THE JACQUET-LANGLANDS CORRESPONDENCE FOR $\text{GL}_2(\mathbb{Q}_p)$

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1. Introduction

Let $p$ be a prime number, $D$ a quaternion algebra over $\mathbb{Q}_p$. In these notes, all representations of groups are over the field $\mathbb{C}$ of complex numbers.

**Theorem 1.1 (Jacquet-Langlands).** For any continuous irreducible finite-dimensional representation $\sigma$ of $D^\times$, there is a unique essentially square-integrable irreducible smooth representation $\pi$ of $\text{GL}_2(\mathbb{Q}_p)$ such that for any $g \in D^\times \smallsetminus \mathbb{Q}_p^\times$ we have $\text{tr} \sigma(g) = -\Theta_\pi(g')$, where $g' \in \text{GL}_2(\mathbb{Q}_p)$ has the same trace and determinant as $g$ and $\Theta_\pi$ is the Harish-Chandra character of $\pi$. Moreover any $\pi$ corresponds to a unique $\sigma$.

We will explain later what “essentially square-integrable” means. Let us simply mention that all these representations of $\text{GL}_2(\mathbb{Q}_p)$ have infinite dimension. In fact we will classify representations of $\text{GL}_2(\mathbb{Q}_p)$ as follows: principal series (quite explicit), “special” (also quite explicit, and essentially square-integrable), and supercuspidal (also essentially square-integrable). We will define the Harish-Chandra character $\Theta_\pi$ even later, it plays the role of the trace function of $\pi$, but since we are considering infinite-dimensional representations, defining the trace is not obvious.

We will see that supercuspidal representations are the most well-behaved representations of $\text{GL}_2(\mathbb{Q}_p)$, i.e. that they behave much like representations of a compact group. Among irreducible smooth representations of $\text{GL}_2(\mathbb{Q})$, they are however the least explicit and most mysterious ones. One can see the Jacquet-Langlands correspondence as a classification of all supercuspidal representations $\pi$ of $\text{GL}_2(\mathbb{Q}_p)$ by seemingly simpler finite-dimensional representations of $D^\times$.

However, this is not the true motivation for this theorem. The theorem should be seen as a consequence of the Langlands correspondence for $\text{GL}_2(\mathbb{Q}_p)$ and $D^\times$. The following theorem is the most difficult part of this correspondence (we will prove the easier part concerning reducible Galois representations, in the first part of this course).

**Theorem 1.2 (Jacquet-Langlands, Gel’fand-Graev, Tunnell, Kutzko).** There is a “natural” bijection between isomorphism classes of irreducible representations of $D^\times$ of dimension $> 1$ (resp. irreducible supercuspidal representations of $\text{GL}_2(\mathbb{Q}_p)$) and having central character $\mathbb{Q}_p^\times \to \mathbb{C}^\times$ of finite order, and isomorphism classes of continuous irreducible 2-dimensional representations of the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.

Finite-dimensional continuous representations of $\text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ over $\mathbb{C}$ factor through $\text{Gal}(F/\mathbb{Q}_p)$ for some finite Galois extension $F/\mathbb{Q}$, so the Galois representations occurring above are also relatively concrete objects. In fact for $p > 2$ it is not too difficult to explicitly classify all such Galois representations (essentially because the restriction to the wild ramification subgroup, which is a $p$-group, cannot be irreducible). Characterizing the correspondences (i.e. giving “natural” a precise meaning) is not straightforward: one has to introduce invariants on both sides (L-functions and $\epsilon$ factors), so the Langlands correspondence is not as obviously natural as the Jacquet-Langlands correspondence.

There is an analogous Langlands correspondence for representations of $D^\times$. Although it is probably possible to deduce the Jacquet-Langlands correspondence from
the Langlands correspondences for both $\text{GL}_2(\mathbb{Q}_p)$ and $D^\times$ after proving the latter, this is not the path that we will follow. The Jacquet-Langlands correspondence generalizes to smooth irreducible representations of $\text{GL}_n(F)$ ($F$ a local non-Archimedean field) and $n$-dimensional “Galois representations”. One of the two known strategies to prove the local Langlands correspondence for $\text{GL}_n(F)$ for $F/\mathbb{Q}$ ([HT01], later simplified in [Sch13]; see [Hen00] for a different proof) is global (“Ihara-Langlands-Kottwitz method”, which is also the main method to find $\ell$-adic representations of the absolute Galois group of a number field attached to an automorphic representation in the étale cohomology of a Shimura variety) and uses the Jacquet-Langlands correspondence as an input. An essential global ingredient that occurs in the proof of the Jacquet-Langlands correspondence ([JL70], [DKV84]) and in the Ihara-Langlands-Kottwitz method is the Arthur-Selberg trace formula. Following [JL70] and [DKV84], the goal of this course is to prove the Jacquet-Langlands correspondence for $\text{GL}_2(\mathbb{Q}_p)$ using the simple trace formula.

More generally, one can (try to) formulate local and global Langlands correspondences for arbitrary connected reductive groups $G$ over local or global fields (conjectural in general). On the Galois side, these involve “Galois representations” taking values in the Langlands dual group $^L G$ (for split $G$, this is a complex reductive group whose Dynkin diagram is dual to that of $G$, see [Bor79] for a proper definition). Assuming these conjectures, whenever we have two connected reductive groups $G$ and $H$ and a morphism $^L H \to ^L G$, we have a relation between representations (automorphic in the global setting) of $H$ and $G$. In many cases, this relation (“Langlands functoriality”, although by no means functorial in the categorical sense!) can be formulated without referring to Galois representations, and in some cases it can even be proved. Some cases of Langlands functoriality can be proved using (some version of) the Arthur-Selberg trace formula. Such results are needed to construct Galois representations.

2. Smooth representations of $\text{GL}_2(\mathbb{Q}_p)$

We begin the study of smooth representations of the locally profinite topological group $G = \text{GL}_2(\mathbb{Q}_p)$. Many tools work just as well for general reductive groups, adding the (non-trivial) combinatorics of Weyl groups etc. References: [Cas], [Ren10].

2.1. Decompositions. Let $K_0 = \text{GL}_2(\mathbb{Z}_p)$. Note that this is exactly the set of $g \in G$ such that $g(\mathbb{Z}_p^2) = \mathbb{Z}_p^2$.

Lemma 2.1. Let $K$ be a compact subgroup of $G$. There exists $g \in G$ such that $K \subset gK_0 g^{-1}$.

Proof. The group $G$ acts transitively on the set of lattices in $\mathbb{Q}_p^2$ (defined as sub-$\mathbb{Z}_p$-modules of finite type and maximal rank, that is rank 2). The statement is equivalent to the existence of a lattice $L \subset \mathbb{Q}_p^2$ such that for any $k \in K$, $k(L) = L$. Because $K$ is compact, the image of $\text{det} : K \to \mathbb{Q}_p^\times$ is compact, and so for any $k \in K$ such that $k(L) \subset L$ we actually have $k(L) = L$. So we have to show that there is a lattice $L$ stable under $K$. Let $L_0$ be any lattice, for example $\mathbb{Z}_p^2$. There is an open subgroup $K' \subset K$ such that $L_0$ is stable under $L_0$ (if $L_0 = \mathbb{Z}_p^2$ we can take $K' = K \cap K_0$). Let $L = \sum_{g \in K/K'} g(L_0)$. It works! \qed
Denote by $N$ the subgroup
$$\left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| x \in \mathbb{Q}_p \right\}$$
of $\text{GL}_2(\mathbb{Q}_p)$. Denote by $T$ the subgroup of diagonal matrices in $G$, and $B = TN \simeq T \rtimes N$ the Borel subgroup of upper triangular matrices.

**Lemma 2.2.** We have the Iwasawa decomposition $G = BK_0$.

**Proof.** We use the same interpretation of $G/K_0$ as in the previous lemma, namely as the set of lattices in $\mathbb{Z}_p^2$: the coset $gK_0$ is identified to the lattice in $\mathbb{Q}_p^2$ admitting the columns of $g$ as a basis. The lemma is equivalent to the claim that for any lattice $L \subset \mathbb{Q}_p^2$, there is a basis $(e_1, e_2)$ of $L$ such that the second coordinate of $e_1$ is zero. Denote by $(f_1, f_2)$ the standard basis of $\mathbb{Q}_p^2$. Pick a basis $e_1$ of $L \cap \mathbb{Q}_p f_1$. Pick a basis $e_2$ of the lattice $L/(L \cap \mathbb{Q}_p f_1)$ in $\mathbb{Q}_p^2/\mathbb{Q}_p f_1 \simeq \mathbb{Q}_p f_2$. Let $e_2 \in L$ be any preimage of $e_2$. It works! □

Note that the multiplication map $B \times K_0 \to G$ is not injective, since $B \cap K_0$ is an open (in particular non-trivial) subgroup of $B$.

**Theorem 2.3.** We have the Cartan decomposition
$$G = \bigsqcup_{a, b, c, d \in \mathbb{Z}} K_0 \text{diag}(p^a, p^b) K_0.$$

**Proof.** This is equivalent to the following: for any lattices $L_1 = g_1(\mathbb{Z}_p^2)$ and $L_2 = g_2(\mathbb{Z}_p^2)$ in $\mathbb{Q}_p^2$, there is a unique pair of integers $a \geq b$ such that there exists a basis $(e, f)$ of $L_1$ (the columns of $g_1 k_1$ for $k_1 \in K_0$) such that $(p^a e, p^b f)$ is a basis of $L_2$ (the columns of $g_2 k_2$ for some $k_2 \in K_0$): the “relative position” of $L_1$ and $L_2$ is given by the double coset $K_0 g_1^{-1} g_2 K_0$. Indeed $G$ acts transitively on $G/K$, so that $G \backslash (G/K \times G/K) \simeq K \backslash G/K$. This statement on lattices is a particular case of the structure theorem of finitely generated modules over principal ideal domains (applied to $L_2/p^N L_1$ where $N \in \mathbb{Z}$ is large enough so that $p^N L_1 \subset L_2$). □

Let $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in G$.

**Lemma 2.4.** We have the Bruhat decomposition $G = B \sqcup BwN$, where the natural map $B \times N \to BwN$ is an isomorphism of algebraic varieties.

**Proof.** We have
$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b & a \\ c & 0 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} b & bd - a \\ c & cd \end{pmatrix}$$
and solving the equation is easy. □

It will sometimes be more convenient to translate the Bruhat decomposition on the right by $w^{-1}$ so that the “big open cell”, which is the complement of a single
coset in $B\backslash G$, contains $1 \in G$. Let $N$ be the subgroup $\begin{pmatrix} 1 & 0 \\ Q_p & 1 \end{pmatrix}$ of $G$, so that $N = wNw^{-1}$. Then we have $G = B\overline{N} \sqcup Bw^{-1}$.

These three decompositions generalize to general linear groups of arbitrary dimension (only the Bruhat decomposition requires a more clever proof), and even to connected reductive groups over $Q_p$ (the choice of $K_0$ is delicate if the group is not reductive over $Z_p$).

2.2. Smooth representations of $G$.

**Definition 2.5.** A smooth representation of $G$ is a complex vector space $V$ together with a group representation $\pi : G \to \text{GL}(V)$ such that for any $v \in V$, the stabilizer of $v$ in $G$ is an open subgroup. Denote $\text{Rep}(G)$ the category of smooth representations of $G$.

A smooth representation $(V, \pi)$ is admissible if for any open subgroup $K$ of $G$, the space $V^K$ of $K$-invariants has finite dimension.

The contragredient $(\tilde{V}, \tilde{\pi})$ of a smooth representation $(V, \pi)$ of $G$ is the space of $K_0$-finite linear forms $\tilde{v} : V \to \mathbb{C}$, i.e. the space of smooth vectors in the dual representation. We denote the pairing between $\tilde{v} \in \tilde{V}$ and $v \in V$ by $\langle v, \tilde{v} \rangle$.

A typical smooth representation of $G$ has infinite dimension. As the example $g \mapsto \begin{pmatrix} 1 & \log |\det(g)| \\ 0 & 1 \end{pmatrix}$ shows, even finite-dimensional representations of $G$ may fail to be semisimple. Later we shall see less obvious examples.

We do however have Schur’s lemma for irreducible smooth representations of $G$.

**Lemma 2.6.** Let $(V, \pi)$ be an irreducible smooth representation of $G$, and let $\phi \in \text{Hom}_G(V, V)$ be an endomorphism. Then $\phi$ is multiplication by a scalar.

**Proof.** We claim first that the complex vector space $\text{Hom}_G(V, V)$ has countable dimension. Let $v \in V \setminus \{0\}$, and let $K$ be a compact open subgroup of $G$ such that $v \in V^K$. Then $V = \sum_{g \in G/K} \mathbb{C}f(g)v$. The Cartan decomposition implies that $G/K$ is countable, so that $V$ has countable dimension, and any $G$-equivariant map $V \to V$ is determined by the image of $V$ so $\text{End}_G(V)$ also has countable dimension.

Now we claim that there exists $\lambda \in \mathbb{C}$ such that $\varphi - \lambda \text{Id}_V \notin \text{GL}(V)$. Otherwise we would obtain a morphism of algebras $\mathbb{C}(X) \to \text{End}_G(V)$, which would be injective since the LHS is a field, and since $\mathbb{C}(X)$ does not have countable dimension over $\mathbb{C}$ (e.g. the vectors $((X - \lambda)^{-1})_{\lambda \in \mathbb{C}}$ are linearly independent) this gives a contradiction.

So for some $\lambda \in \mathbb{C}$ we have $\ker(\phi - \lambda) \neq 0$ or $\text{im}(\phi - \lambda) \neq V$, and $\ker(\phi - \lambda)$ and $\text{im}(\phi - \lambda)$ are subrepresentations of $G$. By irreducibility of $V$, in the first case we have $\ker(\phi - \lambda) = V$ and in the second case we have $\text{im}(\phi - \lambda) = 0$.

There is an easier proof under the assumption that $V$ is admissible. Later we will show that in fact any irreducible smooth representation of $G$ is admissible.

Let $Z = \{\text{diag}(x, x) | x \in \mathbb{Q}_p^\times\}$ be the center of $G$.

**Corollary 2.7.** If $(V, \pi)$ is an irreducible smooth representation of $G$ then there exists a (unique) smooth character $\omega_\pi : Z \to \mathbb{C}^\times$ such that for any $z \in Z$, $\pi(z) = \omega_\pi(z)\text{Id}_V$. 

Using representation theory of finite groups it is easy to show that for any smooth representation \((V, \pi)\) of \(G\), we have a canonical decomposition into isotypical components \(V = \bigoplus \tau V_\tau\) where the sum is over irreducible representations \(\tau\) of \(K_0/K\) for varying open distinguished subgroups \(K\) of \(K_0\) (these have finite index in \(K_0\) so \(\tau\) has finite dimension) and \(V_\tau = \text{Hom}_{K_0}(\tau, V) \otimes \mathbb{C} \tau\). (exercise: any continuous irreducible finite-dimensional representation of a profinite topological group factors through a quotient by a distinguished open subgroup. We will not need this fact.) We see that \(\tilde{V} = \bigoplus \tau \text{Hom}_{\mathbb{C}}(V_\tau, \mathbb{C})\). The following lemma follows easily.

**Lemma 2.8.** Let \((V, \pi)\) be a smooth representation of \(G\). The following are equivalent

1. \(V\) is admissible,
2. \(\tilde{V}\) is admissible,
3. the natural (always injective) map \(V \to \tilde{V}\) is surjective.

We fix (until further notice) a left Haar measure on \(G\). Recall that it is unique up to multiplication by \(R_{>0}\). Recall the Riesz–Markov–Kakutani representation theorem: for a locally compact Hausdorff topological space \(X\), Radon measures correspond bijectively to positive linear functionals on \(C_c(X)\), the space of continuous compactly supported functions on \(X\). With this formulation of measure theory, in the case of a locally profinite topological space such as \(G\), one can construct the Haar measure concretely as follows (see [Bou63, Chapitre 7, §1.6] for details in a more general setting):

- Choose \(\text{vol}(K_0) \in \mathbb{R}_{>0}\) arbitrarily,
- For any open subgroup \(K \subset K_0\), define \(\text{vol}(K) = |K_0/K|^{-1} \text{vol}(K_0)\).
- For any \(f \in C_c(G)\), choose \(K\) as above such that \(f\) is right \(K\)-invariant, and let \(\int_G f = \text{vol}(K) \sum_{g \in G/K} f(g)\). Exercise: this does not depend on the choice of \(K\)!
- Extend \(\int_G\) to \(C_c(G)\) by continuity (approximate any continuous compactly supported function by smooth ones).

This concrete definition can also be used to check that \(G\) is unimodular, i.e. this left Haar measure is also right-invariant (Definition 2.23 is useful for this). Later we will give an explicit “differential” definition of the Haar measure on \(G\), and this will show that \(G\) is unimodular.

One can define Haar measures on the closed (also locally profinite) subgroups \(B, T, N\) in the same way. The groups \(T\) and \(N\) are commutative and so also unimodular, but \(B\) is not: if \(db\) is a left Haar measure on \(B\) then for any \(f \in C_c^\infty(B)\),

\[
\int_B f(ba)db = \delta_B(a) \int_B f(b)db
\]

where \(\delta_B \begin{pmatrix} x & u \\ 0 & y \end{pmatrix} = |x/y|\). This implies that \(\delta_B(b)db\) is a right Haar measure on \(B\).

By default Haar measures will be left Haar measures in these notes.
Let $\mathcal{H}(G)$ be the space of smooth (this means locally constant in this setting) compactly supported functions $f : G \to \mathbb{C}$. This is a convolution algebra:

$$(f * f')(x) = \int_G f(y)f'(y^{-1}x)dy = \int_G f(xy)f'(y^{-1})dy$$

Checking that this product is associative is left as an exercise. This algebra is not commutative and has no unit (that would be a Dirac distribution . . . ), but it has lots of idempotents: for any compact open subgroup $K$ of $G$, $e_K := \text{vol}(K)^{-1}1_K$ is an idempotent. Similarly, it is easy to define an idempotent $e_K,\tau$ for any finite-dimensional irreducible representation $\tau$ of $K/K'$ where $K'$ is an open distinguished subgroup of $K$, but we shall not need this. Note that for any idempotent $e \in \mathcal{H}(G)$ we have a unital subalgebra $e\mathcal{H}(G)e$ of $\mathcal{H}(G)$. We denote $\mathcal{H}(G, K) = e_K\mathcal{H}(G)e_K$, the space of functions $K \backslash G/K \to \mathbb{C}$ having finite support.

If $(V, \pi)$ is a smooth representation then $\mathcal{H}(G)$ acts on $V$ by the formula

$$\pi(f)v := \int_G f(g)\pi(g)v \, dg$$

Note that there is no analysis and very little measure theory involved: let $K$ be a compact open subgroup such that $v \in V^K$ and $f$ is right $K$-invariant. Then $\pi(f)v = \sum_{g \in G/K} \text{vol}(K)f(g)\pi(g)v$ where only finitely many terms are non-zero since $f$ has compact support.

Conversely, if $V$ is an $\mathcal{H}(G)$-module such that $\sum Ke_K V = V$, one can endow $V$ with a smooth action of $G$: for $v \in V^K$ and $g \in G$ let $\pi(g)v = \text{vol}(K)^{-1}e_{gK}v$ (exercise: this does not depend on $K$ and defines a group action). Thus smooth representations of $G$ are equivalent to smooth $\mathcal{H}(G)$-modules.

**Lemma 2.9.** Let $K$ be a compact open subgroup of $G$. The functor $V \rightsquigarrow V^K = \pi(e_K)V$, from smooth representations of $G$ to representations of $\mathcal{H}(G, K)$, induces a bijection between irreducible smooth representations $V$ of $G$ such that $V^K \neq 0$ and simple $\mathcal{H}(G, K)$-modules.

**Proof.**

(1) First we check that if $(V, \pi)$ is an irreducible smooth representation of $G$ then $V^K$ is a simple $\mathcal{H}(G, K)$-module. Let $M' \subset V^K$ be a non-zero sub-$\mathcal{H}(G, K)$-module. Then $V = \sum_{g \in G} \pi(g)M' = \mathcal{H}(G)M'$ and so

$$V^K = (\mathcal{H}(G)M')^K = e_K\mathcal{H}(G)M' = e_K\mathcal{H}(G)e_KM' = \mathcal{H}(G, K)M' = M'.$$

(2) Let $M$ be a $\mathcal{H}(G, K)$-module, consider the $\mathcal{H}(G)$-module $F_0(M) = \mathcal{H}(G) \otimes_{\mathcal{H}(G, K)} M$. In general this smooth representation of $G$ is too big (i.e. it happens that for some simple $\mathcal{H}(G, K)$-modules $M$, $F_0(M)$ is not irreducible). Consider the linear map $\phi_M : F_0(M) \to M$, $f \otimes m \mapsto e_Kfem$, which restricts to a morphism of $\mathcal{H}(G, K)$-modules $F_0(M)^K \to M$. It is clearly surjective, but it can have a non-trivial kernel. Let $W(M) = \{v \in F_0(M)| e_K\mathcal{H}(G)v = 0\}$, that is the largest subrepresentation $W$ of $F_0(M)$ such that $W^K = 0$. Then $W(M) \subset \ker \phi_M$, and by exactness of “taking $K$-invariants” the natural map $F_0(M)^K \to (F_0(M)/W(M))^K$ is an isomorphism. So we have a functor
$F : M \rightsquigarrow F_0(M)/W(M)$ and a natural transformation $\phi_T : F(?)^K \rightarrow \cdot$. The map $\phi_M : F(M)^K \rightarrow M$ is clearly surjective, and it is also injective:

$$(\mathcal{H}(G) \otimes_{\mathcal{H}(G,K)} M)^K = (e_K \mathcal{H}(G)) \otimes_{\mathcal{H}(G,K)} M = e_K \mathcal{H}(G)e_K \otimes_{\mathcal{H}(G,K)} M \simeq M.$$ (3) Now assume that $M$ is a simple $\mathcal{H}(G,K)$-module. We claim that $F(M)$ is an irreducible representation of $G$. With notation as above, let $V_1$ be a subrepresentation of $F(M)_0$ such that $W(M) \subsetneq V_1$. By definition of $W(M)$ we have $V_1^K \neq 0$ so that $\phi_M(V_1^K) \neq 0$. Since $M$ is simple this implies $\phi_M(V_1^K) = M$, and so $V_1$ contains $\mathcal{H}(G)(1 \otimes M) = F_0(M)$.

(4) The outstanding claim that we have to check is that for $V$ an irreducible smooth representation of $G$ such that $V^K \neq 0$, we have a (natural) isomorphism $F(V^K) \simeq V$. There is a natural morphism $\mathcal{H}(G) \otimes_{\mathcal{H}(G,K)} V^K \rightarrow V$, which is surjective since $V^K \neq 0$ and $V$ is irreducible. It clearly factors through $F(V^K)$, and since we have shown that $F(V^K)$ is irreducible we get that the natural morphism $F(V^K) \rightarrow V$ is an isomorphism.

This lemma suggests that we restrict our study to admissible representations, so that we can study only finite-dimensional objects. Unfortunately, beyond a few cases (small index in $K_0$) it is difficult to describe $\mathcal{H}(G,K)$ in a useful manner, so this point of view is rather limited. We will come back to $\mathcal{H}(G,K)$ in special cases later.

2.3. **Parabolic induction and the Jacquet functor.** The easiest way to construct representations of $G$ is to induce representations from smaller subgroups. For these to be “not too big” it is natural to induce from the cocompact (by the Iwasawa decomposition) subgroup $B$. We will be even more specific and induce only representations of $B$ which are trivial on its distinguished subgroup $N$. Non-trivial characters of $N$ are also very interesting, but this is another subject (Whittaker functionals).

**Definition 2.10.** Let $\mu = \mu_1 \otimes \mu_2$ be a smooth character of $T$, that is $\mu(\text{diag}(x_1,x_2)) = \mu_1(x_1)\mu_2(x_2)$ where $\mu_i : \mathbb{Q}_p^\times \rightarrow \mathbb{C}^\times$ is a smooth character. Define $\text{Ind}^G_B \mu$ as the representation of $G$ with underlying space the set of smooth functions $f : G \rightarrow \mathbb{C}$ such that

$$f \left( \begin{pmatrix} x_1 & u \\ 0 & x_2 \end{pmatrix} g \right) = \mu_1(x_1)\mu_2(x_2)|x_1/x_2|^{1/2}f(g)$$

for all $x_1, x_2 \in F^\times$, $u \in F$ and $g \in G$. The action is given by the formula $(g \cdot f)(x) = f(xg)$ for all $g, x \in G$.

Note that $\text{Ind}^G_B \mu$ admits $\mu_1\mu_2$ as a central character, using the obvious identification $\mathbb{Q}_p^\times \simeq \mathbb{Z}$. Note also that for a smooth character $\chi : \mathbb{Q}_p^\times \rightarrow \mathbb{C}^\times$ we have a natural isomorphism $\text{Ind}^G_B(\chi \mu_1 \otimes \chi \mu_2) \simeq \text{Ind}^G_B(\mu_1 \otimes \mu_2) \otimes (\chi \circ \det)$.

**Lemma 2.11.** The representation $\text{Ind}^G_B \mu$ of $G$ is admissible.

**Proof.** Let $K$ be a compact open subgroup of $G$. We may assume that $K \subset K_0$. Since $G = BK_0$ and $K_0/K$ is finite, the set $B\backslash G/K$ is also finite, showing that $\dim_{\mathbb{C}}(\text{Ind}^G_B \mu)^K < +\infty$. \hfill $\square$
One can see that \( \text{Ind}\_B^G \mu \) has infinite (countable by Lemma 2.11) dimension by producing functions as follows. For \( x \in G \) there exists a compact open subgroup \( K \) of \( G \) such that \( \mu \) is trivial on the image of \( xKx^{-1} \cap B \) in \( T \) (any small enough \( K \) works). Then there is a unique \( f \in \text{Ind}\_B^G \mu \) supported on \( BxK \) such that \( f(xk) = 1 \) for \( k \in K \).

The character \( \delta^1/2 \) \( : \text{diag}(x_1, x_2) \mapsto |x_1/x_2|^{1/2} \) was introduced so that \( \text{Ind}\_B^G \mu \) preserves unitarity. Let us briefly explain this (our goal is Corollary 2.14 below). If \( \mu \) is unitary then for \( f \in \text{Ind}\_B^G \mu \) the function \(|f|^2 : G \to \mathbb{C}\) belongs to \( C^\infty(G, B, \delta_B) = \text{Ind}\_B^G \delta^1/2 \), the space of smooth functions \( G \to \mathbb{C} \) such that \( f(bg) = \delta_B(b)f(g) \) for all \( b \in B \) and \( g \in G \). We will construct a \( G \)-invariant “integration map” \( C^\infty(G, B, \delta_B) \to \mathbb{C} \) (if \( B \) was unimodular like \( G \) we would have a \( G \)-invariant integration map \( C^\infty(B \backslash G) \to \mathbb{C} \) given by integration with respect to the \( G \)-invariant quotient measure on \( B \backslash G \); but \( B \) is not unimodular!). We have a map \( \psi : C^\infty_c(G) \to C^\infty(G, B, \delta_B), f \mapsto (g \mapsto \int_B f(bg)db) \). It is easy to check that it is surjective (concretely, use right translates of the Bruhat decomposition; see [Bou63, Ch. 7, §2, Lemme 1] for a general proof). We want to define an “integration map” \( C^\infty(G, B, \delta_B) \to \mathbb{C} \) so that it maps \( \psi(f) \) to \( \int_G f(g)dg \), so we have to check that \( \int_G f(g)dg = 0 \) if \( \psi(f) = 0 \). For the general case see [Bou63, Ch. 7, §2, Proposition 3]. In our particular case it is more concrete to use the Iwasawa decomposition and prove the following integration formula.

**Lemma 2.12.** Choose Haar measures on the unimodular groups \( G, T, N \) and \( K_0 \) so that \( \text{vol}_G(K_0) = \text{vol}_B(B \cap K_0) \text{vol}_{K_0}(K_0) \). Then for any \( f \in C^\infty_c(G) \) we have

\[
\int_G f(g) dg = \int_{T \times N \times K_0} f(tk) dt dn dk = \int_{N \times T \times K_0} f(n tk) \delta_B^{-1}(t) dt dn dk.
\]

**Proof.** First note that via the homeomorphism \( T \times N \simeq B \), \( (t, n) \mapsto tn \), the product of Haar measures on \( T \) and \( N \) is a left Haar measure on \( B \). The second equality has nothing to do with \( G \) or \( K_0 \) and is a consequence of the definition of the modulus character for \( B \) since \( dn \, dt \) is a right Haar measure on \( B = NT \). So it is enough to prove the formula \( \int_G f(g) dg = \int_{B \times K_0} f(bk) db \, dk \) where a left Haar measure on \( B \) is used.

Consider the map

\[
\phi : C^\infty_c(B \times K_0) \longrightarrow C^\infty_c(G)
\]

\[
F \mapsto \left( bk \mapsto \int_{B \cap K_0} F(bh, k^{-1}h) \, dh \right)
\]

where we have written an arbitrary element of \( G \) as \( bk \) for \( b \in B \) and \( k \in K_0 \) using the Iwasawa decomposition. It is surjective because \( B \cap K_0 \) is compact, in fact it has a natural section (consider \( F \) factoring through \( (b, k) \mapsto bk^{-1} \)). Moreover \( F \mapsto \int_G \phi(F)(g) \, dg \) is left \( B \times K_0 \)-invariant because the Haar measure on \( G \) is invariant under left multiplication by \( B \) and right multiplication by \( K_0 \), so it coincides with \( F \mapsto \int_{B \times K_0} F(b, k) \, db \, dk \) up to a scalar. The scalar is computed by taking \( f \) to be the characteristic function of \( K_0 \).

**Corollary 2.13.** The map \( C^\infty(G, B, \delta_B), f \mapsto \int_{K_0} f(k) \, dk \) is \( G \)-invariant.
Proof. Any \( f \in C^\infty(G, B, \delta_B) \) can be written as \( g \mapsto \int_B \alpha(bg)db \) for some \( \alpha \in C^\infty_c(G) \), so
\[
\int_{K_0} f(k) \, dk = \int_{K_0} \int_B \alpha(bk) \, db \, dk = \int_G \alpha(g) \, dg
\]
and \( (g \cdot f)(x) = \int_B \alpha(bxg)db \), so the assertion follows from the fact that the Haar measure on \( G \) is right \( G \)-invariant. \( \square \)

Corollary 2.14. If \( \mu \) is unitary then \( \text{Ind}_B^G \mu \) has a natural \( G \)-invariant Hermitian inner product, defined by
\[
\|f\|_2^2 = \int_{K_0} |f(k)|^2 \, dk.
\]
Corollary 2.13 also shows that \( \int_{K_0} \) gives an intertwining operator \( \text{Ind}_B^G \delta_B^{-1/2} \to \mathbb{C} \) where the target is endowed with the trivial representation of \( G \). It is easy to see that this linear form is non-zero (as explained above there are non-zero non-negative functions on the left-hand side).

