{Q}_p, x \mapsto x^2)$ and a two-dimensional non-degenerate quadratic space of discriminant -1, which is then split. If $0 \in S$ then (V, q) admits a split non-degenerate two-dimensional quadratic space as a factor and we reach the same conclusion.

Otherwise pick $b \in S$, in particular $b \in \mathbb{Q}_p^\times \smallsetminus \mathbb{Q}_p^{\times,\square}$. Denote $E = \mathbb{Q}_p(\sqrt{b})$ as usual. Then (V, q) is isomorphic to the direct sum of $(\mathbb{Q}_p, x \mapsto bx^2)$ and a two-dimensional non-degenerate quadratic space of discriminant $-b$, which is isomorphic to $(E, \lambda N_{E/\mathbb{Q}_p})$ with $\lambda \in \mathbb{Q}_p^\times$. If $\lambda \in N_{E/\mathbb{Q}_p}(E^\times)$ then $1 \in S$, a contradiction. So we have $\lambda \in \mathbb{Q}_p^\times \smallsetminus N_{E/\mathbb{Q}_p}(E^\times)$ and it remains to check that S is equal to $\mathbb{Q}_p^\times \smallsetminus \mathbb{Q}_p^{\times,\square}$. Let $a \in \mathbb{Q}_p^\times \smallsetminus \mathbb{Q}_p^{\times,\square}$. If $a \in b\mathbb{Q}_p^{\times,\square}$ then we already know that a belongs to S. Otherwise the quadratic form on $\mathbb{Q}_p^2$

$$(x, y) \longmapsto ax^2 - by^2$$

has discriminant $-b' = -ab$ which is neither equal to -1 nor to $-b$ modulo $\mathbb{Q}_p^{\times,\square}$, and so

$$S' := \{ax^2 - by^2 \,|\, (x, y) \in \mathbb{Q}_p^2 \smallsetminus \{(0, 0)\}\}$$

is a coset for $N_{F/\mathbb{Q}_p}(F^\times)$ where $F = \mathbb{Q}_p(\sqrt{b'})$ is not isomorphic to E. In particular S' is not contained in $N_{E/\mathbb{Q}_p}(E^\times)$, and so there exists $(x, y) \in \mathbb{Q}_p^2$ and $t \in E$ such that we have

$$ax^2 = by^2 + \lambda N_{E/\mathbb{Q}_p}(t).$$

We know that x cannot be equal to 0 (otherwise we would have $0 \in S$) and so dividing by x^2 we conclude $a \in S$. $\square$

Corollary E.5. *Up to isomorphism there is a unique non-split quaternion algebra D over $\mathbb{Q}_p$. For any quadratic extension E of $\mathbb{Q}_p$ there exists an embedding of $\mathbb{Q}_p$-algebras $E \to D$.*

Proof. The first statement follows directly from the previous theorem and the relation between quaternion algebras and quadratic spaces explained in the previous section.

Let D be a non-split quaternion algebra over $\mathbb{Q}_p$, and let E be a quadratic extension of $\mathbb{Q}_p$. Choose $x \in E \smallsetminus \mathbb{Q}_p$ such that $b := x^2$ belongs to $\mathbb{Q}_p$. In particular b is not a square in $\mathbb{Q}_p$ and $E = \mathbb{Q}_p(x)$. By the previous theorem there exists $y \in D$ such that $\operatorname{tr} y = 0$ and $\det y = -b$, i.e.

$$y^2 = -yy^* = b.$$

There is thus a unique morphism of $\mathbb{Q}_p$-algebras $E \to D$ mapping x to y. $\square$

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