Definition 2.15. The Steinberg representation \( \text{St} \) of \( G \) is the subrepresentation of \( \text{Ind}_B^G \delta_B^{-1/2} \) consisting of all functions \( f \) satisfying \( \int_{K_0} f(k) \, dk = 0 \).

Later we will prove that the Steinberg representation is irreducible. Dually (this duality will not be explained in these notes ...), we have an embedding \( \mathbb{C} \to \text{Ind}_B^G \delta_B^{-1/2} \) (constant functions) and we will see later that \( \text{St} \) is also realized as the cokernel of this map.

Definition 2.16. Let \( (V, \pi) \) be a smooth representation of \( N \). Let \( V(N) = \sum_{n \in N} (\pi(n) - 1)V \). Let \( V_N = V/V(N) \), the space of coinvariants under \( N \), i.e. the largest quotient of \( V \) on which \( N \) acts trivially. This defines a functor \( \text{Rep}(N) \to \text{Vec} \). Restricting to smooth representations of \( B \), using that \( N \) is distinguished in \( B \) we obtain a functor \( \text{Rep}(B) \to \text{Rep}(T) \).

Define the (normalized) Jacquet functor \( \text{Res}_B : \text{Rep}(B) \to \text{Rep}(T) \) by \( \text{Res}_B V = \delta_B^{-1/2} \otimes V_N \).

If \( (V, \pi) \) is smooth representation of \( G \) we say that it is supercuspidal if \( \text{Res}_B V = 0 \) (equivalently if \( V_N = 0 \)).

Note that if \( (V, \pi) \) is a smooth representation of \( G \) then \( V_N \) is naturally endowed with an action of \( T \), since \( N \) is invariant under \( T \). It is not difficult to check that this representation of \( T \) is smooth. So we can also see the Jacquet functor as a functor from the category of smooth representations of \( G \) to the category of smooth representations of \( T \).

Note that any compact open subgroup of \( N \) is of the form
\[
\left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \bigg| x \in \mathbb{F}_p^n \mathbb{Z}_p \right\}
\]
for some \( n \in \mathbb{Z} \). In particular \( N \) is the union of its compact open subgroups. For \( N_c \) such a subgroup and \( (V, \pi) \) a smooth representation of \( N \) we let \( V(N_c) \) be the space of \( v \in V \) such that \( \int_{N_c} \pi(n)vdn = 0 \). This notation is justified by the following lemma.

Lemma 2.17. \( \text{(1) For any smooth representation of } N, V(N) = \bigcup_{N_c} V(N_c) \) where the union is over all compact open subgroups \( N_c \) of \( N \).
(2) The functor $\text{Res}_B$ is exact, i.e. for any short exact sequence $0 \to V_1 \to V_2 \to V_3 \to 0$ in $\text{Rep}(N)$, the sequence $0 \to \text{Res}_B V_1 \to \text{Res}_B V_2 \to \text{Res}_B V_3 \to 0$ is also exact.

Proof. (1) For $n_1, \ldots, n_k \in N$ choose $N_c$ containing all $n_i$'s, then it is clear that for any $v_1, \ldots, v_k \in V$ we have $\int_{N_c} \pi(n) \sum_{i=1}^k (\pi(n_i) - 1)v_i dn = 0$.

Conversely suppose that $v' \in V(N_c)$. Let $N'_c \subset N_c$ be an open subgroup fixing $v'$. Then $0 = \int_{N_c} \pi(n)v'dn = \text{vol}(N'_c) \sum_{n \in N_c/N_c} \pi(n)v'$ and mapping both sides of this equation to $V_N$ we obtain $\text{vol}(N_c)v' \in V(N)$.

(2) The only non-formal part of this statement is the injectivity of $\text{Res}_B V_1 \to \text{Res}_B V_2$, but this follows easily from the previous point.

Although we will not use it much in this generality, note that $\text{Ind}_B^G$ can also be promoted to a functor $\text{Rep}(T) \to \text{Rep}(G)$. The following theorem (Frobenius reciprocity) is easy to prove but fundamental.

**Theorem 2.18.** The Jacquet functor $\text{Res}_B: \text{Rep}(G) \to \text{Rep}(T)$ is left adjoint to $\text{Ind}_B^G: \text{Rep}(T) \to \text{Rep}(G)$. That is, for any smooth representation $(\pi, V)$ (resp. $(\sigma, W)$) of $G$ (resp. $T$) we have a natural isomorphism

$$\text{Hom}_G(V, \text{Ind}_B^G W) \simeq \text{Hom}_T(\text{Res}_B V, W).$$

**Proof.** The morphism is defined as composition with $\text{Ind}_B^G W \to W$, $f \mapsto f(1)$. The inverse morphism $\text{Hom}_T(\text{Res}_B V, W) \to \text{Hom}_G(V, \text{Ind}_B^G W)$ is $\varphi \mapsto (x \mapsto \varphi(\pi(x)v))$.

**Corollary 2.19.** Let $(V, \pi)$ be an irreducible smooth representation of $G$. Assume that $\text{Res}_B V \neq 0$ (equivalently, $V_N \neq 0$). Then $V$ embeds in a representation induced from a character of $T$.

**Proof.** First we show that the representation $\text{Res}_B V$ of $T$ is generated by finitely many vectors. Choose $v \in V \setminus \{0\}$ and let $K$ be a compact open subgroup of $G$ fixing $v$. There exists a finite $R \subset G$ such that $G = \text{BRK}$. Then $V = \{\sum_{i \in I} \lambda_i \pi(b_i r_i k_i)v | I \text{ finite}, b_i \in B, r_i \in R \text{ and } k_i \in K\}$ and we see that $\{\pi(r)v | r \in R\}$ generates $\text{Res}_B V$.

Now assume that $\text{Res}_B V \neq 0$, then we can find an irreducible quotient of $\text{Res}_B V$: assume that $v_1, \ldots, v_n$ generate $V$ and that $V$ is not irreducible, apply Zorn's lemma to the set of subrepresentations of $V$ which do not contain $\{v_1, \ldots, v_n\}$. By Schur's lemma a smooth irreducible representation of $T$ is one-dimensional (on which $T$ acts by a character, of course), and we can apply the previous theorem.

In particular any irreducible smooth non-supercuspidal representation is admissible.

The Bruhat decomposition is useful to study the restriction of $\text{Ind}_B^G \mu$ to $N$. 

Lemma 2.20. The morphism of $\mathbb{C}$-vector spaces

$$\text{Ind}_B^G \mu \longrightarrow C^\infty(Q_p, \mathbb{C}) \oplus \mathbb{C}$$

$$f \longmapsto \left( x \mapsto f \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right), f(1) \right)$$

is injective and its image is the space of pairs $(F, v)$ such that there exists a compact subset $C$ of $Q_p$ containing $0$ such that for any $x \in Q_p \setminus C$ we have

$$F(x) = [\delta_B^{1/2}(\mu)(\text{diag}(x^{-1}, x))]v.$$

Proof. Injectivity is clear. To characterize the image, note that for $k >> 0$ we have $f\left( \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} \right) = f(1)$ for any $y \in p^k\mathbb{Z}_p$, and if $y \neq 0$ we have

$$\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} y^{-1} & 1 \\ 0 & y \end{pmatrix} w \begin{pmatrix} 1 & y^{-1} \\ 0 & 1 \end{pmatrix}$$

and take $x = y^{-1}$. \qed

2.4. The geometric lemma. Let $V$ be a complex line on which $T$ acts by $\mu = \mu_1 \otimes \mu_2$. We now compute $W = \text{Res}_B \text{Ind}_B^G V$. Let $W_1$ be the subspace of functions supported on $BwN$, i.e. functions $f$ such that $f(1) = 0$. This space is naturally isomorphic to the space of smooth compactly supported functions $N \rightarrow V$. By Lemma 2.20 we have a short exact sequence of smooth representations of $B$

$$0 \rightarrow W_1 \rightarrow \text{Ind}_B^G V \rightarrow W_2 \rightarrow 0$$

with $W_2 = \delta_B^{1/2} \otimes V$. Since $N$ acts trivially on $W_2$ we have $W_2(N) = 0$ and $\text{Res}_B W_2 = V$. Now $W_1(N)$ is the kernel of $\varphi : W_1 \rightarrow V, f \mapsto \int_N f(wn)dn$. This morphism is easily seen to be surjective, and for $f \in W_1$ and $b \in B$, writing $b = ut$ with $t \in T$ and $u \in N$ we have

$$\varphi(b \cdot f) = \int_N f(wnb)dn = \int_N f(wtw^{-1}wt^{-1}nut)dn$$

$$= \delta_B^{1/2}(t^{1/2}) \mu_w(t) \int_N f(wt^{-1}ut)dn = \delta_B(t)^{-1/2} \mu_w(t) \delta_B(t) \varphi(f) = \delta_B(t)^{1/2} \mu_w(t) \varphi(f)$$

and so $\text{Res}_B W_1$ has dimension one and $T$ acts by $\mu_w = \mu_2 \otimes \mu_1$ on it. So we have a short exact sequence

$$0 \rightarrow \mathbb{C}(\mu^w) \rightarrow \text{Res}_B \text{Ind}_B^G \mu \rightarrow \mathbb{C}(\mu) \rightarrow 0.$$

The existence of this short exact sequence is the “geometric lemma” for $G$. We can be more precise and completely determine $\text{Res}_B \text{Ind}_B^G \mu$ (Proposition 2.21 below). Since $T$ is commutative, if $\mu_w \neq \mu$ (i.e. if $\mu_1 \neq \mu_2$) this short exact sequence splits: choose $t \in T$ such that $\mu(t) \neq \mu_w(t)$ and consider $\ker(t - \mu(t)) \text{Res}_B \text{Ind}_B^G V$.

We now consider the case $\mu_w = \mu$, and show that in this case the short exact sequence does not split. The sequence splits if and only if there exists $f \in \text{Ind}_B^G V$
such that \( f(1) \neq 0 \) and for any \( b \in B, b \cdot f - \delta_B^{1/2}(b)\mu(b)f \in (\text{Ind}_B^G V)(N) = W_1(N) = \ker \varphi \). We can take \( b = t \in T \), then

\[
(t \cdot f) \left( \begin{array}{cc} 1 & x \\ 0 & 1 \end{array} \right) - \delta_B^{1/2}(t)\mu(t)f \left( \begin{array}{cc} 1 & x \\ 0 & 1 \end{array} \right) = \delta_B^w(t)^{1/2}\mu^w(t)F(xt_1^{-1}t_2) - \delta_B^{1/2}(t)\mu(t)F(x)
\]

It is clear that \( b \cdot f - \delta_B^{1/2}(b)\mu(b)f \in W_1 \), so for \( k >> 0 \) we have

\[
\varphi(f) = \int_{p^{-k} \mathbb{Z}_p} \delta_B^w(t)^{1/2}\mu^w(t)F(xt_1^{-1}t_2) - \delta_B^{1/2}(t)\mu(t)F(x) \ |dx|
= \delta_B(t)^{1/2} \left( \mu^w(t) \int_{p^{-k} \mathbb{Z}_p} F(x) \ |dx| - \mu(t) \int_{p^{-k} \mathbb{Z}_p} F(x) \ |dx| \right)
\]

so under our assumption that \( \mu^w = \mu \), we see that this vanishes for any \( t \in T \) if and only if for \( k >> 0 \) we have vanishing of \( \int_{p^{-k} \mathbb{Z}_p} F(x) \ |dx| \). Using Lemma 2.20 this equals

\[
\int_{p^{-k} \mathbb{Z}_p} |x|^{-1}\mu_1(x)^{-1}\mu_2(x)f(1) \ |dx| = \left( 1 - \frac{1}{p} \right) f(1)
\]

since \( \mu_1 = \mu_2 \).

Let us state what we have just proved.

**Proposition 2.21.** If \( \mu \neq \mu^w \) then \( \text{Res}_B \text{Ind}_B^G \mu \simeq \mu \oplus \mu^w \). If \( \mu^w = \mu \) then we have a short exact sequence

\[
0 \to \mu \to \text{Res}_B \text{Ind}_B^G \mu \to \mu \to 0
\]

which does not split.

**Corollary 2.22.** If \( \mu_1/\mu_2 \) is unitary, that is if \( |\mu_1(p)/\mu_2(p)| = 1 \), then \( \text{Ind}_B^G \mu \) is irreducible.

**Proof.** Up to twisting by an unramified character we can assume that \( \mu_1 \) and \( \mu_2 \) are unitary. Then \( \text{Ind}_B^G \mu \) admits a \( G \)-invariant Hermitian inner product, and in particular it is semi-simple, so it is irreducible if and only if the vector space \( \text{Hom}_C(\text{Ind}_B^G \mu, \text{Ind}_B^G \mu) \) has dimension one. By Theorem 2.18 this space is isomorphic to \( \text{Hom}_T(\mathbb{C}(\mu), \text{Res}_B \text{Ind}_B^G \mu) \) and we can conclude with the above computation of \( \text{Res}_B \text{Ind}_B^G \mu \). \( \square \)

### 2.5 Supercuspidal representations.

**Definition 2.23.** Let \( I \) be a compact open subgroup of \( G \). We say that \( I \) has an Iwahori factorization if, denoting \( \mathcal{N}_I = \mathcal{N} \cap I, T_I = T \cap I \) and \( N_I = N \cap I \), the product map \( N_I \times T_I \times \mathcal{N}_I \to I \) is surjective.

Of course this map is always injective, and it is clear that it is always a homeomorphism onto an open subset of \( I \).
Remark 2.24. (1) Let \( T^- \) be the set of \( t \in T \) such that \( |t_1/t_2| \leq 1 \). In [Cas, §1.4] there is an extra condition in the definition of Iwahori factorization, namely that for any \( t \in T^- \) we have \( tN_1t^{-1} \subset N_1 \) and \( t^{-1}N_1t \subset N_1 \). It is easy to check that this condition is automatically satisfied in the particular case of \( \text{GL}_2(\mathbb{Q}_p) \), due to the particularly simple classification of closed subgroups of \( \mathbb{Q}_p \).

(2) Taking inverses, we also have \( N_I \times T_I \times N_I \approx I \).

Example 2.25. (1) The Iwahori subgroup \( \left( \mathbb{Z}_p^\times, \mathbb{Z}_p, p\mathbb{Z}_p, \mathbb{Z}_p^\times \right) \) has an Iwahori factorization (and is the reason for this terminology).

(2) For any integer \( i \geq 1 \) the subgroup \( K_i = \left( 1 + p^i\mathbb{Z}_p, p^i\mathbb{Z}_p, 1 + p^i\mathbb{Z}_p \right) \) also satisfies this condition.

We will use a more compact notation for in the second case: \( \overline{N}_i = K_i \cap N \), \( T_i = T \cap K_i \) and \( N_i = N \cap K_i \). For \( i < 0 \) we can also define \( \overline{N}_i = \left( \begin{array}{cc} 1 & 0 \\ p^i\mathbb{Z}_p & 1 \end{array} \right) \) and \( N_i = \left( \begin{array}{cc} 1 & p^i\mathbb{Z}_p \\ 0 & 1 \end{array} \right) \).

Note however that \( K \) does not admit an Iwahori factorization, in fact one can check that \( I \) is maximal among compact open subgroups admitting an Iwahori factorization. For our purpose in this section we will only use the fact that there are arbitrarily small compact open subgroups of \( G \) having an Iwahori factorization. We will come back to the Iwahori case later.

Lemma 2.26. Let \( I \) be a compact open subgroup of \( G \) admitting an Iwahori factorization. For any \( f \in C^\infty(I) \), we have the integration formula.

\[
\int_{N_I \times T_I \times N_I} f(ntn) d\overline{n} dt dn = \int_I f(g) dg.
\]

Proof. The proof is similar to the proof of the integration formula for the Iwasawa decomposition (Lemma 2.12), only simpler because the decomposition is a bijection. The pullback to \( \overline{N}_I \times T_I \times N_I \) of the Haar measure on \( I \) (restriction of the Haar measur on \( G \)) is clearly left \( \overline{N}_I \)-invariant and right \( N_I \)-invariant. It is also left and right \( T_I \)-invariant because any \( t \in T_I \) normalizes \( \overline{N}_I \) and \( N_I \) and preserves their Haar measures because \( T_I \) is compact. \( \square \)

Theorem 2.27. Let \( (\pi, V) \) be a smooth representation of \( G \). The following are equivalent:

(1) \( (\pi, V) \) is supercuspidal.

(2) For any \( v \in V \) and \( \tilde{v} \in \tilde{V} \), there is a compact subset \( C \) of \( G/Z \) such that \( \langle \pi(g)v, \tilde{v} \rangle = 0 \) for all \( g \in G \) such that \( \overline{\pi} \in G/Z \setminus C \).

(3) For any \( v \in V \) and any compact open subgroup \( K \) of \( G \), there is a compact subset \( C \) of \( G/Z \) such that \( \pi(e_K)\pi(g)v = 0 \) for all \( g \in G \) such that \( \overline{\pi} \in G/Z \setminus C \).
Proof. Let us prove that (2) implies (3) first. Let $G' = \{ g \in G \mid \det g \in \mathbb{Z}_p^* \}$, an open subgroup of $G$ which contains all compact subgroups of $G$. Note that $ZG'$ has finite index (two) in $G$, and that $G' \to G/Z$ is proper. Let $v \in V$ and $K \subset G$ a compact open subgroup of $G$. Let $W$ be the sub-vector space of $V^K$ generated by $\pi(e_K)\pi(g)v$ for $g \in G'$. Let $S$ be a subset of $K \setminus G'$ such that $(\pi(e_K)\pi(g)v)_{g \in S}$ is a basis of $W$. There exists $\tilde{v} \in \tilde{V}^K$ such that for any $g \in S$, \( \langle \pi(e_K)\pi(g)v, \tilde{v} \rangle = 1 \). By the assumption (2) applied to the pair $(v, \tilde{v})$, the set $S$ is finite, i.e. $\dim_{\mathbb{C}} W < +\infty$. The restriction map $\tilde{V}^K = \text{Hom}_G(V^K, \mathbb{C}) \to \text{Hom}_G(W, \mathbb{C})$ is surjective, and so there is a finite family $(\tilde{v}_i)_{1 \leq i \leq k}$ of elements of $\tilde{V}^K$ such that $\{ w \in W \mid \langle w, \tilde{v}_1 \rangle = \cdots = \langle w, \tilde{v}_k \rangle = 0 \} = \{ 0 \}$. Now applying assumption (2) to all pairs $(v, \tilde{v}_i)$ shows that the function $G' \to W$, $g \mapsto \pi(e_K)\pi(g)v$ has compact support. Let $t = \text{diag}(p, 1) \in G$. Since no assumption was made on $v$ in the above argument, it also applies to $\pi(t)v$ and the map $G' \to V$, $g \mapsto \pi(e_K)\pi(g)v$ also has compact support. Since $Z$ is central in $G$ ($Z$ being the center $\ldots$) and $G = ZG' \sqcup ZG't$ we get that $G \to V$, $g \mapsto \pi(e_K)\pi(g)v$ has compact support modulo $Z$, i.e. there exists $C$ as in (3).

Now let us show that (3) implies (1). Let $v \in V$. Pick $i \geq 1$ such that $v \in V^{K_i}$. We still denote $t = \text{diag}(p, 1)$, so that $t^{-1}N_it \subset N_i$, in fact $t^{-1}N_it = N_{i+1}$. By assumption and the Cartan decomposition, for $m >> 0$ we have $\pi(e_{K_i})\pi(t^m)v = 0$. By the integration formula (Lemma 2.26) and passing $t^m$ to the left, this can be written
\[
\pi(t^m)\pi(e_{t^{-m}N_im})\pi(e_{t^{-m}N_im})v = 0.
\]
Since $v$ is fixed by $N_i \supset t^{-m}N_it^m$ and by $T_i = t^{-m}T_it^m$ we have $\pi(e_{t^{-m}N_im})v = 0$, i.e. $v \in V(N_{i-m})$.

Finally we show that (1) implies (2). Assume that (2) does not hold, then there exists $v \in V$ and $\tilde{v} \in \tilde{V}$ such that $g \mapsto \langle \pi(g)v, \tilde{v} \rangle$ does not have compact support modulo $Z$. There exists $i \geq 1$ such that $K_i$ fixes both $v$ and $\tilde{v}$. Let $R_i \subset K_0$ be a set of representatives for the quotient $K_0/K_i$. By the Cartan decomposition $G = \sqcup_{m \geq 0} K_0t^mK_0Z$ we have that for infinitely many $m \geq 0$ there exists $z_m \in Z$ and $r_m, r'_m \in R_i$ such that $\langle \pi(r_m^{t^m}r'_mz_m)v, \tilde{v} \rangle \neq 0$. Up to replacing $(v, \tilde{v})$ by $(\pi(r')v, \pi(r^{-1})\tilde{v})$ for some $r, r' \in R_i$, we can assume that there are infinitely many $m \geq 0$ such that there exists $z_m \in Z$ such that $\langle \pi(t^mz_m)v, \tilde{v} \rangle \neq 0$. Since $K_i$ fixes $\tilde{v}$ we have $\langle \pi(z_m)\pi(e_{K_i})\pi(t^m)v, \tilde{v} \rangle = \langle \pi(t^mz_m)v, \tilde{v} \rangle$ and so $\pi(e_{K_i})\pi(t^m)v \neq 0$ for infinitely many $m \geq 0$. Arguing as above we obtain that $\pi(e_{N_{i-m}})v \neq 0$ for infinitely many $m \geq 0$, and so $v \notin V(N)$.

Corollary 2.28. Any irreducible supercuspidal representation is admissible.

Proof. Pick any $v \in V \setminus \{ 0 \}$ and a compact open subgroup $K$ of $G$ which fixes $v$. The sub-vector space $W \subset V^K$ generated by $\pi(e_K)\pi(g)v$ for $g \in G$ has finite dimension by (3) above (here Schur’s lemma is used). Since $V$ is irreducible we have $W = \pi(e_K)V = V^K$.

Remark 2.29. Together with Lemma 2.11 and Corollary 2.19 (and the fact that taking $K$-invariants is an exact functor), this shows that any irreducible smooth representation of $G$ is admissible.
Definition 2.30. Let \((V, \pi)\) be a smooth representation of \(G\), and assume that it has a \((\text{unique and smooth})\) central character \(\omega_\pi : Z \to \mathbb{C}^\times\). We say that \(\pi\) is square-integrable (or part of the discrete series) if \(\omega_\pi\) is unitary and for any \(v \in V\) and \(\tilde{v} \in \tilde{V}\), \(\int_{G/Z} |\langle \pi(g)v, \tilde{v} \rangle|^2 dg < +\infty\).

We say that \(\pi\) is essentially square-integrable if there exists \(s \in \mathbb{R}_{>0}\) (unique) such that \(|\det|^s \otimes \pi\) is square-integrable.

Any irreducible supercuspidal representation is essentially square-integrable.

Lemma 2.31. If \((V, \pi)\) is an irreducible smooth representation of \(G\) with unitary central character \(\omega_\pi\) (Corollary 2.7) then it is square-integrable if and only if there exist non-zero \(v_0 \in V\) and \(\tilde{v}_0 \in \tilde{V}\) such that \(\int_{G/Z} |\langle \pi(g)v_0, \tilde{v}_0 \rangle|^2 dg < +\infty\).

Proof. The set of \(v \in V\) such that \(g \mapsto |\langle \pi(g)v, \tilde{v}_0 \rangle|^2\) is integrable is stable under \(G\) (by right invariance of the Haar measure) and is a sub-vector space of \(V\) (using the Cauchy-Schwarz inequality). Since it contains \(v_0 \neq 0\), it equals \(V\). Repeating this argument for \(\tilde{V}\) allows to conclude. \(\square\)

Lemma 2.32. Any irreducible square-integrable representation is unitarizable, i.e. admits a \(G\)-invariant hermitian inner product. Moreover the \(G\)-invariant hermitian inner product is unique up to \(R_{>0}\).

Proof. Pick \(\tilde{v} \in \tilde{V} \setminus \{0\}\). Then \(v \mapsto (g \mapsto \langle \pi(g)v, \tilde{v}\rangle)\) gives a \(G\)-equivariant embedding of \(V\) into \(L^2(G, \omega_\pi)\), the space of measurable functions \(G \to \mathbb{C}\) such that \(f(zg) = \omega_\pi(z)f(g)\) for all \(z \in Z\) and \(g \in G\) and \(\int_{G/Z} |f(g)|^2 dg < +\infty\).

A \(G\)-invariant hermitian pairing \(H(\cdot, \cdot)\) on \(V\) can be seen as a \(G\)-equivariant morphism \(\varphi : \overline{V} \to \tilde{V}\), where \(\overline{V} := \mathbb{C} \otimes_C V\) (with \(C \to \mathbb{C}, z \mapsto \overline{z}\)). More precisely, \(H(v_1, v_2) = \langle v_1, \varphi(1 \otimes v_2) \rangle\). In the case of a Hermitian inner product, \(\varphi\) is injective and using admissibility we see that \(\varphi\) is an isomorphism. Essential uniqueness follows from Schur’s lemma. \(\square\)

Remark 2.33. The same argument shows that an irreducible supercuspidal representation of \(G\) can be realized in \(C_c^\infty(G, \omega_\pi)\), the space of smooth functions \(f \to \mathbb{C}\) such that \(f(zg) = \omega_\pi(z)f(g)\) and such that the support of \(f\) is compact modulo \(Z\). In fact it is easy to check that \(f \in C_c^\infty(G, \omega_\pi)\), the subspace of functions satisfying \(\int_N f(xny)dn = 0\) for all \(x, y \in G\).

Proposition 2.34. Let \((V, \pi)\) be an irreducible essentially square-integrable representation of \(G\). There exists a unique \(d_\pi \in \mathbb{R}_{>0}\), called the formal degree of \(\pi\), such that for any \(u, v \in V\) and \(\tilde{u}, \tilde{v} \in \tilde{V}\) we have

\[
\int_{G/Z} \langle \pi(g)u, \tilde{u} \rangle \langle \pi(g^{-1})v, \tilde{v} \rangle dg = d_\pi^{-1} \langle u, \tilde{v} \rangle \langle v, \tilde{u} \rangle.
\]

Observe that the integral is well-defined and converges absolutely.

Proof. Fix \(v\) and \(\tilde{u}\). Then the integral, seen as a function of \((u, \tilde{v})\), defines a \(G\)-invariant pairing on \(V \times \tilde{V}\). By Schur’s lemma it is a complex number times the canonical pairing. The same argument with \(u\) and \(\tilde{v}\) fixed shows that the integral equals \(c_\pi \langle u, \tilde{v} \rangle \langle v, \tilde{u} \rangle\) for some \(c_\pi \in \mathbb{C}\).
It remains to show that $c_\pi \in \mathbb{R}_{>0}$. Up to twisting by a character we can assume that $\pi$ is square-integrable. Pick a $G$-invariant Hermitian inner product $\langle \cdot, \cdot \rangle$ on $V$ (see the previous lemma), which is equivalent to an isomorphism $\varphi : \overline{V} \simeq V$ such that $\langle v, v \rangle := \langle v, \varphi(1 \otimes v) \rangle \in \mathbb{R}_{>0}$ for all non-zero $v \in V$. Taking $\tilde{v} = \varphi(1 \otimes u)$ and $\tilde{u} = \varphi(1 \otimes v)$ for arbitrary $u, v \in V \setminus \{0\}$, the integrand equals

$$\langle \pi(g)u, v \rangle \langle \pi(g^{-1})v, u \rangle = |\langle \pi(g)u, v \rangle|^2$$

which is non-negative, smooth and not identically vanishing, and the right-hand side equals $c_\pi(u, u)(v, v)$, therefore $c_\pi \in \mathbb{R}_{>0}$. \hfill \Box

**Remark 2.35.**  
(1) The constant $d_\pi$ depends on the choice of Haar measure for $G/Z$. The Haar measure $d_xdg$ is canonically associated to $\pi$.

(2) This Proposition (and its proof) are inspired by the same result for irreducible unitary representations of finite (or more generally compact) groups, which are all square-integrable and finite-dimensional. In this simpler case, taking the Haar measure to be a probability measure (and removing the quotient by $Z$) one can check that $d_\pi$ is the dimension of $\pi$. Proposition 2.34 for compact groups has a better known coordinate-free analogue (orthonormality of characters of irreducible representations), but it is not as straightforward to generalize this to the infinite-dimensional case. We will prove an analogous theorem for $G$ later.

**Corollary 2.36.** If $(V, \pi)$ is an irreducible supercuspidal representation of $G$ and $(U, \sigma)$ is a smooth representation of $G$ such that $\sigma(zy) = \omega_x(z)\sigma(g)$ for all $z \in Z$ and $g \in G$ then any non-zero morphism $P : U \to V$ admits a splitting.

**Proof.** Pick $v_0 \in V$ and $\tilde{v}_0 \in \overline{V}$ such that $\langle v_0, \tilde{v}_0 \rangle = d_\pi$. Pick $u_0 \in U$ mapping to $v_0$. Define $s : V \to U$ by $s(v) = \int_{G/Z} \langle \pi(g^{-1})v, \tilde{v}_0 \rangle \sigma(g)u_0dg$. The linear map $s$ is $G$-equivariant: for $h \in G$, using the change of variable $x = h^{-1}g$,

$$s(\pi(h)v) = \int_{G/Z} \langle \pi((h^{-1}g)^{-1})v, \tilde{v}_0 \rangle \sigma(g)u_0 \, dg = \int_{G/Z} \langle \pi(x^{-1})v, \tilde{v}_0 \rangle \sigma(hx)u_0 \, dg = \sigma(h)s(v).$$

To compute the image of $s(v)$ in $V$, take any test vector $\tilde{v} \in \overline{V}$. The previous proposition gives us $\langle P(s(v)), \tilde{v} \rangle = \langle v, \tilde{v} \rangle$ and so $P(s(v)) = v$. \hfill \Box

**Corollary 2.37.** For any smooth character $\mu$ of $T$, the induced representation $\text{Ind}_B^G \mu$ has finite length $\leq 2$, and no constituent is supercuspidal.

**Proof.** First we show that any irreducible subquotient of $\text{Ind}_B^G \mu$ is not supercuspidal. Let $W$ be a subrepresentation of $\text{Ind}_B^G \mu$ and $W'$ an irreducible quotient of $W$. If $W'$ is supercuspidal then by Corollary 2.36 we have a splitting $W' \to W$, so we can see $W'$ as an irreducible supercuspidal subrepresentation of $\text{Ind}_B^G \mu$. But this contradicts Theorem 2.18!

Now consider a finite chain $0 \subset W_1 \subset \cdots \subset W_k = \text{Ind}_B^G \mu$ of representations of $G$. By Zorn’s lemma any quotient $W_i/W_{i-1}$ admits an irreducible subquotient, which is also a subquotient of $V$ and so is not supercuspidal. By exactness of
the Jacquet functor this implies that each \( \text{Res}_B(W_i/W_{i-1}) \neq 0 \). Since we have computed that the representation \( \text{Res}_B \text{Ind}^T_B \mu \) of \( T \) has length 2, we deduce \( k \leq 2 \), in particular \( \text{Ind}^T_B \mu \) has finite length. We may then assume that each constituent \( W_i/W_{i-1} \) is irreducible. We also get that each \( W_i/W_{i-1} \) is not supercuspidal, so that \( \text{Res}_B(W_i/W_{i-1}) \) is either one-dimensional (with action of \( T \) by \( \mu \) or \( \mu^w \)) or two-dimensional equal to \( \text{Res}_B \text{Ind}^T_B \mu \). The latter case occurs if and only if \( \text{Ind}^T_B \mu \) is irreducible.

2.6. Interlude: integration on \( p \)-adic manifolds. We sketch the foundations of \( p \)-adic manifolds (sometimes called \( p \)-adic analytic manifolds) in order to state the “change of variables” formula for measures associated to differential forms (introduced in [Wei82]) that will be useful for intertwining operators and harmonic analysis. We emphasize differences with the Archimedean case. For details (and proper foundations) see [Ser06], [Sch11].

Let \( n \geq 1 \) be an integer. Let \( U \) be an open subset of \( \mathbb{Q}_p^n \), and \( f: U \to \mathbb{Q}_p \) a function. We say that \( f \) is locally analytic at \( x_0 \in U \) if there is a family \( (a_\alpha)_{\alpha \in \mathbb{N}} \) of elements of \( \mathbb{Q}_p \) and \( r > 0 \) such that \(|a_\alpha| r^{-|\alpha|} \) is bounded (notation: \(|\alpha| = \sum \alpha_i\)) and for any \( x \in U \cap D(x_0, r) \) (open disk of radius \( r \)) we have \( f(x) = \sum a_\alpha (x - x_0)^\alpha \) (notation: \( z^\alpha = \prod z_i^{\alpha_i} \)). The same proof as in the complex setting shows that \( f \) is then continuous on \( U \cap D(x_0, r) \) and locally analytic at any point of \( U \cap D(x_0, r) \). The main difference from the complex setting is that any \( U \) is totally disconnected: for example, for any function \( \mathbb{F}_p \to \mathbb{Q}_p \), the composition \( U := \mathbb{Z}_p \to \mathbb{F}_p \to \mathbb{Q}_p \) is locally analytic (even locally constant!) at every point of \( U \).

We have the notion of a locally analytic function \( U \to \mathbb{Q}_p^m \) (coordinate-wise), and the composition of two locally analytic functions is again locally analytic. A locally analytic function is differentiable (obvious definition . . . ) and its differential (taking values in \( \text{Hom}_{\mathbb{Q}_p}(\mathbb{Q}_p^n, \mathbb{Q}_p^m) \simeq \mathbb{Q}_p^{mn} \)) is again locally analytic.

**Theorem 2.38** (Inverse function theorem). Let \( U \) be an open subset of \( \mathbb{Q}_p^n \) and \( x_0 \in U \). Let \( f = (f_1, \ldots, f_n): U \to \mathbb{Q}_p^n \) be a locally analytic function. Assume that the differential of \( f \) at \( x_0 \) is invertible. Then up to replacing \( U \) by an open subset containing \( x_0 \), \( f(U) \) is open in \( \mathbb{Q}_p^n \), \( f \) is injective and its inverse \( f(U) \to U \) is also locally analytic.

**Proof.** The proof is similar to the complex case, only easier because convergence is cheaper in a non-Archimedean setting. Using the usual reductions (translations so that \( x_0 = 0 \) and \( f(x_0) = 0 \), post-composing \( f \) with the inverse of its differential, pre- and post-composing \( f \) with homotheties) we may assume that \( U = \mathbb{Z}_p^n \) and \( f_i(x) = x_i + \sum a_{i,\alpha} x^\alpha \) with \( a_{i,\alpha} \in p\mathbb{Z}_p \), vanishing for \(|\alpha| < 2 \). Seek for \( g \) satisfying the same conditions: \( g_i(y) = y_i + \sum b_{i,\beta} y^\beta \). Solving the equation of formal power series \( f \circ g = \text{Id} \), we see that there is a unique solution. More precisely, by induction on \(|\beta| \) we see that \( b_{i,\beta} = P_{i,\beta}(a_{j,\alpha}, |\alpha| \leq |\beta|) \) where \( P_{i,\beta} \) is a polynomial with coefficients in \( \mathbb{Z}_p \) and vanishing constant term.

Reversing the role of \( f \) and \( g \), we get that they are inverse maps of each other \( \mathbb{Z}_p^n \to \mathbb{Z}_p^n \). □

This local theory allows to define \( p \)-adic manifolds, obtained by gluing open subsets of \( \mathbb{Q}_p^n \) (or \( \mathbb{Z}_p^n \)) using locally analytic maps to change coordinates. More precisely, if \( X \) is a topological space:
• a chart on \( X \) is an open subset \( U \) of \( X \) together with a homeomorphism \( \phi : U \to \phi(U) \) where \( \phi(U) \) is an open subset of \( \mathbb{Q}_p^n \) for some \( n \),

• an atlas on \( X \) is a family of charts covering \( X \) which are pairwise compatible, i.e. such that the transition maps \( \phi' \circ \phi^{-1} : \phi(U \cap U') \to \phi'(U \cap U') \) are locally analytic (exchanging \( U \) and \( U' \) we get that \( n = n' \) if \( U \cap U' \neq \emptyset \)).

We say that two atlases are compatible if their charts are pairwise compatible. This is an equivalence relation, and we get the notion of a \( p \)-adic manifold: a topological space \( X \) with an equivalence class of atlases on \( X \), or equivalently a maximal atlas. Note that the dimension \( n \) is a locally constant function on \( X \). We will only consider \( p \)-adic manifolds of constant dimension.

Clearly a \( p \)-adic manifold is locally compact (if one uses a definition that does not include “Hausdorff”). All examples that we will encounter will also be Hausdorff and paracompact (i.e. every open cover has a refinement that is locally finite; this condition holds if \( X \) is a countable union of compact subsets).

We have the obvious notion of morphism between \( p \)-adic manifolds: continuous maps which are locally analytic in local coordinates given by charts. As in the Archimedean case one can define tangent and cotangent bundles, and tensor, symmetric and exterior powers of these bundles, for example differential \( k \)-forms.

The differential of a morphism is a morphism between tangent bundles. Fibers of a submersion are also \( p \)-adic manifolds (use the inverse function theorem).

Example: for any smooth algebraic variety \( X \) over \( \mathbb{Q}_p \), \( X(\mathbb{Q}_p) \) is naturally endowed with the structure of a \( p \)-adic manifold. This is the case for \( G, B, T, N \), and the group structure is compatible, i.e. multiplication and inversion are morphisms of \( p \)-adic manifolds, so these are \( p \)-adic Lie groups. Of course open subgroups of these, in particular compact open subgroups, are also \( p \)-adic Lie groups.

One may also define submanifolds and quotients of manifolds as in the Archimedean case. If an equivalence relation \( R \subset X \times X \) is a submanifold then the quotient manifold exists (without assuming that \( R \) is a submanifold, there is at most one manifold structure on the quotient such that the projection \( X \to X/R \) is a submersion).

It is easy to see that any compact open subset of \( \mathbb{Q}_p^n \) is a disjoint union of balls, which are isomorphic to \( \mathbb{Z}_p^n \). With this observation and a little argument, one may deduce the non-trivial direction in the following theorem, which is the analogue of the existence of partitions of unity in the Archimedean setting, except much stronger.

**Theorem 2.39.** Let \( X \) be a \( p \)-adic manifold. Assume that \( X \) is Hausdorff. The following are equivalent.

1. \( X \) is paracompact.
2. \( X \) is isomorphic to a disjoint union of balls.

Moreover in (2) the balls can be chosen to refine any given cover of \( X \) by open subsets.

**Proof.** See [Ser06, Part II, Chapter III, Appendix 2, Theorem 1]. \( \square \)
Partitions of unity are the essential technical tool to setup the theory of integration of top degree differential forms on real manifolds. We want to mimic this theory in the non-Archimedean setting, but we are interested in integrating complex-valued functions, whereas differential forms have \( p \)-adic coefficients. So we consider the “norm” of differential forms. Give \( \mathbb{Q}_p \) the Haar measure such that \( \text{vol}(\mathbb{Z}_p^n) = 1 \), and give \( \mathbb{Q}_n \) the product measure, that we denote \( |dx| \cdot |dx_1| \cdot \ldots \cdot |dx_n| \) where \( \psi \) is a locally analytic function, then we may consider the Radon measure \( |\omega| \): for any continuous compactly supported continuous function \( f : U \to \mathbb{C} \),

\[
\int_U f |\omega| := \int_U f(x)|\psi(x)||dx_1| \ldots |dx_n|.
\]

As in the real case one can check that this definition is intrinsic, i.e. is invariant under locally analytic isomorphisms between open subsets of \( \mathbb{Q}_n \). The same kind of arguments are used to prove the following “change of variables formula”, that we state in a slightly generalized form for future use.

**Theorem 2.40.** Let \( \phi : X \to Y \) be a morphism between \( p \)-adic manifolds of constant dimensions such that the differential of \( \phi \) is everywhere invertible (in particular, \( \dim X = \dim Y \)). Assume that the fibers of \( \phi \) have bounded cardinality, and denote \( c_\phi : Y \to \mathbb{Z}_{\geq 0}, \ y \mapsto \text{card}(\phi^{-1}(\{y\})) \). Then for any differential form \( \omega \) on \( Y \) and any function \( f : Y \to \mathbb{C} \) that is integrable with respect to \( |\omega| \), we have

\[
\int_X f \circ \phi |\phi^* \omega| = \int_Y f c_\phi |\omega|.
\]

**Proof.** The proof is similar to the real case, but easier. We only sketch the proof. First, since we are comparing Radon measures, we can reduce to the case where \( f \) is continuous and compactly supported, and even smooth and compactly supported. So we may assume that \( Y \) is compact and \( f \) is constant equal to 1. Using (the proof of) Theorem 2.38 and Theorem 2.39 (giving locally constant partitions of unity!) we can reduce to the following cases:

- \( \phi \) is the restriction to \( p^n \mathbb{Z}_p^n \) of an invertible linear map, or
- \( \phi \) is a homeomorphism between \( \mathbb{Z}_p^n \) and \( \mathbb{Z}_p^n \) given by power series, such that at any point the differential of \( \phi \) belongs to \( 1 + pM_\infty(\mathbb{Z}_p) \).

The second case is trivial. The first case follows from the following lemma. \( \square \)

**Lemma 2.41.** Let \( g \in \text{GL}_n(\mathbb{Q}_p) \), then \( \text{vol}(g(p^n \mathbb{Z}_p^n)) = |\det g| p^{-an} \).

**Proof.** We only sketch the proof. Use the Iwasawa decomposition for \( \text{GL}_n(\mathbb{Q}_p) \) to reduce to the case where \( g \) is upper triangular, then use Fubini to compute the volume. The invariance by translation of the Haar measure on \( \mathbb{Q}_p \) implies that the volume only depends on the diagonal of \( g \). The case \( n = 1 \) is elementary. \( \square \)

**Example 2.42.** On \( G \), denoting \( x = (x_{i,j})_{1 \leq i,j \leq 2} \in G \), the differential form \( \omega = \det(x)^{-2} \wedge_{1 \leq i,j \leq 2} x_{i,j} \) (choose an arbitrary order to take wedges) is both left- and right-invariant (exercise). This gives a “differential” definition of the Haar measure on \( G \), and shows that it is unimodular (so is an arbitrary \( p \)-adic reductive group).
2.7. Reducibility of parabolically induced representations in the non-unitary case: intertwining operators. It remains to look at the case where $\mu_1/\mu_2$ is not unitary, in particular $\mu^w \neq \mu$. Frobenius reciprocity tells us that the representation is indecomposable, so if the induced representation is not irreducible Corollary 2.37 tells us that it has a unique irreducible representation and a unique irreducible quotient.

Note that by Theorem 2.18 and our computation of $\text{Res}_B \text{Ind}_B^G \mu$ we have

$$\dim_C \text{Hom}_G(\text{Ind}_B^G \mu, \text{Ind}_B^G \mu^w) = 1.$$  

We will construct a basis of this space, i.e. a non-zero intertwining operator, more explicitly. Start with $f \in \text{Ind}_B^G \mu$. To produce a function which transform under left action of $B$ by $\mu^w$, in particular left invariant under $N$, it is natural to consider the integral

$$g \mapsto \int_N f(wng)dn \tag{2.1}$$

since for any $t \in T$ we have

$$\int_N f(wntg)dn = \int_N f(wt^{-1}wt^{-1}ntg)dn$$

$$= \int_N \mu^w(t)\delta_B^w(t)^{1/2} f(wt^{-1}ntg)dn$$

$$= \mu^w(t)\delta_B^{1/2}(t) \int_N f(wn'g)dn'$$

where we let $n' = t^{-1}nt$ so that $dn = \delta_B(t)dn'$.

**Lemma 2.43.** If $|\mu_1(p)/\mu_2(p)| < 1$, the integral (2.1) converges absolutely.

**Proof.** Write $n = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$ with $y \in \mathbb{Q}_p$. We have $wnw^{-1} = \begin{pmatrix} 1 & 0 \\ -y & 1 \end{pmatrix}$. We have to show that

$$\sum_{k \geq 0} \int_{p^{-k}\mathbb{Z}_p^\times} \left| f\left( \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} w g \right) \right| |dy| < +\infty. \tag{2.2}$$

As in Lemma 2.20 we write $\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} y^{-1} & 1 \\ 0 & y \end{pmatrix} w \begin{pmatrix} 1 & y^{-1} \\ 0 & 1 \end{pmatrix}$ and for $k_0$ large enough $f$ is constant on $w\begin{pmatrix} 1 & p^{k_0}\mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$ so (2.2) is equivalent to the convergence of

$$\sum_{k \geq k_0} \int_{p^{-k}\mathbb{Z}_p^\times} \left| \mu_1(y)^{-1}\mu_2(y) |y|^{-1} f(w^2g) \right| |dy| = \sum_{k \geq k_0} |\mu_1(p)/\mu_2(p)|^k (1 - 1/p) |f(-g)|.$$

\[\square\]
Removing absolute values in the integral in the proof of the lemma, we also see that if $|\mu_1(p)/\mu_2(p)| < 1$, for $k_0 >>> 0$ (depending on $f$ and a compact subset of $G$ in which $g$ lies) we have

$$
\int_N f(wng)dn = \int_{p^{1-k_0}Z_p} f \left( \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} wg \right) |dy|
+ \frac{(\mu_1/\mu_2)(p)^{k_0}}{1 - (\mu_1/\mu_2)(p)} \int_{Z_p^\times} \mu_1(y)^{-1}\mu_2(y) |dy| f(-g).
$$

Moreover

$$
\int_{Z_p^\times} \mu_1(y)^{-1}\mu_2(y) |dy| = \begin{cases} 1 - 1/p & \text{if } (\mu_1/\mu_2)|_{Z_p^\times} = 1 \\ 0 & \text{otherwise.} \end{cases}
$$

These formulas motivate the following definition which “removes denominators”. Assuming that $|(\mu_1/\mu_2)(p)| < 1$, let

$$(2.3) \quad J_\mu(f)(g) := L(\mu_1/\mu_2)^{-1} \times \int_N f(wng)dn$$

with $L(\mu_1/\mu_2)^{-1} := \begin{cases} 1 - (\mu_1/\mu_2)(p) & \text{if } (\mu_1/\mu_2)|_{Z_p^\times} = 1 \\ 1 & \text{otherwise.} \end{cases}$

Then $J_\mu : Ind_B^G \mu \to Ind_B^G \mu^w$ is an intertwining operator.

To get rid of the assumption $|(\mu_1/\mu_2)(p)| < 1$ we now interpolate (algebraically) induced representations. Because of the decomposition $G = BK_0$ we can identify $Ind_B^G \mu$ with the space of functions $f : K_0 \to \mathbb{C}$ such that $f(tnx) = \mu(t)f(x)$ for any $n \in N_0$, $t \in T_0$ and $x \in K_0$. Note that this space only depends on $\mu|_{T_0}$, but that the action of $G$ really depends on $\mu$.

We now replace the coefficient field $\mathbb{C}$ by $A := \mathbb{C}[X_1^\pm, X_2^\pm]$, and consider the character $\mu_X : T \to A^\times, t \mapsto \mu_1(t_1)X_1^{v_p(t_1)}\mu_2(t_2)X_2^{v_p(t_2)}$. Let $f_X \in Ind_B^G A(\mu_X)$ be the unique function such that $f_X|_K = f|_K$ (a somewhat arbitrary smooth interpolation of $f$).

Define

$$
J_{\mu_X}(f_X)(g) = L(\mu_X, 1/\mu_X, 2)^{-1} \int_{p^{1-k_0}Z_p} f_X \left( \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} wg \right) |dy|
+ \frac{(\mu_1/\mu_2)(p)X_1/X_2)^{k_0}}{1 - (\mu_1/\mu_2)(p)} \int_{Z_p^\times} \mu_1(y)^{-1}\mu_2(y) |dy| f_X(-g)
$$

with $L(\mu_X, 1/\mu_X, 2)^{-1} = 1 - \mu_1(p)X_1\mu_2(p)^{-1}X_2^{-1}$ or 1 as above. The same computation as above shows that this does not depend on $k_0 >>> 0$. Finally define $J_\mu(f)$ as the specialization at $X_1 = X_2 = 1$. It is clear that $J_{\mu_X}$ and $J_\mu$ are $G$-equivariant, and that $J_{\mu_X}(f_X)$ is a smooth function $G \to A$. It is not quite as obvious that $J_\mu$ maps $Ind_B^G \mu$ to $Ind_B^G \mu^w$. Although this could certainly be verified by direct computation, it simply follows from smoothness of $J_{\mu_X}(f_X)$ and the fact that for a Zariski-dense set of specializations of $(X_1, X_2)$ it belongs to $Ind_B^G A(\mu_X)$. 

Proposition 2.44. (1) In the case $\mu|_{T_0} = 1$, let $f_\mu \in \text{Ind}_B^G \mu$ be the unique function such that $f|_{K_0} = 1$. Then $J_\mu(f_\mu) = (1 - p^{-1}(\mu_1/\mu_2)(p))f_{\mu^w}$.

(2) In any case the intertwining operator $J_\mu : \text{Ind}_B^G \mu \to \text{Ind}_B^G \mu^w$ is non-zero.

Proof. (1) For $g \in K_0$ the above formula holds for $k_0 = 0$ or $k_0 = 1$, and we obtain the formula.

(2) If $\mu_1\mu_2^{-1}|_{\mathbb{Q}_p^\times} = 1$ the up to twisting $\mu_1$ and $\mu_2$ by the same smooth character of $\mathbb{Q}_p^\times$ we may assume that $\mu|_{T_0} = 1$, and if moreover $(\mu_1/\mu_2)(p) \neq p$, we are done by the previous point.

Otherwise take $f$ supported on $B\left(\frac{1}{\mathbb{Z}_p}, 0\right)$ and constant non-zero on $\left(\frac{1}{\mathbb{Z}_p}, 1\right)$. Evaluate at $g = w^{-1}$, then the above formula holds with $k_0 = 1$ and the second term vanishes while the first one does not (one can also go back to the original integral and observe that the integrand is compactly supported).

Lemma 2.45. Assume that $\mu \neq \mu^w$. Then the representation $\text{Ind}_B^G \mu$ is reducible (i.e. has length 2 by Corollary 2.37) if and only if $J_{\mu^w} \circ J_\mu = 0$.

Proof. Assume first that $\text{Ind}_B^G \mu$ has length two, then it is not semisimple since we have computed $\text{End}_G(\text{Ind}_B^G \mu) = \mathbb{C}$, and so it has a unique irreducible quotient $Q_\mu$, and a unique irreducible subrepresentation $S_\mu$. By the geometric lemma, $\text{Res}_B Q_\mu \in \{\mu, \mu^w\}$, and $\text{Res}_B S_\mu$ is the other character in this set. If $\text{Res}_B Q_\mu = \mu$ then $Q_\mu$ embeds in $\text{Ind}_B^G \mu$ (see Corollary 2.19), but this implies $Q_\mu \simeq S_\mu$, which is a contradiction since they have distinct Jacquet modules. So $Q_\mu$ embeds in $\text{Ind}_B^G \mu^w$, and by comparing Jacquet modules we see that $\text{Ind}_B^G \mu^w$ is not irreducible either, so it also has length 2 and $S_{\mu^w} \simeq Q_\mu$, and by symmetry $Q_{\mu^w} \simeq S_\mu$. Now it is clear that the image of $\text{ker} J_\mu = S_\mu$ and $\text{im} J_\mu = S_{\mu^w}$, so that $J_{\mu^w} \circ J_\mu = 0$.

Conversely, if $J_{\mu^w} \circ J_\mu = 0$ then since both $J_\mu$ and $J_{\mu^w}$ are non-zero we get that at least one of $\text{Ind}_B^G \mu$ or $\text{Ind}_B^G \mu^w$ is not irreducible, and the previous argument shows that in fact both are reducible.

Lemma 2.46. The composition of intertwining operators $J_{\mu^w} \circ J_\mu$ is a scalar.

Proof. This follows from our computation $\text{End}_G(\text{Ind}_B^G \mu) = \mathbb{C}$, thanks to the geometric lemma.

Corollary 2.47. If $\mu_1/\mu_2|_{\mathbb{Z}_p^\times} = 1$ then $\text{Ind}_B^G \mu$ is reducible if and only if $(\mu_1/\mu_2)(p) \in \{p, p^{-1}\}$.

Proof. Up to twisting $\mu_1$ and $\mu_2$ by the same character, we may assume that $\mu|_{T_0} = 1$. By Lemma 2.46 and the first point of Proposition 2.44, $J_{\mu^w} \circ J_\mu = (1 - p^{-1}(\mu_1/\mu_2)(p))(1 - p^{-1}(\mu_2/\mu_1)(p))\text{Id}_{\text{Ind}_B^G \mu}$, and by Lemma 2.45 the statement follows.

In the ramified case, we are left with a computation.
Proposition 2.48. Assume that $\mu_1/\mu_2\mid_{\mathbb{Z}_p^\times} \neq 1$. Let $r \geq 1$ be the smallest integer satisfying $\mu_1/\mu_2|_{1+p^r\mathbb{Z}_p} = 1$. Then $J_{\mu^w} \circ J_{\mu} = p^{-r}(\mu_1/\mu_2)(-1)$. In particular, $\text{Ind}_B^G \mu$ is irreducible.

Proof. By Lemma 2.46 it is enough to compute $J_{\mu^w} \circ J_{\mu}$ on some non-zero $f \in \text{Ind}_B^G \mu$. Again take $f$ supported on $B \left( \frac{1}{\mathbb{Z}_p}, 0 \right)$, constant on $\left( \frac{1}{\mathbb{Z}_p}, 0 \right)$ with $f(1) = 1$. Since the second term in the formula defining $J_{\mu}(f)$ vanishes in the ramified case, we "simply" have

\[ (J_{\mu^w} \circ J_{\mu})(f)(g) = \int_{p^{1-k_0}\mathbb{Z}_p} J_{\mu^w}(f) \left( \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} w \right) \left| dz \right| = \int_{p^{1-k_0}\mathbb{Z}_p} \int_{p^{1-k_1}\mathbb{Z}_p} f \left( \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} w \left( \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} w \right) \right) \left| dy \right| \left| dz \right|. \]

for any large enough integer $k_0$, and any large enough integer $k_1$ (which may be taken uniformly on $z \in p^{1-k_0}\mathbb{Z}_p$ since this set is compact, but note that $k_1$ depends on $k_0$). We may and do assume that $k_1 > 0$. We compute

\[ \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} w \left( \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix} w \right) = \begin{pmatrix} -1 & z \\ -y & yz - 1 \end{pmatrix} \]

and since it is enough to compute at $g = 1$, we want to write this matrix (assuming that it lies in the support of $f$) as

\[ \begin{pmatrix} c^{-1} & b \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} = \begin{pmatrix} c^{-1} + bu & b \\ cu & c \end{pmatrix}. \]

We write the above integral in terms of the new variables $(c, u)$. We may restrict to $y \neq 0$ and so $u \neq 0$, since the subvariety corresponding to $y = 0$ has measure 0. Then the change of variable is well-defined $(y = -cu$ and $z = -(1 + c^{-1})u^{-1}$) and injective, and we will see that it has everywhere invertible differential as well.

Let us first compute the Jacobian of the change of variables. We find

\[ dy = -udc - cdw, \quad dz = \frac{dc}{c^2u} + \frac{(1 + c)du}{cu^2}, \quad dy \wedge dz = \frac{du \wedge dc}{u}. \]

We have to determine the set $S$ of pairs $(c, u) \in \mathbb{Q}_p^\times \times (\mathbb{Z}_p \setminus \{0\})$ mapping to $(y, z) \in (p^{1-k_0}\mathbb{Z}_p)^2$, before we can compute $\int_S \mu_1(c)^{-1}\mu_2(c)|c|^{-1}|u|^{-1}du\left|dc\right|$. This condition on $(y, z)$ is equivalent to $(A)$ $v(cu) \geq 1 - k_1$ and $(B)$ $v(1 + c^{-1}) \geq 1 - k_0 + v(u)$. For a given $u \in \mathbb{Z}_p \setminus \{0\}$ we consider the set of $c \in \mathbb{Q}_p^\times$ such that these two conditions are satisfied.

- If $0 \leq v(u) \leq k_0 - 1$ then condition (B) is equivalent to $v(c^{-1}) \geq v(u) + 1 - k_0$, so that conditions (A) and (B) together are equivalent to $k_0 - 1 - v(u) \geq v(c) \geq 1 - k_1 - v(u).$ For any $k \in \mathbb{Z}$ we have $\int_{p^k\mathbb{Z}_p} \mu_1(c)^{-1}\mu_2(c)|c|^{-1}|dc| = 0.$

- If $v(u) > k_0 - 1$ then condition (B) is equivalent to $c \in -1 + p^{v(u) + 1 - k_0}\mathbb{Z}_p,$ and so condition (A) reads $v(u) \geq 1 - k_1$, which is automatically satisfied since $k_1 > 0$. We have

\[ \int_{-1 + p^{v(u) + 1 - k_0}\mathbb{Z}_p} \mu_1(c)^{-1}\mu_2(c)|dc| = \begin{cases} 0 & \text{if } v(u) < k_0 - 1 + r \\
p^{k_0 - 1 - v(u)}(\mu_1/\mu_2)(-1) & \text{if } v(u) \geq k_0 - 1 + r. \end{cases} \]
Thus
\[
\int_S \mu_1(c)^{-1}\mu_2(c)|c|^{-1}|u|^{-1}|du||dc| = \int_{|u| \geq k_0^{-1}+r} (\mu_1/\mu_2)(-1)p^{k_0-1} |du| = p^{-r}(\mu_1/\mu_2)(-1).
\]

We can finally state the classification of non-supercuspidal representations of $G$.

**Theorem 2.49.** Any irreducible smooth non-supercuspidal representation of $G$ is isomorphic to exactly one of the following:

- $\text{Ind}_B^G \mu$ for $\mu_1/\mu_2 \notin \{|\cdot|^{\pm 1}\}$, with $\{\mu,\mu^w\}$ uniquely determined,
- $\chi \circ \det$ for some continuous character $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$,
- $(\chi \circ \det) \otimes \text{St}$ for some continuous character $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$.

**Proof.** Existence of an isomorphism was proved above, and uniqueness follows from consideration of the Jacquet module. 

**Remark 2.50.** We can define the local Langlands correspondence in an ad hoc manner for principal series. One of the main results of local class field theory is the existence of a natural isomorphism $\text{rec} : (W_{\mathbb{Q}_p})^{ab} \cong \mathbb{Q}_p^\times$ where $W_{\mathbb{Q}_p} \subset \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ is the Weil group of $\mathbb{Q}_p$. Therefore, it is natural to declare that an irreducible $\text{Ind}_B^G \mu$ corresponds to the reducible semi-simple two-dimensional representation $\mu_1 \circ \text{rec} \oplus \mu_2 \circ \text{rec}$. It is not as clear what one should do for the one dimensional and Steinberg representations. It turns out that it is natural to associate $(\chi \cdot |\cdot|^{1/2}) \circ \text{rec} \oplus (\chi \cdot |\cdot|^{-1/2}) \circ \text{rec}$. Later we will prove that the Steinberg representation is square-integrable, although it is clearly not supercuspidal. It turns out that it is natural to introduce the Weil-Deligne group $W_{\mathbb{Q}_p} := W_{\mathbb{Q}_p} \times \text{SL}_2(\mathbb{C})$. Of course the above representations of $W_{\mathbb{Q}_p}$ can simply be seen as representations of $W_{\mathbb{Q}_p}$ which are trivial on the second factor. The Langlands parameter of $\chi \circ \det \otimes \text{St}$ is $\chi \circ \text{rec} \otimes \nu_2$ where $\nu_2$ is the irreducible algebraic 2-dimensional representation of $\text{SL}_2(\mathbb{C})$. More generally, to formulate the local Langlands correspondence for $\text{GL}_n(F)$, $F$ a non-Archimedean local fields, one should consider n-dimensional semi-simple continuous representations of $W_{\mathbb{Q}_p}$ which are algebraic (i.e. polynomial or equivalently, holomorphic) on the factor $\text{SL}_2(\mathbb{C})$.

**2.8. The Iwahori-Hecke algebra and the Steinberg representation.** Recall that the Iwahori subgroup $I$ of $K_0$ is the preimage under $K_0 \to \text{GL}_2(\mathbb{F}_p)$ of the upper-triangular Borel subgroup. We now study the Iwahori-Hecke algebra $\mathcal{H}(G,I)$, which will be useful to study the Steinberg representation because we will see that $\text{St}^I \neq 0$.

Denote $T_0 = T \cap K_0$. In particular $T_0 \subset I$. Let $\widehat{W} = N_G(T)/T_0$ be the extended affine Weyl group, it surjects onto $W = N_G(T)/T = \{1,w\}$, with kernel $T/T_0 \cong \mathbb{Z}^2$.

The natural map $N_K(T_0)/T_0 \to W$ is an isomorphism and gives a splitting of $\widehat{W} \to W$, realizing $\widehat{W}$ as $T/T_0 \rtimes W$.

**Proposition 2.51 (Affine Bruhat decomposition).** $G = \bigsqcup_{x \in \widehat{W}}IxI$. 

Proof. First note that $G/I$ parametrizes pairs $(L, D)$ where $L$ is a lattice in $\mathbb{Q}_p^2$ and $D \subset L/pL$ is an $\mathbb{F}_p$-line: the coset $gI$ corresponds to $(\mathbb{Z}_p e_1 \oplus \mathbb{Z}_p e_2, \langle e_1 \rangle)$ where $e_1, e_2$ are the columns of $g$.

So we have to show that for any $(L_1, D_1)$ and $(L_2, D_2)$ as above, there exists a basis $(e, f)$ of $L_1$ and $a, b \in \mathbb{Z}$ such that $D_1 = \langle e \rangle$, $(p^a e, p^b f)$ is a basis of $L_2$ and $D_2 = \langle p^a e_1 \rangle$ or $\langle p^b e_2 \rangle$, and that the pair $(a, b) \in \mathbb{Z}^2$ is unique (this equivalent statement of uniqueness requires a bit of head scratching ...). Thanks to the Cartan decomposition we know that there is a basis $(e, f)$ of $L_1$ and (unique) integers $a \geq b$ such that $(p^a e, p^b f)$ is a basis of $L_2$. It is clear that we may substitute $e + \mu f$ for $e$, for any $\mu \in \mathbb{Z}_p$. Since any line in $L_1/pL_1$ is generated by $f$ or by $e + \mu f$ for some $\mu \in \mathbb{F}_p$, we obtain that up to swapping $e$ and $f$ (which does not preserve the condition $a \geq b$) we may assume that $D_1 = \langle e_1 \rangle$.

- If $b \leq a$, we may substitute $f + p^{a-b} \mu e$ for $f$ where $\mu \in \mathbb{Z}_p$ is arbitrary, since $p^b (f + p^{a-b} \mu e) = p^b f + p^a \mu e$. By the same argument as above, if $D_2 \neq \langle p^a e \rangle$ we can reduce to $D_2 = \langle p^b f \rangle$.

- If $b > a$, we may substitute $e + p^{b-a} \mu f$ for $e$ where $\mu \in \mathbb{Z}_p$ is arbitrary, and as in the previous case this allows us to achieve $D_2 = \langle p^a e \rangle$ if $D_2 \neq \langle p^b f \rangle$.

Uniqueness can be seen on this argument and is left as an exercise. □

We will use the following more precise description of the double cosets contained in $K_0$.

**Proposition 2.52.** We have a decomposition

$$K_0 = I \sqcup I w I = I \sqcup \bigsqcup_{y \in \mathbb{F}_p} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} w I.$$

Note that the last term makes sense!

**Proof.** Observe that $K_1 \subset I \subset K_0$ with $K_1$ distinguished in $K_0$ and $K_0/K_1 = \text{GL}_2(\mathbb{F}_p)$. The assertion follows immediately from the Bruhat decomposition in $\text{GL}_2(\mathbb{F}_p)$. □

Note that $I x I = \bigsqcup_k k x I$ where $k$ ranges over representatives of $I/I \cap x I x^{-1}$. In particular $\text{vol}(I x I) = |I/I \cap x I x^{-1}| \text{vol}(I)$. If $x = \text{diag}(p^a, p^b)$ then $x I x^{-1} = \left( \begin{array}{cc} \mathbb{Z}_p^\times & p^{-b} \mathbb{Z}_p^\times \\ p^{a-b} \mathbb{Z}_p & \mathbb{Z}_p^\times \end{array} \right)$ and $I \cap x I x^{-1} = \left( \begin{array}{cc} \mathbb{Z}_p^\times & p^{\max(0, a-b)} \mathbb{Z}_p^\times \\ p^{1+\max(b-a, 0)} \mathbb{Z}_p & \mathbb{Z}_p^\times \end{array} \right)$. We easily deduce $|I/I \cap x I x^{-1}| = p^{a-b}$. 

Before we consider elements in $\widetilde{W} \setminus T/T_0$, define $\tilde{w} = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix}$. This element is interesting because $\tilde{w} I \tilde{w}^{-1} = I$. Modulo $T_0$ we have $x := \text{diag}(p^a, p^b)w = \text{diag}(p^a, p^{b-1}) \tilde{w}$ and so $|I/I \cap x I x^{-1}| = p^{a-b+1}$.

**Definition 2.53.** Let $l : \widetilde{W} \to \mathbb{Z}_{\geq 0}$ be the length function on $\tilde{w}$, defined by $|I/I \cap x I x^{-1}| = p^{l(x)}$. 

Denote by $[IxI] \in \mathcal{H}(G, I)$ the element supported on $IxI$ mapping $x$ to $\text{vol}(I)$. Then
\begin{equation}
[IxI][IyI] = \text{vol}(I)^{-1} \sum_{k \in I/I \cap Ix^{-1}k' \in I/I \cap IyI^{-1}} 1_{kzk'yI}
\end{equation}
and we see that $[IxI][IyI]$ is a finite sum of finitely many $[IzI]$, including $z = xy$ at least once. In particular $[IxI][IyI] - [IxyI]$ is a non-negative function. This shows that if $\text{vol}(I) \text{vol}(IyI) = \text{vol}(IxyI)$ then $[IxI][IyI] = [IxI]$. We will use this observation to prove the following description of $\mathcal{H}(G, I)$.

**Proposition 2.54.** Let $S = [IwI]$ and $T = [IwI]$. Then $T$ is invertible and $S, T^{\pm 1}$ generate the algebra $\mathcal{H}(G, I)$ and satisfy the following relations: $T^2S = ST^2$, $(S - p)(S + 1) = 0$.

**Proof.** Since $\tilde{w}$ normalizes $I$ we have $T^2 = [I\tilde{w}^2I]$ and $\tilde{w}^2 = \text{diag}(p, p)$ is central in $G$. This shows both that $T^2$ is central in $\mathcal{H}(G, I)$ and that $T$ is invertible in $G$, with $T^{-1} = T[I\text{diag}(p^{-1}, p^{-1})I]$. To compute $S^2$ we use formula (2.4), and the elementary observation that $S^2$, like $S$, is supported on the subgroup $K_0 = I \cap IwI$.

The elements $\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$, for $y \in \mathbb{Z}_p$ representing $\mathbb{F}_p$, are representatives of $I/I \cap IwI$ (see Proposition 2.52), and
\[
\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} w \begin{pmatrix} 1 & y' \\ 0 & 1 \end{pmatrix} w = - \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -y' & 1 \end{pmatrix} = - \begin{pmatrix} 1 & y'y' \\ -y' & 1 \end{pmatrix}
\]
and we get that $S^2 = peI + \lambda S$ ($p$ corresponds to pairs $(y, y')$ with $y' = 0$) for some integer $\lambda \geq 0$. Integrating over $G$ (i.e. considering the action in the trivial representation ...), we obtain $p^2 = p + p\lambda$, and so $\lambda = p - 1$ and $S^2 = (p - 1)S + peI$.

We now prove a stronger assertion than required, namely that any $[IxI]$ can be written as a product of $T^{\pm 1}$’s and $S$’s, by induction on $l(x)$. It is clear that the elements of length 0 are exactly $Z \sqcup Z\tilde{w}$, and these are powers of $T$. If $x \in \tilde{W}$ is such that $l(x) > 0$, we distinguish two cases.

- If $x = \text{diag}(p^a, pb)$ then $a \neq b$. If $a < b$ then $l(xw) = b - a = l(x) + 1$, so that $l(x) = l(xw^{-1}) + l(w)$. This implies $[IxI] = [IxwI]S$. If $a > b$ then $l(wx) = a - b = l(x) - 1$ and similarly $[IxI] = S[IwxI]$.
- If $x \in \tilde{W} \backslash T/T_0$ we may multiply $x$ by $\tilde{w}$ to reduce to the previous case.

**Remark 2.55.** This proposition holds with coefficients $\mathbb{Z}$ instead of $\mathbb{C}$. Although we will not need it, one can show that the proposition gives a full presentation of $\mathcal{H}(G, I)$. For generalizations see the original paper [IM65], and a more modern exposition [HKP10] using a different approach and including many other results, such as Bernstein’s presentation.

Now consider the $\mathcal{H}(G, I)$-modules $(\text{Ind}_B^G \mu)^I$. We see functions in $\text{Ind}_B^G \mu$ as functions on $K_0$. 

\[\Box\]
From the decomposition $K_0 = I \sqcup N_0 w I$ (i.e. the Bruhat decomposition for $GL_2(\mathbb{F}_p)$) and the fact that $T_0 \subset I$ is normalized by $w$, it follows that $(\text{Ind}^G_B \mu)^I = 0$ if $\mu|_{T_0} \neq 1$. If $\mu|_{T_0} = 1$ we get an isomorphism $(\text{Ind}^G_B \mu)^I \rightarrow \mathbb{C}^2$, $f \mapsto (f(1), f(w))$. Let us compute the matrices of $S$ and $T$ in the corresponding basis $\mathcal{B} = (b_1, b_2)$ of $(\text{Ind}^G_B \mu)^I$.

\[
(\tilde{w}f)(1) = f \left( \begin{pmatrix}
0 & 1 \\
p & 0
\end{pmatrix} \right) = f \left( \begin{pmatrix}
-1 & 0 \\
0 & p
\end{pmatrix} \right) = \mu_2(p)p^{1/2}f(w)
\]

\[
(\tilde{w}f)(w) = f \left( \begin{pmatrix}
-p & 0 \\
0 & 1
\end{pmatrix} \right) = \mu_1(p)p^{-1/2}f(1)
\]

so the matrix of $T$ is

\[
\left( \begin{array}{cc}
0 & \mu_1(p)p^{-1/2} \\
\mu_2(p)p^{1/2} & 0
\end{array} \right).
\]

\[
([IwI]f)(1) = \sum_{y \in \mathbb{F}_p} f \left( \begin{pmatrix}
1 & y \\
0 & 1
\end{pmatrix} \right) = pf(w)
\]

\[
([IwI]f)(w) = \sum_{y \in \mathbb{F}_p} f \left( w \begin{pmatrix}
1 & y \\
0 & 1
\end{pmatrix} \right) = \sum_{y \in \mathbb{F}_p} f \left( \begin{pmatrix}
-1 & 0 \\
y & -1
\end{pmatrix} \right) = f(1) + (p-1)f(w)
\]

since \( \begin{pmatrix} -1 & 0 \\ y & -1 \end{pmatrix} \in K_0 \) and it belongs to $I$ if and only if $y \in p\mathbb{Z}_p$. So the matrix of $S$ is

\[
\left( \begin{array}{cc}
0 & 1/p^{-1} \\
1 & 1
\end{array} \right).
\]

Let $Q = \left( \begin{array}{cc}1 & p^{-1} \\ -1 & 1 \end{array} \right)$, so that $Q^{-1}\text{Mat}(S, \mathcal{B})Q = \text{diag}(-1, p)$. We compute

\[
Q^{-1}\text{Mat}(T, \mathcal{B})Q = \frac{p^{1/2}}{p+1} \begin{pmatrix} -\mu_2(p) - \mu_1(p) & \mu_1(p) - p^{-1}\mu_2(p) \\ p\mu_2(p) - \mu_1(p) & \mu_2(p) + \mu_1(p) \end{pmatrix}.
\]

We recover the fact that $\text{Ind}^G_B \mu$ is reducible if $(\mu_1/\mu_2)(p) \in \{p^{-1}\}$. More importantly, we have computed the action of $\mathcal{H}(G, I)$ on the line $\text{St}^I$: $S$ and $T$ both act by $-1$.

**Proposition 2.56.** The Steinberg representation $\text{St}$ is square-integrable.

**Proof.** By Lemma 2.31 it is enough to check that one non-zero matrix coefficient is square-integrable. Let $v \in \text{St}^I$ and $\tilde{v} \in \tilde{\text{St}}^I$ (both lines) be such that $\langle v, \tilde{v} \rangle = 1$. Then

\[
\int_{G/Z} |\langle \text{St}(g)v, \tilde{v} \rangle|^2 dg = \sum_{x \in W/Z} |\langle \text{St}(x)v, \tilde{v} \rangle|^2 \text{vol}(IxIZ/Z).
\]

Since $\tilde{W} = T/T_0 \sqcup (T/T_0)\tilde{w}$ and $\text{St}(\tilde{w})v = -v$ this equals

\[
2 \sum_{a \in \mathbb{Z}_{>0}} |\langle \text{St}(\text{diag}(p^a, 1))v, \tilde{v} \rangle|^2 \text{vol}(I\text{diag}(p^a, 1)IZ/Z).
\]

Note that $\text{vol}_{G/Z}(IxIZ/Z) = \text{vol}_{Z}(Z \cap I)^{-1} \text{vol}_{G}(IxI)$ (exercise!). Now $[\text{diag}(p^a, 1)I] = (ST)^a$ fixes $v$ and so

\[
\langle \text{St}(\text{diag}(p^a, 1))v, \tilde{v} \rangle = \frac{\text{vol}(I)}{\text{vol}(I\text{diag}(p^a, 1)I)} = p^{-a}.
\]
Finally $\sum_{a \geq 0} p^{-2a} p^a < +\infty$.

**Remark 2.57.** A similar argument shows that the trivial irreducible representation of $G$ is not square-integrable.

**Proposition 2.58.** If $(V, \pi)$ is a supercuspidal representation of $G$ then $V^I = 0$.

**Proof.** We go back to the computation to prove (3) implies (1) in Theorem 2.27, this time with the Iwahori subgroup $I$ instead of a very small $K_n$. Let $v \in V^I$, then for large enough $a \in \mathbb{Z}$ we have $\pi(e_I)\pi(\text{diag}(p^a, 1))v = 0$, and since this equals $\text{vol}(I\text{diag}(p^a, 1)I)^{-1}\pi((ST)^a)v$ we obtain that the action of $ST$ on $V^I$ is nilpotent. But $ST$ is invertible in $\mathcal{H}(G, I)$: we already know that $T$ is invertible, and $S(S - p + 1) = pe_1$. □

**Remark 2.59.** Pushing this argument further, one can prove Casselman’s criterion for square-integrability (see [Cas, Theorem 4.4.6] or [Ren10, Théorème VII.1.2]): one can read whether a representation of $G$ is square-integrable on its Jacquet module. This generalizes to arbitrary connected reductive groups as well (but classifying representations as in Theorem 2.49 becomes a very complicated combinatorial problem for general groups).

2.9. The unramified Hecke algebra and the Satake isomorphism.

**Lemma 2.60.** Fix Haar measures on $G$ and $B$. The map

$$\mathcal{H}(G, K_0) \longrightarrow \mathcal{H}(B, B \cap K_0)$$

$$f \longmapsto \frac{\text{vol}(K_0)}{\text{vol}(B \cap K_0)} f|_B$$

is a morphism of Hecke algebras.

**Proof.** First note (exercise) that $\mathcal{H}(B)$ is indeed an associate algebra for the convolution product defined using a left Haar measure, even though $B$ is not unimodular.

We use the $BK_0$ integration formula 2.12. For $f_1, f_2 \in \mathcal{H}(G, K_0)$ and $b \in B$,

$$(f_1 * f_2)(b) = \int_G f_1(g)f_2(g^{-1}b) \, dg$$

$$= \int_B \int_{K_0} f_1(ak)f_2(k^{-1}a^{-1}b) \, dk \, da$$

$$= \text{vol}_{K_0}(K_0) \int_B f_1(a)f_2(a^{-1}b) \, da$$

where $\text{vol}_{K_0}$ is the volume with respect to the chosen Haar measure on $K_0$. Recall that the Haar measures on $G$, $K_0$ and $B$ are chosen to be compatible for the integration formula, and this compatibility is equivalent to $\text{vol}_B(B \cap K_0) \text{vol}_{K_0}(K_0) = \text{vol}_G(K_0)$ (apply the integration formula to the characteristic function of $K_0$). Multiplying both sides of the above equation by $\text{vol}_{K_0}(K_0)$ shows that the map $\mathcal{H}(G, K_0) \rightarrow \mathcal{H}(B, B \cap K_0)$ preserves $*$.

**Lemma 2.61.** Choose Haar measures on $T$ and $N$, determining a left Haar measure on $B = TN$ (the product measure). Then the map $\phi : \mathcal{H}(B) \rightarrow \mathcal{H}(T)$, $f \mapsto (t \mapsto \int_N f(tn) \, dn)$ is a morphism of algebras.
Proof. We compute \( \phi(f_1 \ast f_2)(t) \) for \( t \in T \):

\[
\int_B \int_N f_1(b)f_2(b^{-1}tn) \, db \, dn = \int_T \int_N \int_N f_1(xu)f_2(u^{-1}x^{-1}tn) \, du \, dx \, dn
\]

\[
= \int_T \int_N \int_N f_1(xu)f_2(x^{-1}t(t^{-1}xu^{-1}x^{-1}tn) \, dn \, du \, dx
\]

\[
= \int_T \int_N f_1(xu)\phi(f_2)(x^{-1}t) \, dx
\]

\[
= \int_T \phi(f_1)(x)\phi(f_2)(x^{-1}t) \, dx.
\]

\[\square\]

In particular we obtain by composition a morphism of unital algebras \( \mathcal{H}(G, K_0) \to \mathcal{H}(T, T_0) \simeq \mathbb{C}[T/T_0] \) (a group algebra because \( T \) is commutative). For reasons explained below, it is useful to twist this morphism by \( \delta_B^{-1/2} \).

Definition 2.62. Normalize the Haar measures on \( G, T \) and \( N \) so that \( K_0, T_0 \) and \( N_0 \) all have measure 1. The Satake transform \( \text{Sat} : \mathcal{H}(G, K_0) \to \mathcal{H}(T, T_0) = \mathbb{C}[T/T_0] \) is the morphism of unital algebras defined by \( \text{Sat}(f)(t) = \delta_B^{-1/2}(t) \int_{N_0} f(tn) \, dn \).

Theorem 2.63 (Satake). The Satake transform takes values in \( \mathbb{C}[T/T_0]^W \) and induces an isomorphism \( \mathcal{H}(G, K_0) \simeq \mathbb{C}[T/T_0]^W \).

Proof. The fact that the image of \( \text{Sat} \) is contained in \( \mathbb{C}[T/T_0]^W \) will be proved later (Lemma 3.3), since it is natural to use orbital integrals for this (of course there will be no circular argument . . . ). Note that this invariance property is the reason for the normalisation by \( \delta_B^{-1/2} \).

Granting this, we are left to show that the image of \( \text{Sat} \) contains \( \mathbb{C}[T/T_0]^W \) and that \( \text{Sat} \) is injective. By the Cartan decomposition the characteristic functions \( f_{a,b} \) of the sets \( K_0 \text{diag}(p^a, p^b) K_0 \) with \( a \geq b \) form a basis of \( \mathcal{H}(G, K_0) \). Similarly we have a basis \( e_{a,b} = [\text{diag}(p^a, p^b)] + [\text{diag}(p^b, p^a)] \) of \( \mathbb{C}[T/T_0] \). Write \( \text{Sat}(f_{a,b}) = \sum_{a' \geq b} \lambda(a, b, a', b') e_{a', b'} \). It is clear that \( \lambda(a, b, a, b) \in \mathbb{R}_{>0} \), and that \( \lambda(a, b, a', b') = 0 \) if \( a' + b' \neq a + b \) (consider determinants).

Fix \( d \in \mathbb{Z} \). Then Lemma 2.64 below shows that for any integer \( c \geq 0 \),

\[
(2.5) \quad \text{Sat}
\bigg[ \bigoplus_{a \geq b \atop a+b=d \atop a-b \leq c} \mathbb{C} f_{a,b} \bigg] \subset \bigoplus_{a \geq b \atop a+b=d \atop a-b \leq c} \mathbb{C} e_{a,b}
\]

and with the observation that \( \lambda(a, b, a, b) \neq 0 \), one easily shows by induction on \( c \geq 0 \) that (2.5) is an equality. \[\square\]

Lemma 2.64. Let \( t = \text{diag}(x, y) \in T \) and \( c \geq 0 \), then for \( n = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \) we have \( tn \in K_0 \text{diag}(p^a, p^b) K_0 \) for some \( a, b \) satisfying \( b + c \geq a \geq b \) if and only if

\[
|v(x) - v(y)| \leq c \quad \text{and} \quad v(u) \geq \frac{-c - v(x) + v(y)}{2}.
\]
Proof. If \( u \in \mathbb{Z}_p \) or \( uxy^{-1} \in \mathbb{Z}_p \) then \( tn = tnt^{-1}t \in K_0 t K_0 \). Otherwise \( tn \) belongs to \( K_0 \text{diag}(xu, u^{-1}y) K_0 \):
\[
\begin{pmatrix}
x & xu \\
0 & y \\
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
1 & 0 \\
y(xu)^{-1} & -1 \\
0 & y \\
1 & -u^{-1} \\
\end{pmatrix}
\begin{pmatrix}
xu & 0 \\
0 & yu^{-1} \\
\end{pmatrix}.
\]
So we find that \( tn \in K_0 \text{diag}(p^a, p^b) K_0 \) for some \( a \geq b \) satisfying \( a - b \leq c \) if and only if

1. \( v(u) \geq \min(0, v(y) - v(x)) \) and \( |v(x) - v(y)| \leq c \), or

2. \( v(u) < \min(0, v(y) - v(x)) \) and \( |v(x) - v(y) + 2v(u)| \leq c \).

It is easy to check that in case (2) we have \( v(x) - v(y) + 2v(u) \leq -|v(x) - v(y)| \) (distinguish cases: \( v(x) \geq v(y) \) or not), and we get that in both cases \( |v(x) - v(y)| \leq c \). We also get that \( 2v(u) + v(x) - v(y) \geq -c \) in case (2). This inequality is easy to deduce from (1) as well (again, distinguish according to the sign of \( v(x) - v(y) \)). We just proved that (1) and (2) imply the conditions in the statement of the lemma.

Conversely, if \( |v(x) - v(y)| \leq c \) and \( v(u) \geq (-c - v(x) + v(y))/2 \) then \( v(x) - v(y) + 2v(u) \geq -c \), and we are left to check that \( v(x) - v(y) + 2v(u) \leq c \) if \( v(u) < \min(0, v(y) - v(x)) \). Again, this is easy to see by distinguishing according to the sign of \( v(x) - v(y) \).

\( \square \)

Remark 2.65. (1) In particular, \( \mathcal{H}(G, K_0) \) is commutative. This can also be proved by observing that the anti-automorphism \( g \mapsto \chi g \) of \( G \) preserves the Cartan decomposition.

(2) One can check that the Satake isomorphism can be defined over \( \mathbb{Z}[p^{1/2}] \), and is still an isomorphism over this ring.

Definition 2.66. We say that an irreducible smooth representation \((V, \pi)\) is unramified if \( V^{K_0} \neq 0 \).

The Satake isomorphism gives a simple description of all unramified representations of \( G \): by Lemma 2.9 they correspond bijectively to \( \mathbb{C} \)-algebra morphisms \( \mathcal{H}(G, K_0) \to \mathbb{C} \) (any simple finite-dimensional \( \mathcal{H}(G, K_0) \)-module is one-dimensional since \( \mathcal{H}(G, K_0) \) is commutative). More precisely, writing \( \mathbb{C}[T/T_0] = \mathbb{C}[X_1^{\pm 1}, X_2^{\pm 2}] \) where \( X_1 \) (resp. \( X_2 \)) corresponds to \( \text{diag}(p, 1) \) (resp. \( \text{diag}(1, p) \)), we have \( \mathbb{C}[T/T_0]^W = \mathbb{C}[X_1 + X_2, (X_1 X_2)^{\pm 1}] \). Therefore characters of \( \mathcal{H}(G, K_0) \) are parametrized by pairs \((x_1, x_2) \in (\mathbb{C}^\times)^2 \) up to permutation \((x_1, x_2) \mapsto (x_2, x_1)\).

Proposition 2.67. Any unramified representation of \( G \) is isomorphic to

- \( \chi \circ \det \) for some unramified character \( \mathbb{Q}_p^\times \to \mathbb{C}^\times \), or
- \( \text{Ind}^G_B \mu \) for some unramified character \( \mu = \mu_1 \otimes \mu_2 \) such that \( \mu_1(p)/\mu_2(p) \not\in \{p^{\pm 1}\} \).

Proof. Since \( I \subset K_0 \), Proposition 2.58 implies that for any supercuspidal \((V, \pi)\) we have \( V^{K_0} = 0 \). By the classification theorem 2.49, we are left to consider \( (\text{Ind}^G_B \mu)^{K_0} \). Using the Iwasawa decomposition, we see that this space vanishes if \( \mu \) is ramified, and is one-dimensional if \( \mu \) is unramified. \( \square \)
The explicit comparison of the two classifications of unramified representations (i.e. the relation \( x_i = \mu_i(p) \) up to the action of the Weyl group) is left as an exercise.

**Remark 2.68.**
(1) The definition of the Satake morphism and proof that it is an isomorphism are easier than the complete classification. This becomes even more true for groups more complicated than \( \text{GL}_2 \).

(2) The phenomenon that \( \text{Ind}_B^G \mu \) is reducible in exceptional cases is not visible on the Satake isomorphism.

3. Harmonic analysis

We start doing harmonic analysis in the following sense: relating conjugacy classes in \( G \) (more precisely, orbital integrals of functions on \( G \), defined below) to traces \( \text{tr} \pi(f) \) for \( \pi \) an admissible representation of \( G \) and \( f \in \mathcal{H}(G) \) (which makes sense because the image of \( \pi(f) \) has finite dimension).

3.1. Conjugacy classes in \( G \). The classification of conjugacy classes in \( G \) is a special case of the classification of \( \text{GL}_n(k) \)-orbits under conjugation on \( M_n(k) \) where \( k \) is a commutative field (deduced from the structure theorem for finitely generated \( k[X] \)-modules, in the special case of torsion modules). There are four types of conjugacy classes:

- central elements, i.e. \( Z \),
- non-semisimple elements, i.e. elements conjugated to \( z \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) for some (uniquely determined) \( z \in Z \),
- hyperbolic (or split) semisimple regular elements, conjugated to \( \text{diag}(x, y) \in T \) for \( x \neq y \), uniquely determined up to the action of \( N_G(T)/T = W \), that is up to \( (x, y) \mapsto (y, x) \),
- elliptic semisimple regular elements, determined by an irreducible characteristic polynomial \( X^2 + aX + b \in \mathbb{Q}_p[X] \) with \( b \neq 0 \) (explicitly, take the companion matrix). These can be grouped according to the quadratic extension \( E \) of \( \mathbb{Q}_p \) splitting this polynomial as follows. Choose an isomorphism of \( \mathbb{Q}_p \)-vector spaces \( \psi : \mathbb{Q}_p^2 \simeq E \), then any \( x \in E^\times \) defines \( m_x \in \text{Aut}_{\mathbb{Q}_p}(E) \) (multiplication by \( x \)), and \( \psi^{-1} \circ m_x \circ \psi \in G \) is elliptic semisimple regular if and only if \( x \in E \setminus \mathbb{Q}_p \). The subgroup \( T' = \{ \psi^{-1} \circ m_x \circ \psi \mid x \in E \} \) of \( G \) is called an anisotropic (or elliptic) maximal torus of \( G \). Note that \( T'/Z \simeq E^\times /\mathbb{Q}_p^\times \) is compact. Denoting \( \text{Gal}(E/\mathbb{Q}_p) = \{ 1, \sigma \} \), it is easy to check that \( N_G(T')/T'' = Z/2Z \), the non-trivial element being represented by \( \psi^{-1} \circ \sigma \circ \psi \).

We denote by \( G_{rs} \) the set of semisimple regular elements of \( G \). For \( T' \) a maximal torus of \( G \) (elliptic or conjugated to our “standard” split torus \( T \)) we will denote by \( T'_{G_{rs}} = T' \setminus Z \) the subset of regular elements.

Central or non-semisimple elements of \( G \) (i.e. \( G \setminus G_{rs} \)) form a closed subset of \( G \) (in fact, Zariski-closed because they are the solutions of the equation \( \text{tr}^2 = 4 \det \)
of measure 0. Indeed, $Z$ is a sub-$p$-adic manifold of $G$ of dimension $1 < 4 = \dim G$, and it is easy to check that the differential of $\text{tr}^2 - 4 \det$ does not vanish at any point of $G \setminus (Z \cup G_{rs})$ so this subset of $G$ is a submanifold of dimension 3.

For $g \in G$ let $D(g) = 4 - \det(g)^{-1} \text{tr}(g)^2$, so that $G \setminus G_{rs}$ is also the vanishing locus of $D$. It is not difficult to compute that for $T'$ a maximal torus of $G$ and $g \in T'$ we have

$$D(g) = \text{det} (1 - \text{Ad}(g) | \text{Lie}(G)/\text{Lie}(T')).$$

3.2. Orbital integrals.

**Definition 3.1.** For $\gamma \in G$ and $f \in C_c^\infty(G)$, define the orbital integral of $f$ at $\gamma$ as

$$O_{\gamma}(f) := \int_{G_{\gamma} \setminus G} f(g^{-1} \gamma g) \, dg$$

where $G_\gamma$ is the centralizer of $\gamma$ in $G$, provided the integral converges absolutely.

**Remark 3.2.**

1. We have a well-defined right $G$-invariant quotient measure on $G_\gamma \setminus G$ because $G$ and $G_\gamma$ are both unimodular (we will give a “differential” definition of this measure in the proof of Theorem 3.9). Note that the orbital integral depends on choices of Haar measures on $G$ and $G_\gamma$. Via the bijection $G_\gamma \setminus G \cong G/G_\gamma$, $g \mapsto g^{-1} G_\gamma$, the quotient measures are identified and so we also have $O_{\gamma}(f) = \int_{G/G_\gamma} f(g \gamma g^{-1}) \, dg$.

2. For $g \in G$, denoting $f^g : h \mapsto f(ghg^{-1})$, we have $O_{\gamma}(f^g) = O_{\gamma}(f)$.

3. If $h \in G$ then, using the isomorphism $\text{Ad}(h) : G_{h^{-1} \gamma h} \to G_\gamma$ to match Haar measures on these two groups, we have $O_{\gamma}(f) = O_{h^{-1} \gamma h}(f)$: use the measure-preserving bijection $G_\gamma \setminus G \cong G_{h^{-1} \gamma h} \setminus G$, $g \mapsto h^{-1} g$.

If $\gamma$ is semisimple we will show that the integrand is smooth and compactly supported (Lemma 3.4 below). More precisely, let $K$ be a compact open subgroup such that $f$ is bi-$K$-invariant. We will show that there are finitely many double cosets $[g] = G_\gamma g K \subset G$ such that $g^{-1} \gamma g$ belongs to the support of $f$, and so the integrand in Definition 3.1 is smooth and compactly supported and

$$(3.1) \quad O_{\gamma}(f) = \sum_{[g] \in G_\gamma \setminus G/K} f(g^{-1} \gamma g) \frac{\text{vol}(K)}{\text{vol}(G_\gamma \cap g K g^{-1})}.$$  

Note that these statements are trivial if $\gamma$ is central, so we will consider semi-simple regular $\gamma$’s.

First look at the case where $\gamma$ is regular semisimple hyperbolic. We can assume that $\gamma \in T$, so that $G_\gamma = T$, and by the integration formula for the Cartan decomposition (Lemma 2.12) and using the expression of $O_{\gamma}(f)$ as an integral over $G/G_\gamma$ we have

$$O_{\gamma}(f) = \int_{N \times K_0} f(k^{-1} n^{-1} \gamma nk) \, dn \, dk.$$
Denote $\gamma = \text{diag}(x, y)$ and $n = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$. Then $n^{-1} \gamma n = \gamma \begin{pmatrix} 1 & (1 - y/x)u \\ 0 & 1 \end{pmatrix}$ and we obtain

$$O_\gamma(f) = \left|1 - \frac{y}{x}\right|^{-1} \int_{K_0 \times N} f(k^{-1} \gamma nk) \, dk \, dn = \left|\frac{x}{y}\right|^{1/2} \left|\frac{x}{y} - 1\right|^{-1/2} \left|\frac{y}{x} - 1\right|^{-1/2} \int_{K_0 \times N} f(k^{-1} \gamma nk) \, dk \, dn$$

$$(3.2) = |D(\gamma)|^{1/2} \delta_B^{1/2}(\gamma) \int_{K_0 \times N} f(k^{-1} \gamma nk) \, dk \, dn.$$

This shows that the sum on the right-hand side of (3.1) has finitely many non-vanishing terms. We also recognize the formula defining the Satake morphism. This allows us to prove the remaining statement in Theorem 2.63.

**Lemma 3.3.** For any $f \in \mathcal{H}(G, K_0)$, $\text{Sat}(f) \in \mathcal{H}(T, T_0)$ is invariant under the Weyl group $W = \{1, w\}$ of $T$.

**Proof.** Any coset in $T/T_0$ contains a regular element $\gamma$, and $\text{Sat}(f)(\gamma) = |D(\gamma)|^{1/2} O_\gamma(f)$ by 3.2. By the third point in Remark 3.2, this is invariant under $w$. □

**Lemma 3.4.** For any $f \in C_c^\infty(G)$ and any semisimple $\gamma \in G$, the sum on the right-hand side of (3.1) has finitely many non-zero terms.

More generally, if $T'$ is a maximal torus of $G$ and $C_{T'} \subset T'_{G-\text{reg}}$ is compact then there is a finite subset $X(f, C_{T'})$ of $T' \setminus G/K$ such that for any $\gamma \in C_{T'}$, the set of $[g] \in T' \setminus G/K$ such that $g^{-1} \gamma g \in \text{supp}(f)$ is contained in $X(f, C_{T'})$. In particular, $T'_{G-\text{reg}} \to \mathbb{C}$, $\gamma \mapsto O_\gamma(f)$ is smooth.

**Proof.** We only prove the first part, the second has the same proof, only with more bookkeeping. This is obvious for $\gamma \in Z$, and we have already checked this for a split regular semisimple $\gamma$, so we can assume that $\mathbb{Q}_p[\gamma] := E$ is a quadratic extension of $\mathbb{Q}_p$. Let $C$ be the (compact) support of $f$. Since $G_{\gamma}/Z$ is compact, we have to show that the set of $g \in G/Z$ such that $g^{-1} \gamma g \in C$ is compact. We will reduce to the split case. Let $T'_E$ be the centralizer of $\gamma$ in $G_E = \text{GL}_2(E)$, $K_{0,E} = \text{GL}_2(O_E)$. Choose a unipotent subgroup $N_E \subset G_E$ normalized by $T'_E$ (there are two choices). We may replace $C$ by the larger but still compact set $K_{0,E}C K_{0,E}$. In the decomposition $G_E = T'_E N_E K_{0,E}$, $g = t n k$ is such that $g^{-1} \gamma g \in C$ then $n$ belongs to a compact subset $N_{E,C}$ of $N_E$. Denote $\{1, \sigma\} = \text{Gal}(E/\mathbb{Q}_p)$, then $g \in G_E$ belongs to $G$ if and only if $\sigma(g) = g$, so $t^{-1} \sigma(t)$ is forced to belong to a compact subset of $T'_E$. There are two natural isomorphisms $T'_E \simeq (E')^2$. Choose one such isomorphism $\phi$. Then $\phi(\sigma(\phi^{-1}(t_1, t_2))) = (\sigma(t_2), \sigma(t_1))$, so considering the image of $t^{-1} \sigma(t)$ by $\phi$ we see that $t$ is forced to belong to a subset of $T'_E$ which is compact modulo $Z$. □

**Remark 3.5.**

1. A similar argument works for semisimple elements in arbitrary reductive groups, see [HC70, Lemma 19, p.52].

2. For non-semisimple elements it is not true that the right-hand side of (3.1) has finitely many non-vanishing terms, but the integral defining the orbital integral does converge absolutely. For $\text{GL}_2$ this is an exercise; for arbitrary reductive groups it is a theorem of Ranga Rao and Deligne (see [RR72]).
(3) For $\omega$ a smooth character of $Z$, the same arguments apply to orbital integrals of smooth $\omega$-equivariant functions on $G$ which have compact support modulo $Z$.

Orbital integrals will show up naturally in the trace formula. But right now we will compute these in a special case. This will allow us to estimate them in general, and such estimates will be useful when we study characters of admissible representations of $G$.

Let $\gamma$ be a semisimple regular element of $G$. Assume that $\gamma$ is compact, i.e. the sequence $(\gamma^n)_{n \in \mathbb{Z}}$ is bounded, equivalently its closure is compact. Equivalently, $\gamma$ is conjugated to an element of $K_0$ (recall that any compact group acting continuously on a finite-dimensional $\mathbb{Q}_p$-vector space stabilizes a lattice). We will compute $O_\gamma(e_{K_0})$.

If $\gamma$ is hyperbolic, i.e. if its eigenvalues are in $\mathbb{Q}_p$, then we may assume that $\gamma \in T_0$ and $G_\gamma = T$. Since $e_{K_0}$ is bi-$K_0$-invariant, taking the Haar measure on $T$ such that $\text{vol}(T_0) = 1$ (3.2) gives $O_\gamma(e_{K_0}) = |D(\gamma)|^{-1/2}$.

Consider now the case where $\gamma$ is elliptic, i.e. $E := \mathbb{Q}_p[\gamma] \subset M_2(\mathbb{Q}_p)$ is a quadratic extension of $\mathbb{Q}_p$. Notice that the set of $[g] \in G_\gamma \backslash G/K_0$ such that $g^{-1}\gamma g \in K_0$ maps (by $g \mapsto g^{-1}\gamma g \ldots$) bijectively onto the set of $K_0$-conjugacy classes $[\gamma]$ in $K_0$ having the same characteristic polynomial as $\gamma$. Moreover this set can be described more intrinsically as the set of isomorphism classes of $\mathbb{Z}_p[\gamma]$-modules $L$ which are free of rank 2 over $\mathbb{Z}_p$. Note that $\mathbb{Z}_p[\gamma]$ is an order, i.e. a sub-$\mathbb{Z}_p$-algebra of the ring of integers $\mathcal{O}_E$ of finite index in $\mathcal{O}_E$. The subgroup $G_\gamma \cap gK_0g^{-1}$ is identified, by conjugation by $g$, to the subgroup of elements of $E^\times$ stabilizing $L$, which is also the group of automorphisms of the $\mathbb{Z}_p[\gamma]$-module $L$. This subgroup contains $\mathbb{Z}_p[\gamma]^\times$ and is contained in $E^\times$. A better formulation would be that two groupoids are isomorphic:

- the set with left $G_\gamma$-action consisting of all $gK_0 \in G/K_0$ such that $g^{-1}\gamma g \in K_0$,
- the groupoid of $\mathbb{Z}_p[\gamma]$-modules which are finite free of rank 2 over $\mathbb{Z}_p$ (equivalently, finite torsion-free $\mathbb{Z}_p[\gamma]$-modules which become one-dimensional over $E$ after $E \otimes_{\mathbb{Z}_p[\gamma]} \cdot$).

This reformulation implies

$$O_\gamma(e_{K_0}) = \sum_{[L]} \text{vol}(\{\lambda \in \mathcal{O}_E^\times | \lambda L = L\})^{-1}.$$  

The following lemma gives an explicit representative in each isomorphism class.

**Lemma 3.6.** Let $L$ be a $\mathbb{Z}_p[\gamma]$-lattice which is free of rank two over $\mathbb{Z}_p$. Then there is an isomorphism of $\mathbb{Z}_p[\gamma]$-modules $\phi : L \simeq \phi(L)$ with $\mathbb{Z}_p[\gamma] \subset \phi(L) \subset \mathcal{O}_E$. Moreover $\phi(L)$ is uniquely determined by the isomorphism class of $L$ and $\phi(L)$ is a $\mathbb{Z}_p$-algebra (i.e. it is stable under multiplication), and the group of automorphisms of the $\mathbb{Z}_p[\gamma]$-module $L$ is $L^\times$.

**Proof.** Choose an isomorphism of $E$-vector spaces $\phi : \mathbb{Q}_p \otimes_{\mathbb{Z}_p} L \simeq E$. Since $L$ is $p$-torsion free, $L$ embeds in $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} L$ and we can see $L$ as embedded in $E$ via $\phi$. 

The $\mathcal{O}_F$-module $\mathcal{O}_F\phi(L) \subset E$ is of the form $\varpi_E^i\mathcal{O}_E$ for some $i \in \mathbb{Z}$, where $\varpi_E$ is a uniformizer of $E$. Up to composing $\phi$ with multiplication by $\varpi_E^{-1}$, we may assume that $\mathcal{O}_E\phi(L) = \mathcal{O}_E$. In particular $L$ contains an element $u \in \mathcal{O}_E^*$. Up to composing $\phi$ with multiplication by $u^{-1}$ we can also assume that $1 \in \phi(L)$, and so $\mathbb{Z}_p[\gamma] \subset \phi(L) \subset \mathcal{O}_E$. This shows existence; for uniqueness we will really use the fact that $[E:\mathbb{Q}_p] = 2$.

Let $x \in \mathcal{O}_E$ be such that $\mathcal{O}_E = \mathbb{Z}_p[x]$. Then any sub-$\mathbb{Z}_p$-module $L$ of $\mathcal{O}_E$ of rank 2 and containing $\mathbb{Z}_p$ is of the form $\mathbb{Z}_p \oplus \mathbb{Z}_p p^n x$, where $n$ is determined by the index $[\mathcal{O}_E/L] = p^n$. Moreover we see that $L$ is also an order, i.e. it is stable under multiplication.

Now if $\mathbb{Z}_p[\gamma] \subset L, L' \subset \mathcal{O}_E$ are $\mathbb{Z}_p[\gamma]$-modules, any isomorphism (of $\mathbb{Z}_p[\gamma]$-modules) $L \cong L'$ is multiplication by some $t \in E^\times$ ($t$ is the image of $1 \in L$). Since $L$ and $L'$ both generate $\mathcal{O}_E$ as an $\mathcal{O}_E$-module we have $t \in \mathcal{O}_E^*$, which implies $[\mathcal{O}_E/L] = [\mathcal{O}_E/L']$ and so $L = L'$. We also see that $t \in L' = L$ and considering the inverse morphism, $t^{-1} \in L$, so $t \in L$. □

We deduce

$$O_\gamma(e_{K_0}) = \sum_{Z_p[\gamma] \subset L \subset \mathcal{O}_E} \text{vol}(L^\times)^{-1}.$$ 

If $L \subset \mathcal{O}_E$ or $L = \mathcal{O}_E$ with $E/\mathbb{Q}_p$ ramified then $L^\times = \mathbb{Z}_p^\times \times \mathbb{Z}_p p^n x$ ($n$ as above). If $E/\mathbb{Q}_p$ is unramified then $O_\gamma^\times = \{(a + bx) \mid (a, b) \in \mathbb{Z}_p^2 \setminus (p\mathbb{Z}_p)^2\}$. Defining $m \in \mathbb{Z}_{\geq 0}$ by $|\mathcal{O}_E/Z_p[\gamma]| = p^m$, we obtain

$$O_\gamma(e_{K_0}) \text{vol}(O_\gamma^\times) = \begin{cases} (1 + p + \cdots + p^m) = \frac{p^{m+1} - 1}{p - 1} & \text{if } E/\mathbb{Q}_p \text{ is ramified}, \\ (1 + (1 + p^{-1})(p + \cdots + p^m)) = \frac{p^{m+1} + p^{m-2}}{p - 1} & \text{if } E/\mathbb{Q}_p \text{ is unramified}. \end{cases}$$

**Proposition 3.7.** There are constants $C > c > 0$ such that for any $\gamma \in G_{rs}$ we have $C|D(\gamma)|^{-1/2} \geq O_\gamma(e_{K_0}) \geq c|D(\gamma)|^{-1/2}$. 

Note that this makes sense even though the orbital integrals depend on choices of measures: there are finitely many conjugacy classes of maximal tori in $G$, and we may fix a Haar measure on each maximal torus, as well as a Haar measure on $G$. Different choices only affect the constants $C$ and $c$.

**Proof.** For $\gamma \in G_{rs}$ not conjugated to an element of $K_0$ it is clear that $O_\gamma(e_{K_0}) = 0$, so we may restrict to $\gamma \in G_{rs} \cap K_0$. For $\gamma$ split this is clear by the above computation $O_\gamma(e_{K_0}) = |D(\gamma)|^{-1/2}$. For $\gamma$ elliptic it remains to relate the integer $m$ appearing in the explicit formula above to $|D(\gamma)|^{1/2}$. To this end write $\gamma = p^m x + y$ with $x$ as above and $y \in \mathbb{Z}_p$. Then $D(\gamma) = (\gamma/\sigma(\gamma) - 1)(\sigma(\gamma)/\gamma - 1)$, and since $\gamma$ is compact we have $v(D(\gamma)) = 2v(\gamma - \sigma(\gamma)) = 2(m + v(x - \sigma(x)))$. □

**Corollary 3.8.** Let $f \in \mathcal{H}(G)$. Then there exists $C > 0$ such that for any $\gamma \in G_{rs}$ we have $|O_\gamma(f)| \leq C|D(\gamma)|^{-1/2}$.

As in Proposition 3.7 the precise constant depends not only on $f$, but also on choices of Haar measures. As before we fix Haar measures on maximal tori of $G$.

**Proof.** Let $X = \{g \in \text{supp}(f) \mid D(g) = 0\}$, a compact subset of $G$. For any $g \in X$, there exists $z \in Z$ and $h \in G$ such that $g \in zhK_0h^{-1}$ (in fact $K_0$ could be replaced by
any neighbourhood of 1 in $G$). By compactness of $X$ there is a finite family $(z_i, h_i)_{i \in I}$ such that $X \subset \bigcup_{i \in I} z_i h_i K_0 h_i^{-1}$. Therefore there exists $c_1 > 0$ and $f_{rs} \in C_c(G_{rs}, \mathbb{R}_{\geq 0})$ (for example, supported on $\text{supp}(f) \sim \bigcup_{i \in I} z_i h_i K_0 h_i^{-1}$) such that

$$|f| \leq f_{rs} + \sum_{i \in I} c_1 \text{vol}(K_0)^{-1} z_i h_i K_0 h_i^{-1}.$$ 

By Lemma 3.4 for any maximal torus $T'$ of $G$ the function $T'_{G-\text{reg}} \to \mathbb{C}$, $\gamma \mapsto O_\gamma(f_{rs})$ is smooth, and since $|D|^{-1}$ is bounded on the support of $f_{rs}$ it is also compactly supported. So $\gamma \mapsto O_\gamma(f_{rs})$ is simply bounded on maximal tori of $G$. By Proposition 3.7 there exists a constant $C > 0$ such that for any $\gamma \in G_{rs}$,

$$O_\gamma(\text{vol}(K_0)^{-1} z_i h_i K_0 h_i^{-1}) = O_{\gamma z_i^{-1}}(e_{K_0}) \leq C|D(\gamma)|^{-1/2}.$$ 

We obtain

$$|O_\gamma(f)| \leq \left( \sup_{\gamma \in G_{rs}} (|D(\gamma^\prime)|^{1/2}O_\gamma(f_{rs})) + |I|c_1C \right)|D(\gamma)|^{-1/2}. \qedhere$$

### 3.3. The Weyl integration formula

For $T'$ a maximal torus of $G$ we have a map $\phi_{T'} : T'_{G-\text{reg}} \times T' \setminus G \to G_{rs}$, $(t, \dot{g}) \mapsto g^{-1}tg$. Then $(t, \dot{g})$ and $(s, h)$ map to the same point if and only if $hg^{-1} \in N_G(T')$ and $s = hg^{-1}tg^{-1}$. Since $N_G(T')/T' = \mathbb{Z}/2\mathbb{Z}$ we get that each non-empty fiber of $\phi_{T'}$ has two elements. Let $T$ be a set of representatives of maximal tori of $G$, under conjugation by $G$. Note that $T$ is finite.

**Theorem 3.9** (Weyl integration formula). Let $f$ be a measurable function on $G$. Then

$$\int_G f(g) \, dg = \sum_{T \in T} \frac{1}{2} \int_{T' \setminus G_{reg}} |D(t)||O_t(f)\, dt$$

if one side is absolutely convergent (i.e. convergent if we substitute $|f|$ for $f$).

Note that the complex Haar measure $O_t(f)\, dt$ on $T'$ does not depend on the choice of Haar measure on $T'$, since $O_t(f)$ is defined using a quotient measure.

**Proof.** Since $G \setminus G_{rs}$ has measure zero and $G_{rs} = \bigsqcup_{T' \in T} \text{Im}(\phi_{T'})$, by linearity of the formula to be proved we may assume without loss of generality that there exists $T' \in T$ such that $f$ vanishes identically outside $\text{Im}(\phi_{T'})$. We will apply Theorem 2.40 to $\phi_{T'}$. We can write $dg |\omega_G|$ where $\omega_G \in \Omega_{\dim G}(G)$ is left (and right) $G$-invariant and non-zero, and so $\omega_G$ corresponds to a non-zero element in $\Lambda^{\dim G}(T_1 G)^* = \Lambda^{\dim G} \text{Lie}(G)^*$. Similarly, we can choose $\omega_{T'}$ corresponding to an element of $\Lambda^{\dim T'} \text{Lie}(T')^*$, inducing a Haar measure on $T'$. We will use these to define a right $G$-invariant $\omega_{T' \setminus G} \in \Omega_{\dim G - \dim T'}(T' \setminus G)$. For $g \in G$, differentiating the submersion $G \to T' \setminus G$, $h \mapsto T'hg$ gives a short exact sequence

$$0 \to \text{Lie}(T') \to \text{Lie}(G) \to T_g(T' \setminus G) \to 0$$

whose dual gives an isomorphism

$$\iota_g : \Lambda^{\dim G} \text{Lie}(G)^* \simeq \Lambda^{\dim T'} \text{Lie}(T')^* \otimes \mathbb{Q}_p \simeq \Lambda^{\dim G - \dim T'} T_g(T' \setminus G)^*.$$
In particular we have a basis $\omega_{T'\backslash G,g}$ of the $\mathbb{Q}_p$-line $\bigwedge^{\dim G - \dim T'} T_g(T'\backslash G)^*$ such that $t_g(\omega_G) = \omega_{T'} \otimes \omega_{T'\backslash G,g}$. We have to check that it depends on $g$ only via $g \mapsto \dot{g} = T'g$. If $g' = tg$ with $t \in T'$ then we have a commutative diagram

$$
\begin{array}{ccc}
G & \xrightarrow{R_g} & T' \backslash G \\
\downarrow{\text{Ad}(t^{-1})} & & \downarrow{R_g} \\
G & & G
\end{array}
$$

where $\text{Ad}(t^{-1}) : h \mapsto t^{-1}ht$ fixes $1 \in G$ and preserves $T'$. We have $\text{Ad}(t^{-1})^* \omega_G = \det(\text{Ad}(t^{-1}) | \text{Lie}(G))$ and $\text{Ad}(t^{-1})^* \omega_{T'} = \det(\text{Ad}(t^{-1}) | \text{Lie}(T'))$, and so $\omega_{T'\backslash G,g}' = \det(\text{Ad}(t^{-1}) | \text{Lie}(G)/\text{Lie}(T')) \omega_{T'\backslash G,g}$. It is easy to check that

$$
\det(\text{Ad}(t^{-1}) | \text{Lie}(G)/\text{Lie}(T')) = 1
$$

as this can be computed after tensoring with a finite extension of $\mathbb{Q}_p$ splitting $T'$. Therefore there is a unique right $G$-invariant $\omega_{T'\backslash G} \in \Omega^{\dim G - \dim T'}(T'\backslash G)$ which specializes to $\omega_{T'\backslash G,1}$ at $1 = T' \in T'\backslash G$: taking a local section $s : U \rightarrow G$ of $G \rightarrow T'\backslash G$, define $\omega_{T'\backslash G} |_U$ by pulling back $\omega_{T'\backslash G,1}$ along $R_{s(?)^{-1}} : T'\backslash G \rightarrow T'\backslash G$. Note that we recover the existence of the quotient measure $|\omega_{T'\backslash G}|$ on $T'\backslash G$ (this is not surprising since the trivial determinant above implies that the modulus characters of $G$ and $T'$ coincide on $T'$).

Let $t \in T'$ and $g \in G$, defining $\dot{g} = T'g \in T'\backslash G$. We want to compute the dual of

$$
\bigwedge^{\dim G} d_{(t,g)} \phi_{T'} : \left( \bigwedge^{\dim T'} T_t(T') \right) \otimes_{\mathbb{Q}_p} \left( \bigwedge^{\dim G - \dim T'} T_g(T'\backslash G) \right) \rightarrow \bigwedge^{\dim G} T_g(T'\backslash G),
$$

in the bases $\omega_{T'}$, $\omega_{T'\backslash G}$ and $\omega_G$. We have a commutative diagram (vertical maps are isomorphisms)

$$
\begin{array}{ccc}
T' \times T' \backslash G & \xrightarrow{\phi_{T'}} & G \\
\downarrow{R_t \times R_g} & & \downarrow{R_g \circ L_{g^{-1}}} \\
T' \times T' \backslash G & \xrightarrow{\psi_t} & G
\end{array}
$$

where $R_a$ (resp. $L_a$) denotes right (resp. left) multiplication by $a$ and $\psi_t(x, \dot{h}) = h^{-1}xtht^{-1}$. Taking differentials, we get a commutative diagram

$$
\begin{array}{ccc}
T_t(T') \times T_g(T' \backslash G) & \xrightarrow{d_{t,g}(\phi_{T'})} & T_g^{-1}T_g \backslash G \\
\downarrow{d_{1}(R_t) \oplus d_{1}(R_g)} & & \downarrow{d_{1}(R_g \circ L_{g^{-1}})} \\
\text{Lie}(T') \oplus \text{Lie}(G)/\text{Lie}(T') & \xrightarrow{d_{1}(\psi_t)} & \text{Lie} G
\end{array}
$$
Writing $x = \exp \delta = 1 + \epsilon + O(\delta^2)$ for $\delta \in \text{Lie} T'$ and $h = \exp \epsilon = 1 + \epsilon + O(\epsilon^2)$ for $\epsilon$ in a complementary subspace of Lie $T'$ in Lie $G$, we compute $d_{1,1}(\psi_t)(\delta, \epsilon) = \delta + (\text{Ad}(t) - 1)(\epsilon)$. Since $\omega_G$ is invariant under left and right multiplication maps, $\omega_{T'}$ is also invariant under multiplication maps and $\omega_{T' \backslash G}$ is invariant under right multiplication maps, we obtain

$$(\phi^*_T \omega_G)|_{\epsilon, \delta} = \det(\text{Ad}(t) - 1 \mid \text{Lie}(G)/\text{Lie}(T')) \times (\omega_{T'}|_t) \wedge (\omega_{T' \backslash G}|_\delta).$$

The formula now follows from Theorem 2.40 and Fubini’s theorem. \hfill \qed

**Proposition 3.10.** Let $\epsilon > 0$. Any (measurable . . . ) function $G \to \mathbb{C}$ which coincides with $|D|^{1-\epsilon}$ on $G_{rs}$ is locally integrable.

**Proof.** The function is locally smooth on $G_{rs}$ so we really have to show that for any $x \in G \setminus G_{rs}$, there is a neighbourhood $U$ of $x$ in $G$ such that $\int_U |D|^{-1/2} < +\infty$. The function $D$ is invariant by conjugation and by multiplication by $Z$, so we may replace $x$ by $zg x g^{-1}$ for some $g \in G$ and some $z \in Z$. So we may assume that $x \in K_i$ for arbitrary $i > 0$, in particular $x \in K_0$. We simply take $U = K_0$. To show that $e_{K_0} |D|^{-1+\epsilon}$ is integrable, we apply the Weyl integration formula:

$$\text{vol}(K_0)^{-1} \int_{K_0} |D(g)|^{-1+\epsilon} \, dg = \sum_{T' \in T} \frac{1}{2} \int_{G_{\text{reg}}} |D(t)|^\epsilon |O_t(e_{K_0})| \, dt$$

and by Proposition 3.7 we are left to show that for any maximal torus $T'$, $|D|^{-1/2+\epsilon}$ is locally integrable (for the Haar measure on $T'$). For $T' = T$ this amounts to

$$\int_{\mathbb{Z}_p \setminus \{0\}} |1 - x|^{-1+2\epsilon} \, dx \leq \int_{\mathbb{Z}_p \setminus \{0\}} |u|^{-1+2\epsilon} \, du = \text{vol}(\mathbb{Z}_p) \sum_{k \geq 0} p^{-k} p^{k-2\epsilon} < +\infty.$$

For $T'$ anisotropic, corresponding to a quadratic extension $E/\mathbb{Q}_p$, we have

$$\int_{O_E \setminus \mathbb{Z}_p} |x - \sigma(x)|^{-1+2\epsilon} \, dx \leq C \int_{\mathbb{Z}_p \setminus \{0\}} |2ux_0|^{-1+2\epsilon} \, du$$

where $x_0 \in O_E \setminus \mathbb{Z}_p$ is such that $\sigma(x_0) = -x_0$, and we conclude as in the previous case. \hfill \qed

**3.4. Harish-Chandra characters.** We now begin the study of characters of representations of $G$. If $(V, \pi)$ is an admissible representation of $G$ (for example if it is an irreducible smooth representation of $G$) then for any $f \in \mathcal{H}(G)$ the operator $\pi(f) : V \to V$ has image contained in the finite-dimensional subspace $V^K$ for any compact open subgroup $K$ such that $f$ is left $K$-invariant. Thus we can define $\text{tr} \pi(f) = \text{tr} (\pi(f) \mid \pi(f)(V))$, which also equals $\text{tr} (\pi(f) \mid V^K)$ for $K$ as above by the following lemma applied to $W = V^K$ and $A = \pi(f)|_{V^K}$.

**Lemma 3.11.** Let $A$ be an endomorphism of a finite-dimensional vector space $W$ over a field. Then $\text{tr} A = \text{tr}(A|A(W))$.

**Proof.** Left as an exercise. \hfill \qed
Thanks to the theory of finite-dimensional representation of algebras, if \((V_1, \pi_1), \ldots, (V_k, \pi_k)\) are non-isomorphic irreducible smooth representations of \(G\) then the linear forms \(\text{tr} \, \pi_i\) on \(\mathcal{H}(G)\) are linearly independent (this follows from the existence of projection operators, see [Lan02, XVII Theorem 3.7]). In particular the trace of an admissible \emph{semisimple} representation of \(G\) determines the isomorphism class of this representation (exercise . . .).

**Theorem 3.12.** Let \((V, \pi)\) be an irreducible smooth representation of \(G\). Then there is a unique smooth function \(\Theta_\pi : G_{rs} \to \mathbb{C}\) such that, extending \(\Theta_\pi\) arbitrarily to \(G\), \(\Theta_\pi\) is locally integrable on \(G\), and for any \(f \in \mathcal{H}(G)\) we have

\[
\text{tr} \, \pi(f) = \int_G f(g) \Theta_\pi(g) \, dg.
\]

Moreover \(|D|^{1/2} \Theta_\pi\) is bounded on \(G_{rs}\).

**Remark 3.13.** The function \(\Theta_\pi\) does not depend on the choice of Haar measure (the measure occurs in the definition of \(\text{tr} \, \pi(f)\) as well).

The proof is going to be quite long. First we handle the supercuspidal case.

**Lemma 3.14.** Let \((V, \pi)\) be an irreducible supercuspidal representation. Let \(v \in V\) and \(\tilde{v} \in \tilde{V}\) be such that \(\langle v, \tilde{v} \rangle = d_\pi\). Then for any \(f \in \mathcal{H}(G)\),

\[
\dot{g} \mapsto \int_G f(h) \langle \pi(g^{-1} h g) v, \tilde{v} \rangle \, dh
\]

is a smooth compactly supported function on \(G/Z\) and we have

\[
\text{tr} \, \pi(f) = \int_{G/Z} \int_G f(h) \langle \pi(g^{-1} h g) v, \tilde{v} \rangle \, dh \, dg.
\]

**Proof.** Let \((v_i)_i\) be a basis of \(V\) such that each \(v_i\) belongs to a \(K_0\)-isotypic component. Let \((\tilde{v}_i)_i\) be the dual basis of \(\tilde{V}\) (this is well-defined by admissibility of \(V\)). Let \(a_{i,j} = \langle \pi(f) v_i, \tilde{v}_j \rangle\). By admissibility of \(\pi\) (and since \(f\) if bi-

\[
\text{tr} \, \pi(f) = \sum_{i,j} a_{i,j} \langle \pi(g) v_i, \tilde{v}_j \rangle.
\]

This can also be written

\[
\int_G f(h) \langle \pi(g^{-1} h g) v, \tilde{v} \rangle \, dh = \sum_{i,j} a_{i,j} \langle \pi(g) v_i, \tilde{v}_j \rangle \langle \pi(g^{-1} h g) v, \tilde{v} \rangle.
\]

This function of \(\dot{g} \in G/Z\) is visibly smooth and compactly supported, and integrating over \(\dot{g} \in G/Z\), by Schur orthogonality (Proposition 2.34) we get

\[
\int_{G/Z} \int_G f(h) \langle \pi(g^{-1} h g) v, \tilde{v} \rangle \, dg \, dh = \sum_{i,j} a_{i,j} d^{-1}_\pi \langle v, \tilde{v} \rangle \langle v_j, v_i \rangle = \sum_i a_{i,i} = \text{tr} \, \pi(f).
\]
Now we would very much like to swap integral signs in this formula. This is not formal, and in fact wrong!

It is however justified if we restrict to $g$ in the subset $G_{rs}^\mathrm{ell}$ of elliptic regular semisimple elements.

**Lemma 3.15.** Let $f \in C_c^\infty(G)$ and $\psi \in C_c^\infty(G, \omega)$ for some smooth character $\omega : Z \to \mathbb{C}^\times$. Then the function $(g, h) \mapsto f(h)\psi(g^{-1}h g)$ is integrable on $G/Z \times G_{rs}^\mathrm{ell}$ and

$$\int_{G/Z \times G_{rs}^\mathrm{ell}} f(h)\psi(g^{-1}h g) d\hat{g} dh = \int_{G_{rs}^\mathrm{ell}} f(h) \operatorname{vol}(G_h/Z) O_h(\psi) dh.$$

**Proof.** We can assume that $f, \psi$ take values in $\mathbb{R}_{\geq 0}$. Let $T_{cusp}$ be a set of representatives for the $G$-conjugacy classes of elliptic maximal tori in $G$. Now for $h \in G_{rs}^\mathrm{ell}$ we have $\int_{G/Z} \psi(g^{-1}h g) d\hat{g} = \operatorname{vol}(G_h/Z) O_h(\psi)$ because $G_h/Z$ is compact. Now by Corollary 3.8 and Proposition 3.10 the function $h \mapsto O_h(\psi)$, extended by zero on $G \setminus G_{rs}^\mathrm{ell}$, is locally integrable on $G$. Therefore

$$\int_{G_{rs}^\mathrm{ell}} f(h) \int_{G/Z} \psi(g^{-1}h g) d\hat{g} dh = \int_{G_{rs}^\mathrm{ell}} f(h) \operatorname{vol}(G_h/Z) O_h(\psi) dh < \infty.$$

□

This argument does not work for $h \in G_{rs} \setminus G_{rs}^\mathrm{ell}$ because then $\operatorname{vol}(G_h/Z) = +\infty$. But recall from Remark 2.33 that the matrix coefficient $\psi$ is not just any element of $C_c^\infty(G, \omega_\pi)$: it belongs to the subspace $C_c^\infty(G, \omega_\pi)$.

**Lemma 3.16** (Selberg’s principle). Let $\omega : Z \to \mathbb{C}^\times$ be a smooth character. Let $\psi \in C_c^\infty(G, \omega)$, i.e. for any $x, y \in G$ we have $\int_N \psi(xny) dn = 0$. Then for any $h \in G_{rs} \setminus G_{rs}^\mathrm{ell}$ we have $O_h(\psi) = 0$.

**Proof.** It is enough to consider $h \in T_{G-\mathrm{reg}}$. We computed (see (3.2))

$$O_h(\psi) = |D(h)|^{-1/2} \delta_B^{1/2}(h) \int_{K_0 \times N} \psi(k^{-1}h k) dk dn = 0.$$

□

To “swap the two $\int$ signs” in the formula given in Lemma 3.14, we will write the outer integral as a limit over a particular increasing and exhaustive sequence of compact subsets of $G/Z$. For $c \geq 0$ an integer define $C_c = \bigsqcup_{m \leq c} K \operatorname{diag}(p^m, 1)KZ$, so that $C_c/Z$ is a bi-$K$-invariant compact subset of $G/Z$.

**Lemma 3.17.** Let $\omega : Z \to \mathbb{C}^\times$ be a smooth character and $\psi \in C_c^\infty(G, \omega)$. The sequence of functions on $G_{rs} \setminus G_{rs}^\mathrm{ell}$

$$\left( \Theta_{\psi, c} : h \mapsto \int_{C_c/Z} \psi(g^{-1}hg) d\hat{g} \right)_{c \geq 0}$$

converges pointwise as $c \to +\infty$ to a smooth function $\Theta_\psi$, which is invariant under conjugation by $G$. Moreover for any $C' \subset G_{rs}^{\mathrm{hyp}}$ which is relatively compact in $G$, 

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there exists $\kappa(C') > 0$ such that for any $c \geq 0$ we have $|\Theta_{\psi,c}| \leq \kappa(C')|D|^{-1/2}$ on $C$. For $h \in T_{G-\text{reg}}$ we have

$$\Theta_{\psi}(h) = \int_{N \times K_0} \psi(k^{-1}n^{-1}h n k) \min(0, 2v(n)) \, dn \, dk$$

where we have denoted $v(n) = v(u)$ for $n = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \in N$.

Note that this formula for $\Theta_{\psi}(h)$ differs from the formula for $O_h(\psi)$ only by the factor $\min(0, 2v(n))$ in the integrand. This expression is called a weighted orbital integral.

**Proof.** We can replace $\psi$ by $h \mapsto \int_{K_0} \psi(k^{-1}hk) \, dk$ and assume that $\psi$ is invariant under conjugation by $K_0$. Let $h \in G_{\text{hyp}}$, then we can write $h = \alpha^{-1} \text{diag}(a, b) \alpha$ for some $\alpha \in G$ and $a, b \in \mathbb{Q}_p^\times$ satisfying $a \neq b$. We can write $\alpha \in TNK_0$. Since the sets $C_c$ are left $K_0$-invariant the function $h \mapsto \int_{C_c/Z} \psi(g^{-1}hg) \, dg$ is invariant under conjugation by $K_0$ and we can reduce to $\alpha \in TN$, and so $h \in TN$. Since any element of $T$ centralizes $\text{diag}(a, b)$ we can even assume $\alpha \in N$. Let $\hat{g} \in G$ and write $g = \text{diag}(x, 1) \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} k$ with $x \in \mathbb{Q}_p^\times$, $u \in \mathbb{Q}_p$ and $k \in K_0$. Recall from Lemma 2.64 that $g \in C_c$ if and only if $|v(x)| \leq c$ and $v(u) \geq (\psi(x) - c)/2$. Let $(TN)_c$ be the compact open and $Z$-invariant subset of $TN$ consisting of elements satisfying these two conditions. Using the integration formula for the Iwasawa decomposition (Lemma 2.12) and invariance under $K_0$-conjugation of $\psi$, we have

$$\int_{C_c/Z} \psi(g^{-1}hg) \, dg = \int_{(TN)_c/Z} \psi(n^{-1}t^{-1}htn) \, dt \, dn$$

Writing $h = \text{diag}(a, b) \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$ we have

$$n^{-1}t^{-1}htn = \text{diag}(a, b) \begin{pmatrix} 1 & x^{-1}z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & (1 - a^{-1}b)u \\ 0 & 1 \end{pmatrix}.$$ 

There exists an integer $d(\psi)$ and a finite family $(t_j)_{j \in J}$ of elements of $T$, distinct in $T/ZT_0$, such that $\text{supp}(\psi) \subset \bigcup_{j \in J} ZK_0 t_j N_{-d(\psi)}$. Then for any $\beta \in G$ there exists $\delta \in N$ such that $\text{supp}(\psi) \cap \beta N \subset \beta \delta N_{-d(\psi)}$. By cuspidality of $\psi$, for any $i \geq d(\psi)$ and any $\beta \in G$ we have $\int_{p^{-i}p_\beta} \psi \left( \beta \begin{pmatrix} 1 & u' \\ 0 & 1 \end{pmatrix} \right) \, |du'| = 0$ (if the support of $\psi$ is met then the integral is equal to the integral over $\mathbb{Q}_p$). We will apply this to $\beta(h, x) = \text{diag}(a, b) \begin{pmatrix} 1 & x^{-1}z \\ 0 & 1 \end{pmatrix}$.

For fixed $x \in \mathbb{Q}_p^\times$ such that $|v(x)| \leq c$ we integrate over $u \in \mathbb{Q}_p$ satisfying $v(u) \geq (\psi(x) - c)/2$. Using the change of variables $u' = (1 - a^{-1}b)u$ we compute

$$\int_{v(u) \geq (\psi(x) - c)/2} \psi \left( \beta(h, x) \begin{pmatrix} 1 & (1 - a^{-1}b)u \\ 0 & 1 \end{pmatrix} \right) \, |du| =$$

$$|1 - a^{-1}b|^{-1} \int_{v(u') \geq (1 - a^{-1}b) + (\psi(x) - c)/2} \psi \left( \beta(h, x) \begin{pmatrix} 1 & u'' \\ 0 & 1 \end{pmatrix} \right) \, |du'|$$


If \( v(1 - a^{-1}b) + (-v(x) - c)/2 \leq -d(\psi) \) this vanishes. Otherwise, that is if \(-c \leq v(x) < -c + 2v(1 - a^{-1}b) + 2d(\psi)\), denoting \( e = 2v(1 - a^{-1}b) - v(x) - c > -2d(\psi)\) we have

\[
\left| \int_{v(u') \geq e/2} \psi \left( \beta(h, x) \begin{pmatrix} 1 & u' \\ 0 & 1 \end{pmatrix} \right) |du'| \right| \leq \|\psi\|_\infty p^{-e/2}.
\]

Therefore, summing over possible values for \( v(x) \),

\[
\left| \int_{G / Z} \psi(g^{-1}h) d\bar{g} \right| \leq |1 - a^{-1}b|^{-1} \times \|\psi\|_\infty \sum_{e > -2d(\psi)} p^{-e/2}
\]

\[
\leq |1 - a^{-1}b|^{-1} \times p^{-d(\psi)}(1 - p^{-1/2})^{-1}\|\psi\|_\infty.
\]

Recall (see (3.2)) that \(|1 - a^{-1}b|^{-1} = |D(h)|^{-1/2} \delta_B^{1/2}(\text{diag}(a, b))\). Let \( C' \) be a compact subset of \( G \), then there exists \( \eta > 0 \) such that \( \delta_B^{1/2}(\text{diag}(a, b)) \leq \eta \) for any \( h \in C' \cap G_{rs}^{\text{hyp}} \) and any choice of ordering \((a, b)\) for its eigenvalues, and so our sequence of functions is uniformly bounded by constant \( \times |D(\cdot)|^{-1/2} \) on \( C' \cap G_{rs}^{\text{hyp}} \). Thanks to Proposition 3.10 (with \( \epsilon = 1/2 \)) this last function is integrable.

We are left to compute, for a fixed \( h \), the limit of \( \Theta_{\psi, c}(h) \) as \( c \to +\infty \). As observed above we can restrict to \( x \in \mathbb{Q}_p^\times \) satisfying \(-c \leq v(x) < -c + 2v(1 - a^{-1}b) + 2d(\psi)\) and so \( \beta(h, x) \to \text{diag}(a, b) \) uniformly in \( x \).

Therefore

\[
\lim_{c \to +\infty} \int_{C' / Z} \psi(g^{-1}h) d\bar{g} = 
\]

\[
|D(h)|^{-1/2} \delta_B^{1/2}(\text{diag}(a, b)) \sum_{e = -2d(\psi)}^{2v(1 - a^{-1}b)} \int_{v(u') \geq e/2} \psi \left( \text{diag}(a, b) \begin{pmatrix} 1 & u' \\ 0 & 1 \end{pmatrix} \right) |du'|
\]

We have

\[
\int_{v(u') \geq e/2} \psi \left( \text{diag}(a, b) \begin{pmatrix} 1 & u' \\ 0 & 1 \end{pmatrix} \right) |du'| = - \int_{e/2 > v(u') \geq -d(\psi)} \psi \left( \text{diag}(a, b) \begin{pmatrix} 1 & u' \\ 0 & 1 \end{pmatrix} \right) |du'|
\]

and so \(|1 - a^{-1}b| \Theta_{\psi, c}(h)\) is equal to

\[
2v(1 - a^{-1}b) \sum_{e = -2d(\psi)}^{2v(1 - a^{-1}b)} \int_{v(u') \geq e/2} \psi \left( \text{diag}(a, b) \begin{pmatrix} 1 & u' \\ 0 & 1 \end{pmatrix} \right) |du'|
\]

\[
= 2 \sum_{i = -d(\psi)}^{v(1 - a^{-1}b) - 1} (i - v(1 - a^{-1}b)) \int_{v(u') = i} \psi \left( \text{diag}(a, b) \begin{pmatrix} 1 & u' \\ 0 & 1 \end{pmatrix} \right) |du'|
\]

\[
= 2 \sum_{i = -d(\psi)}^{v(1 - a^{-1}b) - 1} (i - v(1 - a^{-1}b))(1 - a^{-1}b) \int_{v(u) = i - v(1 - a^{-1}b)} \psi \left( \text{diag}(a, b) \begin{pmatrix} 1 & (1 - a^{-1}b)u \\ 0 & 1 \end{pmatrix} \right) |du|
\]

\[
= 2 |1 - a^{-1}b| \int_{0 > v(u) \geq -d(\psi) - v(1 - a^{-1}b)} \psi \left( \text{diag}(a, b) \begin{pmatrix} 1 & (1 - a^{-1}b)u \\ 0 & 1 \end{pmatrix} \right) v(u) |du|
\]

Therefore, under the assumption that \( \psi \) is invariant under conjugation by \( K_0 \) we have

\[
\lim_{c \to +\infty} \Theta_{\psi, c}(h) = \int_N \psi(n^{-1}\text{diag}(a, b)n) \min(0, 2v(n)) \, dn.
\]
Recall that we can reduce to this case by averaging over $K_0$, and the formula given in the Lemma follows. The smoothness and invariance by conjugation of $\psi$ follow. \hfill \Box

**Corollary 3.18.** If $(V, \pi)$ is an irreducible supercuspidal representation of $G$ then Theorem 3.12 holds for $\pi$.

**Proof.** This follows from the previous three lemmas, the dominated convergence theorem and Proposition 3.10. \hfill \Box

**Remark 3.19.** This generalizes to arbitrary connected reductive groups: the Harish-Chandra character of an irreducible supercuspidal representation is given by the weighted orbital integral of any matrix coefficient whose value at 1 is the formal degree. See [Art87].

To conclude the proof of Theorem 3.12 we are left to consider non-supercuspidal representations.

**Proposition 3.20.** Let $\mu : T \to \mathbb{C}^\times$ be a smooth character, and consider $(V, \pi) = \text{Ind}_B^G \mu$. Then Theorem 3.12 holds for $\pi$, and $\Theta_\pi$ is the unique $G$-invariant function on $G_{rs}$ which vanishes identically on $G_{rs}^{\text{nil}}$ and such that for any $t \in T_{G-\text{reg}}$ we have $\Theta_\pi(t) = |D(t)|^{-1/2}(\mu(t) + \mu^w(t))$.

Note that since $\text{Ind}_B^G \mu$ may be reducible, it is not strictly speaking Theorem 3.12 that we prove for $\text{Ind}_B^G \mu$, but the statement makes sense.

**Proof.** Recall that we can realize $\text{Ind}_B^G \mu$ as the space of smooth functions $\phi : K_0 \to \mathbb{C}$ such that $\phi(bk) = \mu(b)\phi(k)$ for any $b \in B_0$. For such a $\phi$, $f \in \mathcal{H}(G)$ and $k_1 \in K_0$ we have

$$\pi(f)(\phi)(k_1) = \int_G f(g)\phi(k_1 g) dg = \int_G \phi(g)f(k_1^{-1}g) dg.$$  

Using the integration formula for the Iwasawa decomposition this also equals

$$\int_{K_0 \times B} \phi(b k_2) f(k_1^{-1} b k_2) dk_2 db = \int_{K_0} \phi(k_2) \int_B \mu(b) \delta_B^{1/2}(b) f(k_1^{-1} b k_2) db dk_2 = \int_{K_0} \phi(k_2) \psi(k_1, k_2) dk_2$$

with $\psi(k_1, k_2) = \int_B \mu(b) \delta_B^{1/2}(b) f(k_1^{-1} b k_2) db$ (as usual, using a left Haar measure on $B$). Note that $\psi$ is a smooth function on $K_0 \times K_0$. The operator $I(\psi) : \phi \mapsto (k_1 \mapsto \int_{K_0} \phi(k_2) \psi(k_1, k_2) dk_2)$ is defined for $\phi \in C^\infty(K_0)$, not just on the subspace $\text{Ind}_B^G \mu$ of $B_0$-equivariant functions for $\mu$. For $k_1, k_2 \in K_0$ and $x \in B_0$ we have

$$\psi(x k_1, k_2) = \int_B \mu(b) \delta_B^{1/2}(b) f(k_1^{-1} x^{-1} b k_2) db = \int_B \mu(x b') \delta_B^{1/2}(x b') f(k_1^{-1} b' k_2) db' = \mu(x) \psi(k_1, k_2)$$

using the change of variable $b' = x^{-1} b$. Therefore $I(\psi)$ maps $C^\infty(K_0)$ to $\text{Ind}_B^G \mu$ and coincides with $\pi(f)$ on $\text{Ind}_B^G \mu \subset C^\infty(K_0)$, and so $\text{tr} \pi(f) = \text{tr} I(\psi)$. Since $\psi$ is smooth it is left and right $K_i$-invariant for some $i \geq 1$, and so the image of $I(\psi)$ is contained in $C^\infty(K_0 \backslash G)$, and we can compute $\text{tr} I(\psi)$ on this finite-dimensional
the basis of characteristic functions of cosets of $K_i$ in $K_0$, we get that
\[ \text{tr } I(\psi) = \int_{K_0} \psi(k,k) \, dk \]
and so
\[ \text{tr } \pi(f) = \int_T \int_N \int_{K_0} \mu(t) \delta_B^{1/2}(t) f(k^{-1}tnk) \, dk \, dn \, dt = \int_T \mu(t) |D(t)|^{1/2} O_t(f) \, dt \]
and this equals
\[ \int_{G_{rs}} \Theta_\pi(g) f(g) \, dg = \frac{1}{2} \int_T |D(t)| (\mu(t) + \mu_w(t)) |D(t)|^{-1/2} O_t(f) \, dt \]
where $\Theta_\pi$ is defined as in the Proposition, thanks to the Weyl integration formula.

**Remark 3.21.** This generalizes to arbitrary connected reductive groups, see [vD72].

**Corollary 3.22.** Theorem 3.12 holds for the Steinberg representation, and for any elliptic maximal torus $T'$ of $G$ and any $t \in T'_{G_{rs}}$ we have $\Theta_{St}(t) = -1$.

**Proof.** For $\mu = \delta_B^{1/2}$ the semi-simplification of $\text{Ind}_B^G \mu$ is isomorphic to $1 \oplus \text{St}$ (where 1 denotes the trivial one-dimensional representation of $G$). Obviously the trivial representation satisfies Theorem 3.12 and $\Theta_1 = 1$, so we deduce Theorem 3.12 for the Steinberg representation and the relation $\Theta_{St} = \Theta_{\text{Ind}_B^G \delta_B^{1/2}} - \Theta_1$ on $G_{rs}$.

Of course the Proposition also allows us to compute $\Theta_{St}$ on the split maximal torus $T$.

We have just proved a special case of the Jacquet-Langlands correspondence: the Steinberg representation of $G = \text{GL}_2(\mathbb{Q}_p)$ will correspond to the trivial representation of $D^\times$.

**Remark 3.23.** This strategy of reduction to the supercuspidal case was not successful for arbitrary reductive groups (in general we do not have enough “obvious” cases like the trivial representation). Harish-Chandra [HC99] proved the general case by passing to the Lie algebra instead. This uses the exponential map, so this argument does not apply over positive characteristic local fields.

**Remark 3.24.** Any $\omega^{-1}$-equivariant smooth function with compact support modulo $Z$ can be written as $g \mapsto \int_Z \omega(z)^{-1} f(zg) \, dz$ for some $f \in C_\infty^c(G)$ (this can be shown using local sections of $G \to G/Z$, for example $(\text{SL}_2(\mathbb{Q}_p) \cap K_2) \times Z$ is isomorphic via the multiplication map to a neighbourhood of 1 in $G$). This implies (exercise) that for any $f \in \mathcal{H}(G,\omega^{-1})$, $\text{tr } \pi(f) = \int_{G/Z} f(g) \Theta_\pi(g) \, dg$.

3.5. **Coefficients and pseudo-coefficients.** We push further the argument used in the proof of Corollary 2.36.

**Proposition 3.25.** Let $(V, \pi)$ be an irreducible supercuspidal representation of $G$, and let $\omega_\pi$ be its central character.
(1) Let \((U, \sigma)\) be a smooth representation of \(G\) admitting central character \(\omega_\pi\). For \(v_0 \in \tilde{V}\) and \(u_0 \in U\), the linear map

\[ \phi_{v_0,u_0} : V \longrightarrow U \]

\[ v \mapsto \int_{G/Z} \langle \pi(g^{-1})v, \tilde{v}_0 \rangle \sigma(g)u_0 \, dg \]

is \(G\)-equivariant. In particular, it vanishes identically if \((U, \sigma)\) is irreducible but not isomorphic to \((V, \pi)\). For \((U, \sigma) = (V, \pi)\), \(\phi_{v_0,u_0} = d_\pi^{-1}\langle u_0, \tilde{v}_0 \rangle \text{Id}_V\).

(2) For \(v_0 \in V\) and \(\tilde{v}_0 \in \tilde{V}\) let \(f_{v_0,\tilde{v}_0} \in \mathcal{H}(G, \omega_\pi^{-1})\) be the matrix coefficient (of \(\tilde{V}\)) \(g \mapsto \langle \pi(g^{-1})v_0, \tilde{v}_0 \rangle = \langle v_0, \tilde{\pi}(g)\tilde{v}_0 \rangle\). Then for any irreducible smooth representation \((U, \sigma)\) of \(G\) having central character \(\omega_\pi\),

\[ \text{tr } \sigma(f_{v_0,\tilde{v}_0}) = \begin{cases} 0 & \text{if } \sigma \not\simeq \pi, \\ d_\pi^{-1}\langle v_0, \tilde{v}_0 \rangle & \text{if } \sigma \simeq \pi. \end{cases} \]

Proof. The first point was proved in the proof of Corollary 2.36. Let \((U, \sigma)\) be an irreducible smooth representation of \(G\) having central character \(\omega_\pi\). For \(u \in U\) we have \(\sigma(f_{v_0,\tilde{v}_0})u = \phi_{v_0,u}(v_0)\). The first point shows that \(\sigma(f_{v_0,\tilde{v}_0}) = 0\) if \(\sigma \not\simeq \pi\). The first point also shows that for \(v \in V\) we have \(\pi(f_{v_0,\tilde{v}_0})v = \phi_{v_0,v}(v_0) = d_\pi^{-1}\langle v, \tilde{v}_0 \rangle v_0\) and so \(\text{tr } \pi(f_{v_0,\tilde{v}_0}) = \text{tr } \pi(f_{v_0,\tilde{v}_0} | \mathbb{C} v_0) = d_\pi^{-1}\langle v_0, \tilde{v}_0 \rangle\). \(\square\)

In particular if we take \(v_0 \in V\) and \(\tilde{v}_0 \in \tilde{V}\) such that \(\langle v_0, \tilde{v}_0 \rangle = d_\pi\) then we have produced \(f \in \mathcal{H}(G, \omega_\pi^{-1})\) distinguishing \(\pi\) among all irreducible smooth representations of \(G\) having same central character. Note that for finite (or compact) groups there is a natural choice for such a function, namely the trace of the contragredient of \(\pi\), but this is not a smooth compactly supported function! We would like to have similar functions also for irreducible non-supercuspidal representations of \(G\). It turns out that this is not possible for an irreducible \(\text{Ind}_B^G \mu\), but it is almost possible for the Steinberg representation.

Recall that \(\tilde{w} = \begin{pmatrix} 0 & 1 \\ p & 0 \end{pmatrix} \in G\) normalizes \(I\). Denote \(\tilde{I} = \text{IZ}/Z \sqcup \tilde{w}\text{IZ}/Z\), a compact open subgroup of \(G/Z\). Let \(\text{sign} : \tilde{I} \rightarrow \{\pm 1\}\) be the character which is trivial on \(\text{IZ}/Z\) and maps \(\tilde{w}\) to \(-1\). Define \(f_{EP} \in \mathcal{H}(G/Z)\) as \(e_{K_0Z/Z} - e_{\tilde{I}\text{sign}}\) where \(e_{\tilde{I}\text{sign}}\) is \(\text{vol}(\tilde{I})^{-1}\text{sign}\) (extended by zero outside of \(\tilde{I}\)).

**Proposition 3.26.** For any smooth irreducible representation \((V, \pi)\) of \(G\) having trivial central character, we have

\[ \text{tr } \pi(f_{EP}) = \begin{cases} -1 & \text{if } \pi \simeq \text{St}, \\ 1 & \text{if } \pi \text{ is trivial}, \\ 0 & \text{otherwise}. \end{cases} \]

**Proof.** The function \(f_{EP}\) is bi-\(\text{IZ}/Z\)-invariant and so \(\pi(f) V \subset V^I\), in particular the trace vanishes if \(V^I = 0\). We classified the representations of \(G\) such that \(V^I \neq 0\) in Proposition 2.58 and before Proposition 2.56. The ones having trivial central
character are the characters \( \chi \circ \det \) for \( \chi : \mathbb{Q}_p^\times \to \{ \pm 1 \} \) unramified (there are two such characters), \( \chi \circ \det \otimes \text{St} \) for the same \( \chi \)'s, and the irreducible \( \text{Ind}_B^G \mu \) with \( \mu|_Z = 1 \) (i.e. \( \mu_1 \mu_2 = 1 \)) and \( \mu \) unramified. Note that \( \text{tr} \pi(e_{K_0/Z}) = \dim V^{K_0} \) and \( \text{tr} \pi(e_{\text{sign}}) = \dim \ker([I \tilde{w} I] + 1| V') \). They both equal 1 for \( \pi = \text{Ind}_B^G \mu \), or for \( \chi \circ \det \) where \( \chi \) is unramified of order 2. For the trivial representation, \( \dim V^{K_0} = 1 \) and \( \ker([I \tilde{w} I] + 1| V') = 0 \). For \( \chi \circ \det \otimes \text{St} \) we have \( V^{K_0} = 0 \) and \( \ker([I \tilde{w} I] + 1| V') \) has dimension one (resp. zero) if \( \chi \) is trivial (resp. non-trivial). \( \square \)

So apart from the trivial representation, \( -f_{EP} \) plays the same role for the Steinberg representation as the matrix coefficient for a supercuspidal representation. We call \( -f_{EP} \) a pseudo-coefficient for St.

Inspired by the supercuspidal case, we ask if the orbital integrals of \( f_{EP} \) are related to the Harish-Chandra character of the Steinberg representation.

**Theorem 3.27.** Let \( \gamma \in G_m \). Then

\[
O_\gamma(f_{EP}) = \begin{cases} 
0 & \text{if } \gamma \text{ is hyperbolic,} \\
\vol(G_\gamma/Z)^{-1} & \text{if } \gamma \text{ is elliptic.}
\end{cases}
\]

Exercise: prove the first case using Proposition 3.26 and Proposition 3.20.

For the proof we introduce a geometric tool. Recall that the discrete set \( G/K_0Z \) parametrizes lattices in \( \mathbb{Q}_p^2 \) up to rescaling, and that \( G/IZ \) parametrizes pairs \( (L, D) \) where \( L \) is a lattice in \( \mathbb{Q}_p^2 \) and \( D \subset L/pL \) is an \( \mathbb{F}_p \)-line, again up to rescaling. In particular to such a pair \( (L, D) \) we can associate another lattice: the preimage of \( D \) in \( L \). This motivates the following definition.

**Definition 3.28.** Denote \( \mathcal{V} = G/K_0Z \). Two lattices-up-to-rescaling \([L_1], [L_2] \subset \mathcal{V}\) are neighbours if \( L_1 \) and \( L_2 \) can be chosen so that \( L_2 \subset L_1 \) and \( |L_1/L_2| = p \).

This relation is symmetric \( (pL_1 \subset L_2 \text{ and } |L_2/pL_1| = p^2/p = p) \) and \( [L] \) is never its own neighbour. The neighbours of \([L]\) are naturally in bijection with the set of lines in \( L/pL \). Let \( \mathcal{E} \subset \mathcal{P}(\mathcal{V}) \) be the set of \( \{[L_1], [L_2]\} \) which are neighbours. This new notation suggests that \( \mathcal{V} \) is a set of vertices and \( \mathcal{E} \) is a set of edges, i.e. \( (\mathcal{E}, \mathcal{V}) \) is a graph (in some combinatorial sense). Let \( \mathcal{A} \) be the associated topological space, i.e. \( \mathcal{A} = \bigsqcup_{\{v_1, v_2\} \in \mathcal{V}} [v_1, v_2] / \sim \) where \([v_1, v_2] = [0, 1]\) (with a labelling of its endpoints) and the equivalence relation identifies a vertex \( v \) in all the edges having \( v \) as one of its endpoints.

**Proposition 3.29.** The graph \( (\mathcal{V}, \mathcal{E}) \) is a tree, i.e. it is connected (for any \( v, v' \in \mathcal{V}, \) there exist \( k \geq 2 \) and \( (v_1, \ldots, v_k) \in \mathcal{V}^k \) such that \( v_1 = v, v_k = v' \) and \( \{v_i, v_{i+1}\} \in \mathcal{E} \) for any \( i, \) i.e. a path between \( v \) and \( v' \) and does not contain any cycle (that is, a non-trivial path from \( v \) to \( v' \) such that \( v_1, \ldots, v_{k-1} \) are pairwise distinct).

**Proof.** Recall that \( G \) acts transitively on \( \mathcal{V} \) and \( \mathcal{E} \). In fact the Cartan decomposition says that for any \([L_1] \) and \([L_2] \) in \( \mathcal{V} \), there is a basis \((e, f)\) of \( L_1 \) and integers \( a \geq b \) such that \((p^a e, p^b f)\) is a basis of \( L_2 \). From uniqueness in the Cartan decomposition we get that \( a - b \in \mathbb{Z}_{\geq 0} \) is uniquely determined by the orbit of \([L_1], [L_2]\) under \( G \). Denote \( d([L_1], [L_2]) = a \), then \( d([L_2], [L_1]) = d([L_1], [L_2]) \) and \([L_1] \) and \([L_2] \) are
neighbours if and only if $d([L_1], [L_2]) = 1$. Up to rescaling one or both lattices we may assume that $b = 0$. Then

$$[L_1] = [Z_p e \oplus Z_p f] \leftrightarrow [Z_p p^a e \oplus Z_p f] \leftrightarrow \cdots \leftrightarrow [Z_p p^a e \oplus Z_p f] = [L_2]$$

is a path joining $[L_1]$ and $[L_2]$.

Now assume that $d([L_1], [L_2]) > 0$ (i.e. $[L_2] \neq [L_1]$) and that $[L_3]$ is a neighbour of $[L_2]$ distinct from $[Z_p p^{a-1} e \oplus Z_p f]$. This means that we can take $L_3$ to be the preimage of an $\mathbb{F}_p$-line $D$ in $L_2/pL_2$ distinct from $Z_p p^{a-1} e \oplus Z_p f / L_2$. This means that $L_3/pL_2$ is generated by $f + \lambda p^a e$ for some $\lambda \in \mathbb{Z}_p$. Up to replacing $f$ by $f + \lambda p^a e$ (note that this does not change the path from $[L_1]$ to $[L_2]$: $Z_p p^a e \oplus Z_p (f + \lambda p^a e) = Z_p p^a e \oplus Z_p f$ for $0 \leq i \leq a$), we can assume that $L_3 = Z_p p^{a+1} e \oplus Z_p f$. This shows that $d([L_1], [L_3]) = d([L_1], [L_2]) + 1$, in particular $[L_3] \neq [L_1]$.

We can extend the function $d(\cdot, \cdot)$ to $\mathcal{A}^2$ as follows:

(1) For $\{v_1, v_2\} \in \mathcal{E}$ and $x \in [v_1, v_2]$, identified to $[0, 1]$ as above via $\phi : [0, 1] \simeq [v_1, v_2]$, define $d(\phi(x), v_1) = d(v_1, \phi(x)) = x$.

(2) For $x, y \in \mathcal{A}$ define

$$d(x, y) = \min\{d(x, v_1) + d(v_1, v_2) + d(v_2, y) | \exists v_3, v_4 \in \mathcal{V}, \{v_1, v_3\} \in \mathcal{E}, \{v_2, v_4\} \in \mathcal{E}\}.$$

Exercise: check that this is well-defined, that $d(\cdot, \cdot)$ is a metric on $\mathcal{A}$ (a general fact for any graph), that $G$ acts by isometries, and that for any $x, y \in \mathcal{A}$ there is a unique geodesic in $\mathcal{A}$ from $x$ to $y$, denoted $[x, y]$ (a general fact for trees). Recall that a geodesic is (in this context) a continuous map $f : [0, d(x, y)] \to \mathcal{A}$ such that $f(0) = x$, $f(d(x, y)) = y$ and for any $t_1, t_2 \in [0, d(x, y)]$, $d(f(t_1), f(t_2)) = |t_1 - t_2|$.

The Bruhat-Tits tree (for more general groups, the Bruhat-Tits building, see [BT72] and [BT84]) is a $p$-adic analogue of symmetric spaces in the theory of real Lie groups. The Cartan fixed point theorem gives a geometric proof of conjugacy of maximal compact subgroups in a connected semisimple Lie group. The following lemma is the analogue in the present context (see [BT72, §3.2] for the general case).

**Lemma 3.30.** Let $K$ be a compact subgroup of $G/Z$. Then $A^K \neq \emptyset$. In particular, if $\gamma$ is an elliptic element of $G$ (i.e. if $\gamma \in Z \cup G_{rs}$) then $A^K \neq \emptyset$.

**Proof.** Choose $v_0 \in \mathcal{V}$ arbitrarily. Then $Kv_0 \subset \mathcal{V}$ is finite. Since any closed ball in $\mathcal{A}$ is compact, there exists $x \in \mathcal{A}$ minimizing $\max\{d(x, y) | y \in Kv_0\}$. Let us show that $x$ is unique. Let $x' \in \mathcal{A}$ be a different minimizer, and let $x'' \in [x, x']$ distinct from $x$ and $x'$. For any $y \in \mathcal{A}$, we have $d(x'', y) < \max(d(x, y), d(x', y))$. (Quick and dirty argument: $\mathcal{A} \setminus \{x''\}$ has finitely many connected components, and $x$ and $x''$ lie in different components, so $y$ is not in the same component as $x$ or $x'$, say $x$. Then the geodesic $[x, y]$ goes through $x''$.) This gives a contradiction. So $x$ is unique. For any $k \in K$, $kx$ has the same minimizing property, so $x$ is fixed by $K$.

If $\gamma$ is elliptic then the closure of the subgroup of $G/Z$ generated by $\gamma$ is compact. \qed
Proof of Theorem 3.27. Let \( v_0 = [\mathbb{Z}_p^2] \in \mathcal{V} \). For \( g \in G \), \( g^{-1}g \in K_0Z/Z \) if and only if \( \gamma \) fixes \( gv_0 \in \mathcal{V} \). Note that \( \tilde{I} \) is the stabilizer of \( e_0 = \{[\mathbb{Z}_p^2], [p\mathbb{Z}_p \times \mathbb{Z}_p]\} \in \mathcal{E} \) (observe that \( \tilde{w} \) swaps the two endpoints). Therefore \( g^{-1}g \in \tilde{I} \) if and only if \( \gamma \) fixes \( ge_0 \).

Using these facts, we get

\[
O_\gamma(f_{EP}) = \sum_{v \in G_\gamma \setminus \mathcal{V}^\gamma} \text{vol}(\text{Stab}_{G_\gamma/Z}(v))^{-1} - \sum_{e \in G_\gamma \setminus \mathcal{E}^\gamma} \text{vol}(\text{Stab}_{G_\gamma/Z}(e))^{-1}\text{sign}(\gamma, e)
\]

where \( \text{sign}(\gamma, e) = +1 \) if \( \gamma \) fixes \( e \) pointwise (i.e. if it fixes the endpoints of \( e \)) and \( \text{sign}(\gamma, e) = -1 \) if it swaps the endpoints.

Let us show that \( \mathcal{A}^\gamma \) connected. Let \( x, y \in \mathcal{A}^\gamma \), then \( \gamma([x, y]) \) is a geodesic from \( \gamma x = x \) to \( \gamma y = y \), so it equals \( [x, y] \) and since \( \gamma \) is an isometry every point of \( [x, y] \) is fixed by \( \gamma \).

First we consider the case where \( \gamma \) is elliptic. We have just proved that \( \mathcal{A}^\gamma \neq \emptyset \).

Now \( G_{\gamma}/Z \) is compact so by Lemma 3.4 the set \( \mathcal{A}^\gamma \) is compact, and we can expand the above expression to get

\[
O_\gamma(f_{EP}) = \sum_{v \in \mathcal{V}^\gamma} \text{vol}(G_{\gamma}/Z)^{-1} - \sum_{e \in \mathcal{E}^\gamma} \text{vol}(G_{\gamma}/Z)^{-1}\text{sign}(\gamma, e)
\]

Although it is somewhat artificial, we distinguish two cases:

- If there exists \( e \in \mathcal{E}^\gamma \) such that \( \text{sign}(\gamma, e) = -1 \), then denoting by \( x \) the middle point of \( e \), \( x \) is fixed by \( \gamma \) and \( \gamma \) swaps the two connected components of \( \mathcal{A} \setminus \{x\} \), so \( \mathcal{V}^\gamma = \emptyset \) and \( \mathcal{E}^\gamma = \{e\} \). It follows that \( O_\gamma(f_{EP}) = 1 \).

- Otherwise \( \mathcal{A}^\gamma \) is a subgraph of \( \mathcal{A} \), and we are left to compute the difference between the number of its edges and the number of its vertices (i.e. its Euler characteristic!). Since \( \mathcal{A}^\gamma \) is non-empty and connected, it is also a tree and one can give a simple elementary argument, by induction on the number of vertices (remove a vertex from the boundary, as well as the unique edge containing it; repeat until there is only one vertex left).

We now consider the hyperbolic case. The centralizer \( G_{\gamma}/Z \) is not compact, and \( \mathcal{A}^\gamma \) is compact if and only if it is empty. If \( \mathcal{A}^\gamma = \emptyset \) then the result is obvious, so we might as well assume that it is non-empty. Using the argument above we see that there is no \( e \in \mathcal{E}^\gamma \) such that \( \text{sign}(\gamma, e) = -1 \); in particular \( \mathcal{V}^\gamma \) is infinite. Up to conjugating, we may assume that \( \gamma \in T_{G-reg} \). It is easy to see that \( v(\det \gamma) \) has to be even, i.e. \( \gamma \in ZT_0 \). Obviously the connected subgraph \( \mathcal{X} \) with vertices \( \{p^a\mathbb{Z}_p \times \mathbb{Z}_p | a \in \mathbb{Z}\} \) (an apartment in the terminology of Bruhat and Tits), that we encountered in the proof of Proposition 3.29, is included in \( \mathcal{A}^\gamma \). The element \( t = \text{diag}(p, 1) \) of \( G_{\gamma} = T \) acts simply transitively on the set of vertices of \( \mathcal{X} \). For \( y \in \mathcal{A} \setminus \mathcal{X} \), there is a unique vertex \( x \) of \( \mathcal{X} \) such that \( d(x', y) > d(x, y) \) for any \( x' \in \mathcal{X} \setminus \{x\} \). We call this the projection of \( y \) on \( \mathcal{X} \), denote \( \text{pr}_{\mathcal{X}}(y) \). The fibres of \( \text{pr}_{\mathcal{X}} \) give a partition of \( \mathcal{A}^\gamma \setminus \mathcal{X} \), and \( \tilde{L}^2 \) acts simply transitively on this partition. Since \( \mathcal{A}^\gamma \) is connected, for any \( x \in \mathcal{E} \cap \mathcal{X} \) the subset \( \text{pr}_{\mathcal{X}}^{-1}(\{x\}) \cap \mathcal{A}^\gamma \) is finite. The quotient group \( G_{\gamma}/Zt\mathbb{Z} \) is compact, so the quotient \( \tilde{L}^2 \setminus \mathcal{A}^\gamma \) is finite, and arguing as in the elliptic case we see
that $O_\gamma(f_{EP})$ is proportional to the Euler characteristic of the graph $t^2\backslash A^\gamma$. Now this graph is very simple: $t^2\backslash A^\gamma$ has one vertex, with one edge from this vertex to itself (a loop), and so $t^2\backslash A^\gamma$ is simply obtained by attaching a finite tree to this vertex. The Euler characteristic of this graph is zero: by the same induction as in the elliptic case, we are reduced to the case of a loop, which has one vertex and one edge. □

**Remark 3.31.** (1) This beautiful geometric argument generalizes to an algebro-topologic one for a general connected reductive group, see [Ser71] and [Kot88].

(2) The computation of orbital integrals of $e_{K_0}$ (preceding Proposition 3.7), even of other elements of $H(G, K_0)$, can also be done geometrically using $A$, see [Kot05]. Like the “lattice-theoretic” computation, this is particular to $GL_2$.

To conclude, we have constructed a pseudo-coefficient $f_\pi$ for any essentially square-integrable representation $\pi$, having orbital integrals $\Theta_{\tilde{\pi}}$ on $G_{\text{ell}}$ and vanishing on $G_{\text{hyp}}$ (in fact any pseudo-coefficient satisfies this, but to prove this we would need the very natural fact that orbital integrals vanish if all traces vanish, and this is not obvious . . . ).


**Theorem 3.32.** If $\pi_1$ and $\pi_2$ are irreducible smooth essentially square-integrable representations of $G$ with $\omega_{\pi_1} = \omega_{\pi_2}$, we have

$$\sum_{T' \in T_{G-\text{reg}}/Z} \frac{1}{2} \int_{T'G_{\text{reg}}/Z} |D(t)| \Theta_{\pi_1}(t) \Theta_{\pi_2}(t) \text{vol}(T'/Z)^{-1} \, dt = \begin{cases} 1 & \text{if } \pi_1 \simeq \pi_2 \\ 0 & \text{otherwise}. \end{cases}$$

**Proof.** Recall that we have a pseudo-coefficient $f_{\pi_2} \in C^\infty_c(G, \omega_{\pi_2}^{-1})$.

$$\text{tr} \pi_1(f_{\pi_2}) = \int_{G/Z} f_{\pi_2}(g) \Theta_{\pi_1}(g) \, dg$$

$$= \sum_{T' \in T_{G-\text{reg}}/Z} \frac{1}{2} \int_{T'G_{\text{reg}}/Z} |D(t)| \Theta_{\pi_1}(t) \Theta_{\pi_2}(t) \, dt$$

$$= \sum_{T' \in T_{G-\text{reg}}/Z} \frac{1}{2} \int_{T'G_{\text{reg}}/Z} |D(t)| \Theta_{\pi_1}(t) \Theta_{\pi_2}(t) \text{vol}(T'/Z)^{-1} \, dt.$$ □

**Remark 3.33.** If $\omega_{\pi_1}$ is unitary (which can always be arranged after twisting), then both $\pi_i$’s are unitary and $\pi_1 \simeq \pi_2$ so that (exercise) $\Theta_{\pi_2} = \Theta_{\pi_2}$ and we recover the more familiar “orthogonality of characters” formulation.

3.7. Existence of supercuspidal representations.

**Theorem 3.34.** Let $\omega : Z \to C^\times$ be a smooth character. There exists an irreducible supercuspidal representation of $G$ having central character $\omega$.

**Proof.** Maybe later. □
We now change notations: $G$ will denote linear algebraic groups etc.

4.1. Quaternion algebras and inner forms of $\text{GL}_2$. For simplicity we consider a base field $K$ of characteristic zero (NB: in these notes “field” implies “commutative”). Recall that a quaternion algebra over $K$ is a 4-dimensional central simple algebra over $K$, i.e. a $K$-algebra $D$ such that there exists a finite separable extension $K'/K$ for which $K' \otimes_K D \simeq M_2(K')$. If $D \simeq M_2(K)$ then we say that $D$ is split. Recall that the group of automorphisms of $M_2(K')$ is $\text{PGL}_2(K')$ (via the adjoint action): to prove this, consider the idempotents $e = \text{diag}(1,0)$ and $f = \text{diag}(0,1)$ which satisfy $ef = fe = 0$, and show that any pairs of non-zero idempotents satisfying this relation is conjugated to $(e,f)$. This implies that quaternion algebras over $K$ are classified (up to isomorphism) by the Galois cohomology pointed set $H^1(K, \text{PGL}_2)$. More precisely, choose a finite Galois extension $K'/K$ and an isomorphism $\psi : M_2(K') \simeq K' \otimes_K D$. For $\sigma \in \text{Gal}(K'/K)$, $\sigma$ acts on $K' \otimes_K D$ (in the natural way on $K'$ and trivially on $D$), and $c(\sigma) := \psi^{-1} \circ \sigma \circ \psi \circ \sigma^{-1}$ is an automorphism of the $K'$-algebra $M_2(K')$, i.e. an element of $\text{PGL}_2(K')$. We obtain a 1-cocycle $c : \text{Gal}(K'/K) \to \text{PGL}_2(K')$, and it is easy to check that making different choices for $K'$ and $\psi$ amounts to taking another representative in $H^1(K, \text{PGL}_2)$.

By Galois descent any quaternion algebra $D$ has trace and determinant maps taking value in $K$, since these maps on $M_2(K')$ are invariant by conjugation. The trace map gives us the conjugation map $D \mapsto D$, $x \mapsto x^* := \text{tr}x - x$ which is an anti-automorphism of $D$. Any quaternion algebra $D$ defines a linear algebraic group $G$ over $K$, whose functor of points is as follows: for any (commutative) $K$-algebra $R$, let $G(R) = (R \otimes_K D)^\times$. In particular the base change of $G$ to $K'$ is isomorphic to $\text{GL}_2$. The group $G$ can also be described explicitly using the 1-cocycle $c$ introduced above: $\psi$ induces a natural isomorphism between the functor

$$\begin{align*}
R \rightsquigarrow \{g \in \text{GL}_2(K' \otimes_K R) & \mid \forall \sigma \in \text{Gal}(K'/K), \ Ad(c(\sigma))(\sigma(g)) = g\}\end{align*}$$

and $G$. We call $G$ the inner form of $\text{GL}_2$ associated to $D$ (because $\text{PGL}_2$ is the group of inner automorphisms of $\text{GL}_2$).

Via the adjoint action of $\text{PGL}_2$ on its Lie algebra, which is orthogonal for the Killing form, we have a natural isomorphism of linear algebraic groups $\text{PGL}_2 \simeq \text{SO}_3$, where $\text{SO}_3$ is the split special orthogonal group associated to the split quadratic form $(x,y,z) \mapsto x^2 + yz$ on $K^3$. Thanks to Hilbert’s 90 (more precisely, $H^1(K, \text{GL}_3) = \{1\}$) we see (exercise . . . ) that $H^1(K, \text{SO}_3)$ also classifies isomorphism classes of pairs $(V,q)$ where $V$ is a three-dimensional vector space over $K$ and $q$ is a non-degenerate quadratic form on $K$ of given discriminant. The space $(V,q)$ that corresponds to $D$ is $\ker \text{tr}$, endowed with the quadratic form $x \mapsto xx^*$. Conversely one recovers the quaternion algebra from $(V,q)$ by taking the even part of the Clifford algebra.

Remark 4.1. If $R$ is a commutative ring then $\text{PGL}_2(R)$ is somewhat ambiguously defined. The “correct” definition is not the obvious one $\text{GL}_2(R)/R^\times$. There are (at least) three correct definitions, the first two working in arbitrary dimension:

(1) Write down equations for $\text{PGL}_2$ (hint: writing matrix entries as $x_{i,j}$, the ring of functions on $\text{PGL}_2$ is the degree zero subalgebra of the graded algebra $\mathbb{Z}[x_{i,j}][y]/(y \det((x_{i,j})),1)$,
where $x_{i,j}$ has degree 1 and $y$ has degree $-2$).

(2) Sheafify (on the big Zariski site of Spec $\mathbb{Z}$) the naive definition.

(3) Prove that the kernel of $GL_2 \to SO_3$ is the obvious $GL_1$ (easy), and that this morphism of Zariski sheaves is surjective, i.e. for any local $K$-algebra $R$, the morphism $GL_2(R) \to SO_3(R)$ is surjective (a more interesting exercise).

Note that if one defines $PGL_2$ using one of the first two definitions then one has to check that the first two definitions are equivalent in order to do anything useful. These subtleties will not bother us because we will only be considering points over fields ($\mathbb{Q}$ and its localizations), local rings ($\mathbb{Z}_p$) or $\mathbb{A}$, and in all these cases the map $GL_2(R) \to PGL_2(R)$ is surjective.

It is not difficult to show that for $D$ a non-split quaternion algebra over $K$, the characteristic polynomial $X^2 - (\text{tr } x)X + \det x$ of any element $x$ of $D \setminus K$ is non-split over $K$ (otherwise we could find a non-trivial idempotent in $D$, of the form $ax + b$ for $a, b \in K$). In particular $K[x]$ is a quadratic extension of $K$ and is the centralizer of $x$ in $D$. We will call the subgroup $T$ of $G$ defined by $T(R) = R[x]^\times$ a maximal torus of $G$.

**Lemma 4.2.** Let $D$ be a non-split quaternion algebra. Two elements of $D$ are conjugated by $G(K)$ if and only if they have the same characteristic polynomial.

For any torus $T$ of $G$ we have $N_G(T(K))/T(K) = \mathbb{Z}/2\mathbb{Z}$.

**Proof.** Let $x, y \in D$ have the same characteristic polynomial. If one of them belongs to $K$ then the result is clear. Otherwise they are conjugated in $K' \otimes_K D$ for some finite extension $K'/K$, i.e. there exists $g \in G(K')$ such that $g x g^{-1} = y$. We can assume that $K'/K$ is Galois. Let $T$ be the maximal torus of $G$ which is the centralizer of $x$. Denote $E = K[x]$, a quadratic extension of $K$, so that $T \simeq \text{Res}_{E/K}(GL_1)$. For any $\sigma \in \text{Gal}(K'/K)$ we have $\sigma(g)x\sigma(g)^{-1} = y$ and so $\sigma(g)^{-1}g \in T(K')$, and this defines a 1-cocycle $\text{Gal}(K'/K) \to T(K')$. By Shapiro’s lemma we have $H^1(K, T) \simeq H^1(E, GL_1) = \{1\}$ (by Hilbert 90), so up to replacing $K'$ by a quadratic extension we can find $t \in T(K')$ such that $gt \in G(K)$.

Take $x \in T(K) \setminus K^\times$ and consider $x^{-1}\det x$: it is conjugated to $x$ in $G(K)$, this gives the non-trivial element of the normalizer.

Let us recall the classification results over local and global fields that we shall need.

**Theorem 4.3.**

(1) The pointed set $H^1(\mathbb{R}, PGL_2)$ has two elements, the non-trivial element corresponding to the “usual” quaternion algebra.

(2) For any prime number $p$, $H^1(\mathbb{Q}_p, PGL_2)$ has two elements. Explicitly, the non-trivial element corresponds to the quaternion algebra $D = E \oplus E\varpi$ where $E/\mathbb{Q}_p$ is an unramified quadratic extension, $\varpi^2 = p$ and conjugation by $\varpi$ on $E$ induces the non-trivial element of $\text{Gal}(E/\mathbb{Q}_p)$.

(3) The isomorphism class of a quaternion algebra $D$ over $\mathbb{Q}$ is determined by the finite set $S$ of places of $\mathbb{Q}$ such that $D_v := \mathbb{Q}_v \otimes \mathbb{Q} D$ is not isomorphic to $M_2(\mathbb{Q}_v)$, and this set has even cardinality. Conversely any finite set $S$ of
places of $\mathbb{Q}$ having even cardinality is associated to a quaternion algebra over $\mathbb{Q}$.

The first point is well-known (in fact the classification of quadratic spaces over $\mathbb{R}$ by their signature is well-known in any dimension). The second point can be proved by elementary means (essentially using Hensel’s lemma in the $p$-adic case), and generalized to quadratic spaces of arbitrary dimension. The third point (a special case of the theorem of Hasse-Minkowski, itself a special case of several theorems, many due to Kneser) is harder. See [Ser77, Ch. IV] for an elementary proof (over $\mathbb{Q}$).

We now recall two theorems proved in Gabriel Dospinescu’s course, using a slightly different formulation.

First we mention that for any affine scheme $X = \text{Spec} R$ of finite type over $\text{Spec}(\mathbb{Q})$, the set $X(\mathbb{A})$ of $\mathbb{A}$-points has a natural topology: if we choose $x_1, \ldots, x_n$ generating the $\mathbb{Q}$-algebra $R$ then we have a corresponding embedding $X(\mathbb{A}) \hookrightarrow \mathbb{A}^n$, and we can endow $X(\mathbb{A})$ with the induced topology. Since $X(\mathbb{A})$, being defined by polynomial equations, is closed in $\mathbb{A}^n$, it inherits the property of being Hausdorff and locally compact. The problem is to show that this topology does not depend on the choice of $x_1, \ldots, x_n$. Exercise: prove that this topology coincides with the topology induced by the embedding $X(\mathbb{A}) \hookrightarrow \mathbb{A}^R$ (here $\mathbb{A}^R$ is endowed with the product topology, what else?).

In particular $G(\mathbb{A})$ has a natural topology, making it a locally compact topological group. As in the case of $GL_2$ this topological group is also a restricted product over all places of $\mathbb{Q}$, namely the restricted product of the groups $G(\mathbb{Q}_v)$ with respect to the compact open subgroups $G(\mathbb{Z}_p)$. Note that this makes sense: since it is affine of finite type over $\mathbb{Q}$, we can find equations for the group scheme $G$ over $\mathbb{Z}[1/m]$ for some integer $m > 0$ (a model of $G$, i.e. a scheme $\mathcal{G}$ over $\mathbb{Z}[1/m]$ together with an isomorphism $\mathbb{Q} \times \mathbb{Z}[1/m] \mathcal{G} \simeq G$); then $G(\mathbb{Z}_p)$ is well-defined for all $p$ not dividing $m$. If we consider another model then the two possible definitions of $G(\mathbb{Z}_p)$ coincide for almost all $p$. Concretely, a basis of open neighbourhoods of $1 \in G(\mathbb{A})$ consists of $\prod_{v \in S} U_v \times \prod_{p \in S} G(\mathbb{Z}_p)$, where $S$ is a large enough finite set of places of $\mathbb{Q}$ and $U_v \subset G(\mathbb{Q}_v)$ is an open neighbourhood of $1 \in G(\mathbb{Q}_v)$.

**Remark 4.4.** These considerations were unnecessary for $GL_2$ because this group is naturally defined over $\mathbb{Z}$. Using a cocycle $c \in Z^1(\mathbb{Q}, PGL_2)$ introduced above, we can give a “concrete” model of $G$ as follows. For some finite Galois extension $K/\mathbb{Q}$, the cocycle $c$ is inflated from an element of $Z^1(\text{Gal}(K/\mathbb{Q}), PGL_2(K))$, that we abusively still denote $c$. Let $S$ be a finite set of primes, large enough so that every prime which ramifies in $K/\mathbb{Q}$ is in $S$ and $c$ takes values in $PGL_2(\mathcal{O}_{K,S})$, where $\mathcal{O}_{K,S} = \mathcal{O}_K[1/m]$ with $\mathcal{O}_K$ the ring of integers of $K$ and $m = \prod_{p \in S} p$. For simplicity we also assume that $2 \in S$. Then the functor (4.1) makes sense for $\mathbb{Z}[1/m]$-algebras $R$, giving us a model $\mathcal{G}$ of $G$ over $\mathbb{Z}[1/m]$. For $p \notin S$, choose a place $p$ of $K$ over $p$, and let $c_p \in Z^1(\text{Gal}(K_p/\mathbb{Q}_p), PGL_2(\mathcal{O}_{K,p}))$ (here $\mathcal{O}_{K,p}$ is the ring of integers of the completion $K_p$) be the 1-cocycle obtained by restricting $c$ to $\text{Gal}(K_p/\mathbb{Q}_p)$ and using the projection $\mathbb{Z}_p \otimes \mathbb{Z}[1/m] \mathcal{O}_{K,S} \to \mathcal{O}_{K,p}$. Writing Shapiro’s lemma explicitly, we see that the group scheme $\mathbb{Z}_p \otimes \mathbb{Z}[1/m] \mathcal{G}$ is given by the analogue of (4.1) for $c_p$. But one can show that $H^1(\text{Gal}(K_p/\mathbb{Q}_p), PGL_2(\mathcal{O}_{K,p})) = \{1\}$ (hint: first show that $H^1(k/\mathbb{F}_p, PGL_2(k)) = \{1\}$ for any finite extension $k/\mathbb{F}_p$ using the interpretation with
3-dimensional quadratic spaces, then use the filtration of $GL_2(O_{K,p})$ by congruence subgroups and $H^1(k/F_p, k) = \{0\}$. Thus there is an isomorphism $\mathbb{Z}_p \times \mathbb{Z}GL_2$, well-defined (from $c$) up to composing with conjugation by an element of $PGL_2(\mathbb{Z}_p)$. An alternative way to produce these isomorphisms is to consider orders in quaternion algebras, a maximal order in $M_2(\mathbb{Q}_p)$ being conjugated to $M_2(\mathbb{Z}_p)$.

Note that for $p \notin S$, we similarly have have $c_p \in Z^1(\text{Gal}(K_p/\mathbb{Q}_p), PGL_2(K_p))$, and its cohomology class is trivial if and only if $\mathbb{Q}_p \otimes_{\mathbb{Q}} D$ is split. Therefore under this assumption we get an isomorphism $\mathbb{Q}_p \times \mathbb{Q} G \simeq \mathbb{Q}_p \times \mathbb{Z} GL_2$, well-defined up to composition with conjugation by an element of $GL_2(\mathbb{Q}_p)$.

Let $G_{\text{ad}}$ be the special orthogonal in three variables associated to $D$, an inner form of $PGL_2$ which can be described by twisting $PGL_2$ by $c$ similarly to (4.1). As above we have a natural model $G_{\text{ad}}$ of $G_{\text{ad}}$ over $\mathbb{Z}[1/m]$. Using Remark 4.1 and the descriptions of $G$ and $G_{\text{ad}}$, we see that we have a short exact sequence of sheaves in groups over the big étale site of $\mathbb{Z}[1/m]$:

$$1 \rightarrow GL_1 \rightarrow G \rightarrow G_{\text{ad}} \rightarrow 1.$$ 

Since $H^1(\text{et}, GL_1) \simeq H^1(\text{et}, R, GL_1) \simeq \text{Pic}(R)$ (the Picard group) for any $\mathbb{Z}[1/m]$-algebra $R$, the morphism $G \rightarrow G_{\text{ad}}$ is already surjective on the big Zariski site. In particular, the morphisms $G(\mathbb{Q}) \rightarrow G_{\text{ad}}(\mathbb{Q})$ and $G(\mathbb{Q}_v) \rightarrow G_{\text{ad}}(\mathbb{Q}_v)$ ($v$ any place of $\mathbb{Q}$) are surjective, and for $p \notin S$ so is $G(\mathbb{Z}_p) \rightarrow G_{\text{ad}}(\mathbb{Z}_p)$. This also implies that $G(\mathbb{A}) \rightarrow G_{\text{ad}}(\mathbb{A})$ is surjective.

Recall the following special case of a theorem of Mostow and Tamagawa, proved in Gabriel Dospinescu’s course in the non-adic setting.

**Theorem 4.5.** Let $D$ be a non-split quaternion algebra over $\mathbb{Q}$, and $G$ the corresponding inner form of $GL_2$. Then $G(\mathbb{Q}) \backslash G(\mathbb{A})/Z(\mathbb{A})$ is compact.

Let $E$ be a quadratic extension of $\mathbb{Q}$. Then $(\mathbb{A} \otimes \mathbb{Q} E)^{\times}/\mathbb{A}^{\times}$ is compact.

There are useful variants of this formulation, for example $G(\mathbb{Q}) \backslash G(\mathbb{A})/\mathbb{R}_{>0}$ is also compact, since the map $G(\mathbb{Q}) \backslash G(\mathbb{A})/\mathbb{R}_{>0} \rightarrow G(\mathbb{Q}) \backslash G(\mathbb{A})/Z(\mathbb{A})$ is proper: the fibers are isomorphic to $\mathbb{Q}^{\times} \mathbb{A}^{\times}/\mathbb{R}^{>0} \simeq \prod_p \mathbb{Z}_p^{\times}$.

**Theorem 4.6.** Let $D$ and $G$ be as in the previous theorem. Let $\omega: Z(\mathbb{Q}) \backslash Z(\mathbb{A}) \rightarrow \mathbb{C}^{\times}$ be a continuous unitary character. Let $K_f$ be a compact open subgroup of $G(\mathbb{A}_f)$. Then the unitary representation $L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}), \omega)^{K_f}$ of $G(\mathbb{R})$ decomposes discretely.

**Remark 4.7.** There are variants of this formulation, for example the unitary representation $L^2(G(\mathbb{Q}) \backslash G(\mathbb{A})/R_{>0})$ of $G(\mathbb{A})/R_{>0}$ also decomposes discretely. This statement is more elegant because it does not isolate the Archimedean place of $\mathbb{Q}$ among all places, but the equivalence between the two statements is not trivial and it will be easier for us to work with levels $K_f$.

Recall that the proof relies on a general theorem of Gelfand, Graev and Piatetski-Shapiro: it is enough to show that for any $f \in C_0^c(G(\mathbb{R}), \omega^{-1})$, the operator

$$\rho(f): L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}), \omega)^{K_f} \rightarrow L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}), \omega)^{K_f}$$

$$\phi \mapsto \left( g \mapsto \int_{Z(\mathbb{R}) \backslash G(\mathbb{R})} \phi(gx) f(x) \, dx \right)$$
is compact. In fact it is even Hilbert-Schmidt (definition recalled below, as we will also need this notion).

4.2. Compact, Hilbert-Schmidt and trace-class operators. Let $V$ be a separable Hilbert space. Our convention is that Hermitian inner products are linear in the first variable. Recall that a continuous operator $T : V \to V$ is said to be compact if the image of any ball is relatively compact. Also recall that compact operators form a closed subspace of the space $\mathcal{B}(V)$ of continuous operators on $V$ (for the strong topology). The spectrum $\sigma(T)$ of a compact operator $T$ is such that for any $\epsilon > 0$, \{ $\lambda \in \sigma(T) \mid |\lambda| > \epsilon$\} is finite. We will use the spectral theory of compact operators only in the normal (even self-adjoint semi-positive definite) case. If $T$ is compact and normal then for $\lambda \in \sigma(T) \setminus \{0\}$ the eigenspace $\ker(T - \lambda \operatorname{Id}_V)$ is finite-dimensional, and we have an orthogonal decomposition $V = \bigoplus_{\lambda \in \sigma(T)} \ker(T - \lambda \operatorname{Id})$. Applying this to $T^*T$, we get the following “explicit” characterization of compact operators on Hilbert spaces.

Lemma 4.8. An operator $T : V \to V$ is compact if and only if there exist a set $J$ and orthonormal families $(f_j)_{j \in J}$ and $(g_j)_{j \in J}$ in $V$ and a family $(\lambda_j)_{j \in J}$ such that for any $\epsilon > 0$, \{ $j \in J \mid |\lambda_j| > \epsilon$\} is finite, and for any $v \in V$

\[ Tv = \sum_{j \in J} \lambda_j(v, f_j)g_j. \]

Proof. It is easy to check that for families as in the lemma, the sum converges for the operator norm, i.e. $T$ is a limit of finite rank operators. Therefore such a $T$ is compact. Moreover it is easy to compute $T^*v = \sum_{j \in J} \overline{\lambda_j}(v, g_j)f_j$, and we see that $f_j \in \ker(T^*T - |\lambda_j|^2 \operatorname{Id}_V)$.

Conversely and guided by this computation, take $(f_i)_{i \in I}$ an orthonormal basis of $V$ consisting of eigenvectors for $T^*T$, with eigenvalues $\rho_i \in \mathbb{R}_{\geq 0}$, and let $J = \{ i \in I \mid \rho_i > 0 \}$, $g_j = \rho_j^{-1/2}Tf_j$ and $\lambda_j = \rho_j^{1/2}$ for $j \in J$. \hfill \Box

The proof shows that if we impose $\lambda_j \in \mathbb{R}_{>0}$ then the families are essentially unique (up to reordering and choosing different bases for the eigenspaces of $T^*T$).

Recall that a continuous operator $T : V \to V$ is said to be Hilbert-Schmidt if for some orthonormal basis $(e_i)_{i \in I}$ of $V$ we have $\sum_{i \in I} \|Te_i\|^2 < \infty$. Also recall that any HS operator is compact.

Lemma 4.9. Let $T$ be a Hilbert-Schmidt operator on $V$. Then $\|T\|_{\text{HS}} := \sum_{i \in I} \|Te_i\|^2$ does not depend on the choice of an orthonormal basis $(e_i)_{i \in I}$ of $V$. Moreover $\|T^*\|_{\text{HS}} = \|T\|_{\text{HS}}$, and $\|\cdot\|_{\text{HS}}^2$ defines a Hermitian inner product on the space $\mathcal{B}(V)_{\text{HS}}$ of Hilbert-Schmidt operators on $V$, endowing it with a Hilbert space structure.

Finally, writing a compact operator $T : V \to V$ as in Lemma 4.8, we have that $T$ is Hilbert-Schmidt if and only if $\sum_{j \in J} |\lambda_j|^2 < \infty$.

Proof. Let $(f_j)_{j \in J}$ be another orthonormal basis of $V$. We have

\[ \sum_{i \in I} \|Te_i\|^2 = \sum_{i \in I} \sum_{j \in J} |(Te_i, f_j)|^2 = \sum_{i \in I} \sum_{j \in J} |(e_i, T^*f_j)|^2 = \sum_{j \in J} \|T^*f_j\|^2 \]
and this implies both independence of the choice of basis and \( \|T^*\|_{\text{HS}} = \|T\|_{\text{HS}} \). The rest is easy (that is, left as an exercise): any orthonormal basis \((e_i)_{i \in I}\) identifies \( \mathcal{B}(V)_{\text{HS}} \) with \( \ell^2(I, V) \), by \( T \mapsto (Te_i)_{i \in I} \).

Let \((X, \mu)\) be a separable measured space. Recall that HS operators on \( L^2(X, \mu) \) are identified with elements of \( L^2(X \times X, \mu \times \mu) \): a kernel \( K \in L^2(X \times X, \mu \times \mu) \) defines a Hilbert-Schmidt operator \( T_K : L^2(X, \mu) \to L^2(X, \mu) \) defined by

\[
T_K(f)(x) = \int_X f(y)K(x, y)d\mu(y).
\]

The expression given in Lemma 4.8 amounts to

\[
K(x, y) = \sum_{j \in J} \lambda_j g_j(x)\overline{f_j(y)}
\]

which is a sum of pairwise orthogonal elements of \( L^2(X \times X, \mu \times \mu) \). Exercise: check that \( \|T_K\|_{\text{HS}}^2 = \|K\|^2 \).

**Definition 4.10.** A continuous operator \( T : V \to V \) is trace class if it is compact and for any set \( J \) and any orthonormal families \((e_i)_{i \in I}\) and \((h_i)_{i \in I}\) in \( V \) we have \( \sum_{i \in I} |(Te_i, h_i)| < \infty \).

**Remark 4.11.** This is not the standard definition, as we impose compactness, but this one does not require us to define \( \sqrt{T^*T} \) for arbitrary \( T \in \mathcal{B}(V) \).

**Proposition 4.12.** Let \( V \) be a Hilbert space.

1. A linear combination of trace class operators \( V \to V \) is trace class.
2. The composition of two Hilbert-Schmidt operators is trace class.
3. A continuous operator \( T : V \to V \) is trace class if and only if

\[
\sum_{\rho \in \sigma(T^*T)} \sqrt{\rho} < \infty.
\]

(Equivalently, if \( T \) is compact and if, writing \( T \) as in Lemma 4.8, \( \sum_{j \in J} |\lambda_j| < \infty \) ) In particular, any trace class operator is Hilbert-Schmidt.

4. If \( T \) is trace class then \( \text{tr } T := \sum_{i \in I} (Te_i, e_i) \) does not depend on the choice of an orthonormal basis \((e_i)_{i \in I}\) of \( V \).

**Proof.**

1. Easy.

2. Using Cauchy-Schwarz,

\[
\sum_{i \in I} |(T_1T_2e_i, h_i)| = \sum_{i \in I} |(T_2e_i, T_1^*h_i)| \leq \sqrt{\sum_{i \in I} \|T_1T_2e_i\|^2} \sqrt{\sum_{i \in I} \|T_1^*h_i\|^2}.
\]
(3) Assume that $T$ is of trace class, then $T$ is compact so we can write $T$ as in Lemma 4.8. Taking $e_j = f_j$ and $h_j = g_j$, we see that $\sum_{j \in J} |\lambda_j| < \infty$.

Conversely, if $T$ can be written as in Lemma 4.8 with $\sum_{j \in J} |\lambda_j| < \infty$ then for any orthonormal families $(e_i)_{i \in I}$ and $(h_i)_{i \in I}$ we have

$$\sum_{i \in I} |(Te_i, h_i)| \leq \sum_{i \in I} \sum_{j \in J} |\lambda_j||e_i||f_j||g_j, h_i|$$

$$\leq \sum_{j \in J} |\lambda_j| \left(\sum_{i \in I} |(e_i, f_j)|^2 \right)^{1/2} \left(\sum_{i \in I} |g_j, h_i|^2 \right)^{1/2}$$

$$\leq \sum_{j \in J} |\lambda_j|$$

using the Cauchy-Schwarz inequality.

(4) Writing $T$ as in Lemma 4.8 we have

$$\sum_{i \in I} (Te_i, e_i) = \sum_{i \in I} \sum_{j \in J} \lambda_j(e_i, f_j)(g_j, e_i) = \sum_{j \in J} \lambda_j \sum_{i \in I} (e_i, f_j)(g_j, e_i) = \sum_{j \in J} \lambda_j(g_j, f_j)$$

where the exchange of $\sum$ signs is justified by absolute convergence:

$$\left(\sum_{i \in I} |e_i, f_j)(g_j, e_i)| \right)^2 \leq \left(\sum_{i \in I} |(e_i, f_j)|^2 \right) \left(\sum_{i \in I} |g_j, e_i|^2 \right) = 1.$$

\[\square\]

It seems that the only practical way of showing that an operator is trace class is to write it as a sum of product of Hilbert-Schmidt operators: an easy computation shows that for $T_1, T_2 \in B(V)_{HS}$ we have $\text{tr} T_1 T_2 = (T_2, T_1^*)_{HS}$. Let us make the trace more explicit in the case where $V = L^2(X, \mu)$. We have $T_i = T_{K_i}$ for $K_i \in L^2(X \times X, \mu \times \mu)$ and so $T_1 T_2 = T_{K}$ with $K(x, y) = \int_X K_1(x, z) K_2(z, y) d\mu(z)$ (exercise). Note that this makes sense in $L^2(X \times X, \mu \times \mu)$: the Cauchy-Schwarz inequality gives us

$$|K(x, y)|^2 \leq \left(\int_X |K_1(x, z)|^2 d\mu(z) \right) \left(\int_X |K_2(z, y)|^2 d\mu(z) \right)$$

and so $\|K\|^2 \leq \|K_1\|^2 \|K_2\|^2$. We also deduce that

$$\int_X |K(x, x)| d\mu(x) \leq \int_X \sqrt{\int_X |K_1(x, z)|^2 d\mu(z)} \sqrt{\int_X |K_2(z, x)|^2 d\mu(z)} d\mu(x)$$

$$\leq \|K_1\| \|K_2\|$$

where the last inequality is another application of Cauchy-Schwarz. This shows that the restriction of $K$ to the diagonal in $X \times X$ is well-defined (by $K_1$ and $K_2$, not by $K$ directly!) in $L^1(X, \mu)$. Thinking of the case where $X$ is finite, we guess that
Let $X$ be a locally compact, second-countable topological space and $\mu$ a Radon measure on $X$. If $T_K : L^2(X, \mu) \to L^2(X, \mu)$ is trace class and for almost all $y \in X$ the function $K(\cdot, y)$ is continuous then $x \mapsto K(x, x)$ is integrable with respect to $\mu$ and $\text{tr} T_K = \int_X K(x, x) d\mu(x)$.

**Proof.** As we saw above we can write

$$K(x, y) = \sum_{j \in J} \lambda_j g_j(x) \overline{f_j(y)}$$

with $(f_j)_{j \in J}$ and $(g_j)_{j \in J}$ orthonormal families in $L^2(X, \mu)$, $\lambda_j > 0$ and $\sum_{j \in J} \lambda_j < \infty$. Integrating over $X$, we see that the series $\sum_{j \in J} \lambda_j |g_j(x)|^2$ and $\sum_{j \in J} \lambda_j |f_j(y)|^2$ converge almost everywhere. Write $X$ as the increasing union of compact subsets $(C_k)_{k \geq 1}$. By Lusin’s theorem, for any $k \geq 1$, $j \in J$ and $\epsilon > 0$ there is an open subset $U_k$ of $C_k$ such that $\mu(U_k) < \epsilon$ and $g_j$ and $f_j$ are continuous on $C_k \setminus U_k$. By Egorov’s theorem, for any $k \geq 1$ and any $\epsilon > 0$ there is an open subset $U_k$ of $C_k$ such that $\mu(U_k) < \epsilon$ and $\sum_{j \in I} \lambda_j |g_j|^2$ and $\sum_{j \in J} \lambda_j |f_j|^2$ converge uniformly on $C_k \setminus U_k$. Putting these two results together, we get (exercise) that there is a sequence $(C'_k)_{k \geq 1}$ of compact subsets of $X$ with $C'_k \subset C_k$ and $C'_k \subset C'_{k+1}$, such that for any $k \geq 1$ we have

- $\mu(C_k \setminus C'_k) < 1/k$,
- for any $j \in J$, $f_j$ and $g_j$ are continuous on $C'_k$,
- $\sum_{j \in J} \lambda_j |f_j|^2$ and $\sum_{j \in J} \lambda_j |g_j|^2$ converge uniformly on $C'_k$.

Replacing $C'_k$ be its smallest closed subset of full measure (note that second-countability is used here), we may also assume that $C'_k$ does not admit any proper closed subset of full measure. It follows from the Cauchy-Schwarz inequality that

$$K'(x, y) := \sum_{j \in J} \lambda_j g_j(x) \overline{f_j(y)}$$

converges uniformly on $C'_k \times C'_k$, and so it defines a continuous function on $C'_k \times C'_k$, which coincides with $K$ away from a negligible set. More precisely, let $S_k = \{ y \in C'_k \mid \int_{C'_k} |K(x, y) - K'(x, y)| d\mu(x) > 0 \}$, so that $\mu(S_k) = 0$. For any $y \in C'_k \setminus S_k$, the set of $x$ in $C'_k$ where $K(x, y) \neq K'(x, y)$ has measure zero, and since both $K(\cdot, y)$ and $K'(\cdot, y)$ are continuous it is also open and by construction of $C'_k$ it is empty.
Let $X' = \left( \bigcup_k C'_k \right) \setminus \left( \bigcup_k S_k \right)$. Taking all $k$ into consideration, we get that $K$ coincides with $K'$ on $(X')^2$. We have $\mu(X \times X') \leq \limsup_k \mu(C_k \setminus C'_k) = 0$. Finally

$$\text{tr} T_K = \sum_{j \in J} \lambda_j \int_X g_j(x)f_j(\overline{x}) \, d\mu(x) = \sum_{j \in J} \lambda_j \int_{X'} g_j(x)f_j(\overline{x}) \, d\mu(x) = \int_{X'} K(x, x) \, d\mu(x)$$

where the last equality is given by the dominated convergence theorem (using $|g_j(x)f_j(x)| \leq (|g_j(x)|^2 + |f_j(x)|^2)/2$ and $\sum_{j \in J} \lambda_j < \infty$) and also shows that $x \mapsto K(x, x)$ is integrable.

4.3. The trace formula for anisotropic groups. Let $D$ be a non-split quaternion algebra over $\mathbb{Q}$ and denote by $G$ the corresponding inner form of $\operatorname{GL}_2$.

Recall that a smooth function on $G(\mathbb{A})$ is $f : G(\mathbb{A}) \to \mathbb{C}$ such that for any $g \in G(\mathbb{A})$, there exists $U_\infty$ an open neighbourhood of $g_\infty$ in $G(\mathbb{R})$ and $U_f$ a neighbourhood of $g_f$ in $G(\mathbb{A}_f)$, and $\psi : U_\infty \to \mathbb{C}$ a smooth function, such that $f(x) = \psi(x_\infty)$ for any $x \in U_\infty \times U_f$. Similarly, for any $k \geq 1$ we define functions of class $C^k$ on $G(\mathbb{A})$ (note that these are “smooth”, i.e. locally constant, on the finite adèlic factor $G(\mathbb{A}_f)$). Exercise: show that any $f \in C^k_c(G(\mathbb{A}))$ is a linear combination of functions of the form $\prod_v f_v$ where $f_\infty \in C^k_c(G(\mathbb{R}))$, for any prime number $p$ $f_p \in C^\infty_c(G(\mathbb{Q}_p))$ and for almost all prime numbers $p$, $f_p$ is the characteristic function of $G(\mathbb{Z}_p)$. Similarly, for $\omega : A^\times \to \mathbb{C}^\times$ a continuous character, any function in $C^k_c(G(\mathbb{A}), \omega^{-1})$ is a linear combination of functions of the form $\prod_v f_v$ where for almost all prime numbers $p$, $f_p$ is supported on $G(\mathbb{Z}_p)Z(\mathbb{Q}_p)$ and for any $k \in G(\mathbb{Z}_p)$ we have $f_p(k) = 1$.

Also recall that any Haar measure on $G(\mathbb{A})$ is given by a collection of Haar measures on $G(\mathbb{Q}_v)$ such that for almost all prime numbers $p$, $\operatorname{vol}(G(\mathbb{Z}_p)) = 1$.

We will consider orbital integrals of functions on $G(\mathbb{Q}_v)$, $v$ any place of $\mathbb{Q}$, and $G(\mathbb{A})$. For the case where $v$ is non-Archimedean and $\mathbb{Q}_v \otimes \mathbb{Q} D$ is split we defined and studied orbital integrals in Section 3.2. For any place $v$ such that $\mathbb{Q}_v \otimes \mathbb{Q} D$ is not split, for any $\gamma \in G(\mathbb{Q}_v)$ the quotient $G_{\gamma}(\mathbb{Q}_v)/G(\mathbb{Q}_v)$ is compact and so the theory is easy (of course, explicit computations are not so easy . . . ). For $\operatorname{GL}_2(\mathbb{R})$, the (formal) definition of $O_\gamma(f)$ in Definition 3.1, the computation for $\gamma$ semi-simple regular hyperbolic (3.2), and Lemma 3.4 (showing that for any $f \in C^0_c(G(\mathbb{R})), O_\gamma(f)$ is the integral of a continuous compactly supported function) all adapt. Of course in the real case orbital integrals are almost never finite sums as in (3.1). As in the $p$-adic case we have similar results when $f \in C^0_c(G(\mathbb{R}), \omega^{-1})$. The analogous property in the adèlic setting is the following fact.

**Lemma 4.14.** Let $\omega : A^\times \to \mathbb{C}^\times$ a continuous character. Fix Haar measures on $G_{\text{ad}}(\mathbb{Q}_v)$ such that $\operatorname{vol}(G_{\text{ad}}(\mathbb{Z}_p)) = 1$ for almost all prime numbers $p$. Let $f = \prod_v f_v \in C^k_c(G(\mathbb{A}), \omega^{-1})$ (as discussed above). Let $\gamma \in G(\mathbb{Q})$ be semi-simple. Then for almost all $p$ the set of $[g_p] \in G_{\gamma}(\mathbb{Q}_p)/G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ such that $g_p^{-1} \gamma g_p \in G(\mathbb{Z}_p)Z(\mathbb{Q}_p)$ is simply $\{[1]\}$, and so $\operatorname{vol}(G_{\gamma}(\mathbb{Q}_p)/Z(\mathbb{Q}_p))O_\gamma(f_p) = 1$ for almost all $p$. In particular the function $g \mapsto f(g^{-1} \gamma g)$ in $C^k_c(G_{\gamma}(\mathbb{A})/G(\mathbb{A}))$ is compactly supported, and $O_\gamma(f) = \prod_v O_\gamma(f_v)$.

**Proof.** Left as an exercise, using formula (3.2) and the argument around Lemma 3.6 (for almost all $p$ we have $\mathbb{Z}_p[\gamma] = O_E$).
Recall the following theorem which was stated in Gabriel Dospinescu’s course, that we will not prove either.

**Theorem 4.15** ([DM78]). For any \( f \in C_c^\infty(G(\mathbb{R})) \) there exist \( k \geq 1 \) and \( f_1, g_1, \ldots, f_k, g_k \in C_c^\infty(G(\mathbb{R})) \) such that \( f = \sum_i f_i * g_i \).

**Remark 4.16.** In the applications in this course a weaker result would be enough, with \( g_i \in C_c^k(G(\mathbb{R})) \) for a large enough integer \( k \). This weaker result is easier to prove (although far from trivial: another use of elliptic operators . . . ); see [DL71, §I.1.10] and [War79, Theorem 4.3 and Lemma 4.5]. In fact the reader can check that all consequences of trace formulas that we will prove could be proved by only considering functions of the form \( \sum \gamma, f_i * g_i \) (without using that any smooth function can be written in this manner). In other words, these results are not strictly necessary for the purpose of these notes. However, avoiding them would make the formulation of certain results more complicated, and require more computations.

**Corollary 4.17.** Let \( \omega : Z(\mathbb{Q}) \backslash Z(\mathbb{A}) \to \mathbb{C}^\times \) be a continuous unitary character. Then for any \( f \in C_c^\infty(G(\mathbb{A}), \omega^{-1}) \) there exist \( k \geq 1 \) and \( f_1, g_1, \ldots, f_k, g_k \in C_c^\infty(G(\mathbb{A}), \omega^{-1}) \) such that \( f = \sum_i f_i * g_i \).

**Theorem 4.18.** Fix a Haar measure on \( G_{ad}(\mathbb{A}) \) and a continuous unitary character \( \omega : Z(\mathbb{Q}) \backslash Z(\mathbb{A}) \to \mathbb{C}^\times \). For any \( f \in C_c^\infty(G(\mathbb{A}), \omega^{-1}) \), the operator \( \rho(f) \) on \( L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}), \omega) \) is trace class and

\[
\text{tr} \rho(f) = \sum_{[\gamma]} \iota(\gamma)^{-1} \text{vol}(G_\gamma(\mathbb{Q}) \backslash G(\mathbb{A})) O_\gamma(f)
\]

where the sum is over conjugacy classes of elements \( \gamma \) in \( G_{ad}(\mathbb{Q}) \), \( \iota(\gamma) \) is the index of \( G_\gamma(\mathbb{Q}) \backslash Z(\mathbb{Q}) \) in \( \text{Cent}(\gamma, G_{ad}(\mathbb{Q})) \) (exercise: \( \iota(\gamma) \in \{1, 2\} \), and \( \iota(\gamma) = 2 \) if and only if \( \text{tr} \gamma = 0 \), and only finitely many terms in the sum are non-zero.

Note that the product \( \text{vol}(G_\gamma(\mathbb{Q}) \backslash Z(\mathbb{A})) O_\gamma(f) \) does not depend on the choice of a Haar measure on \( G_\gamma(\mathbb{A})/Z(\mathbb{A}) \). Observe also that \( f \) is bi-\( K_f \)-invariant for some compact open subgroup \( K_f \) of \( G(\mathbb{A}) \), and so we could replace \( L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}), \omega) \) by \( L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}), \omega)^{K_f} \) (same trace), and find ourselves in the setting of Theorem 4.6.

**Proof.** We have

\[
(\rho(f) \phi)(x) = \int_{G_{ad}(\mathbb{A})} \phi(xy) f(y) \ d\gamma
\]

\[
= \int_{G_{ad}(\mathbb{A})} \phi(y) f(x^{-1}y) \ d\gamma
\]

\[
= \int_{G_{ad}(\mathbb{Q}) \backslash G_{ad}(\mathbb{A})} \phi(y) \sum_{\gamma \in G_{ad}(\mathbb{Q})} f(x^{-1} \gamma y) \ d\gamma
\]

\[
= \int_{G_{ad}(\mathbb{Q}) \backslash G_{ad}(\mathbb{A})} \phi(y) K_f(x, y) \ d\gamma
\]

with \( K_f(x, y) = \sum_{\gamma \in G_{ad}(\mathbb{Q})} f(x^{-1} \gamma y) \). For \( x \) and \( y \) in a compact subset \( C \) of \( G(\mathbb{A}) \), there is a finite subset \( F(C, \text{supp}(f)) \) of \( G_{ad}(\mathbb{Q}) \) such that for \( x, y \in C \)
and $\gamma \in G_{\text{ad}}(\mathbb{Q}) \setminus F(C, \text{supp}(f))$ we have $f(x^{-1} \gamma y) = 0$, since $G_{\text{ad}}(\mathbb{Q})$ is discrete in $G_{\text{ad}}(\mathcal{A})$. In particular the function $K_f$ on $(G(\mathbb{Q}) \setminus G(\mathcal{A}))^2$ is continuous. Moreover $K_f(z_1 x, z_2 y) = \omega(z_1 z_2^{-1}) K_f(x, y)$ for $z_1, z_2 \in Z(\mathcal{A})$, so $|K_f|$ induces a bounded function on the compact topological space $(G_{\text{ad}}(\mathbb{Q}) \setminus G_{\text{ad}}(\mathcal{A}))^2$, in particular $|K_f| \in L^2((G_{\text{ad}}(\mathbb{Q}) \setminus G_{\text{ad}}(\mathcal{A}))^2)$. This shows that $\rho(f)$ is Hilbert-Schmidt. To show that it is of trace class, use the Dixmier-Malliavin theorem which expresses $\rho(f)$ as $\sum_i \rho(f_i) \rho(q_i)$ and apply Proposition 4.12. Finally Theorem 4.13 (or the Dixmier-Malliavin expression, see the discussion before Theorem 4.13) shows that

$$\text{tr } \rho(f) = \int_{G_{\text{ad}}(\mathbb{Q}) \setminus G_{\text{ad}}(\mathcal{A})} K_f(x, x) \, dx.$$  

Note that integrability of $K_f$ (and of $|K_f|$, defined analogously even though $|f|$ may not be differentiable ...) can also be checked directly, without using Theorem 4.13: $x \mapsto K_{|f|}(x, x)$ is a continuous function on the compact topological space $G_{\text{ad}}(\mathbb{Q}) \setminus G_{\text{ad}}(\mathcal{A})$. This justifies the inversion of integral signs in the following:

$$\int_{G_{\text{ad}}(\mathbb{Q}) \setminus G_{\text{ad}}(\mathcal{A})} K_f(x, x) \, dx = \sum_{[\gamma]} \int_{G_{\text{ad}}(\mathbb{Q}) \setminus G_{\text{ad}}(\mathcal{A})} f(x^{-1} \delta x) \, dx$$

$$= \sum_{[\gamma]} \int_{G_{\text{ad}}(\mathbb{Q}) \setminus G_{\text{ad}}(\mathcal{A})} \sum_{\alpha \in \text{Cent}(\gamma, G_{\text{ad}}(\mathbb{Q})) \setminus G_{\text{ad}}(\mathcal{Q})} f(x^{-1} \alpha^{-1} \gamma \alpha x) \, dx$$

$$= \sum_{[\gamma]} \int_{G_{\text{ad}}(\mathbb{Q}) \setminus G_{\text{ad}}(\mathcal{A})} \iota(\gamma)^{-1} \sum_{\alpha \in G_{\gamma}(\mathcal{Q}) \setminus G(\mathcal{Q})} f(x^{-1} \alpha^{-1} \gamma \alpha x) \, dx$$

Thus

$$\text{tr } \rho(f) = \sum_{[\gamma]} \iota(\gamma)^{-1} \int_{(G_{\gamma}(\mathcal{Q}) \setminus Z(\mathcal{Q})) \setminus G_{\text{ad}}(\mathcal{A})} f(x^{-1} \gamma x) \, dx$$

$$= \sum_{[\gamma]} \iota(\gamma)^{-1} \int_{G_{\gamma}(\mathcal{Q}) \setminus G(\mathcal{Q})} \int_{(G_{\gamma}(\mathcal{Q}) \setminus Z(\mathcal{Q})) \setminus G_{\gamma}(\mathcal{A})} f(x^{-1} y^{-1} \gamma y x) \, dy \, dx$$

$$= \sum_{[\gamma]} \iota(\gamma)^{-1} \int_{G_{\gamma}(\mathcal{A}) \setminus G(\mathcal{A})} \text{vol}((G_{\gamma}(\mathcal{Q}) \setminus Z(\mathcal{Q})) \setminus G_{\gamma}(\mathcal{A})) f(x^{-1} \gamma x) \, dx$$

$$= \sum_{[\gamma]} \iota(\gamma)^{-1} \text{vol}((G_{\gamma}(\mathcal{Q}) \setminus Z(\mathcal{Q})) \setminus G_{\gamma}(\mathcal{A})) O_{\gamma}(f).$$

Finally we must prove that only finitely many conjugacy classes $[\gamma]$ in $G_{\text{ad}}(\mathbb{Q})$ satisfy $O_{\gamma}(f) \neq 0$. In fact this follows from the proof of continuity of $K_f$ above: choose a compact subset $C \subset G_{\text{ad}}(\mathcal{A})$ which surjects onto $G_{\text{ad}}(\mathbb{Q}) \setminus G_{\text{ad}}(\mathcal{A})$, then on $C \times C$ only finitely many elements on $G_{\text{ad}}(\mathbb{Q})$ contribute to the sum defining $K_f$. Although it is not absolutely necessary, let us give a direct argument which does not use compactness of $G_{\text{ad}}(\mathbb{Q}) \setminus G_{\text{ad}}(\mathcal{A})$. Recall that for any field of characteristic zero $F$, conjugation classes in $G(F)$ are parametrized by trace and determinant. This implies that the map $\nu : \text{tr}^2 / \det : G_{\text{ad}}(F) \to F$ is an invariant of conjugacy classes in $G_{\text{ad}}(F)$. Consider the compact subset $\text{supp}(f)$ of $G_{\text{ad}}(\mathcal{A})$, and its image $\nu(\text{supp}(f))$ in $\mathcal{A}$. Since $\mathbb{Q}$ is discrete in $\mathcal{A}$, the subset $F \cap \nu(\text{supp}(f))$ of $F$ is finite.
Unfortunately the invariant \( \nu \) does not completely characterize conjugacy (if \( D \) was split and \( G = \text{GL}_2 \) a simple example would be \( \text{diag}(-1,1) \) and \( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \)). To conclude we also consider arithmetic invariants of conjugacy classes in \( \text{G}_{\text{ad}}(\mathbb{Q}) \). If \( \overline{g} \in \text{G}_{\text{ad}}(F) \) then for a lift \( g \in \text{G}(F) \) of \( \overline{g} \), the image \( \zeta(g) \) of \( \det g \) in \( F^*/F^{*2} \) does not depend on the choice of the lift \( g \), and is clearly invariant by conjugation. (Identifying \( \text{G}_{\text{ad}} \) with a special orthogonal group as explained in Section 4.1, \( \zeta(g) \) is the spinor norm of \( g \).) There exists a finite set \( S' \) of places of \( \mathbb{Q} \), containing the Archimedean place and all finite places where \( \text{norm of } f \) depend on the choice of the lift \( g \), and is clearly invariant by conjugation. (Identifying \( \text{G}_{\text{ad}} \) with a special orthogonal group as explained in Section 4.1, \( \zeta(g) \) is the spinor norm of \( g \).) Since \( \text{G}(\mathbb{Q}_p) \) is a compact subgroup of \( \text{G}(\mathbb{Q}_p) \) we have \( \det \text{G}(\mathbb{Z}_p) \subset \mathbb{Q}_p^* \) and so a necessary condition for the non-vanishing of \( O_\gamma(f) \) is that \( v_p(\zeta(\gamma)) \in 2\mathbb{Z} \) (note that this parity is well-defined!) for all \( p \) not in \( S \). Since \( S \) is finite this only leaves finitely many possible values for \( \zeta(\gamma) \) in \( \mathbb{Q}^*/\mathbb{Q}^{*2} \), and since we have already seen that \( (\text{tr } \gamma)^2/\det \gamma \) can only take finitely many values, this implies that up to the action of \( \mathbb{Z}(\mathbb{Q}) \), the pair \( (\text{tr } \tilde{\gamma}, \det \tilde{\gamma}) \), where \( \tilde{\gamma} \in \text{G}(\mathbb{Q}) \) lifts \( \gamma \), can only take finitely many values.

Of course this formula is useful in combination with Theorem 4.6. Recall that this theorem gives a canonical orthogonal decomposition

\[
L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}), \omega)^{K_f} = \bigoplus_{\pi_\infty \in \text{G}(\mathbb{R})} \text{Hom}_{G(\mathbb{R})}(\pi_\infty, L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}), \omega)^{K_f}) \otimes \pi_\infty
\]

where each \( \text{Hom}_{\text{G}(\mathbb{R})}(\pi_\infty, L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}), \omega)^{K_f}) \) is finite-dimensional and has an action of \( \mathcal{H}(G(\mathbb{A}_f), K_f, \omega_f^{-1}) \). In particular for any irreducible unitary representation \( \pi_\infty \) of \( \text{G}(\mathbb{R}) \) having central character \( \omega_\infty \),

\[
\lim_{K_f} \text{Hom}_{\text{G}(\mathbb{R})}(\pi_\infty, L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}), \omega)^{K_f})
\]

is an admissible representation of \( \text{G}(\mathbb{A}_f) \) having central character \( \omega_f \). It is endowed with a natural \( \text{G}(\mathbb{A}_f) \)-invariant Hermitian inner product (canonical up to \( \mathbb{R}_{>0} \)), and so it is semi-simple. (Exercise: formulate unitarity of the \( \text{G}(\mathbb{A}_f) \)-action in terms of the Hecke algebra action. Which formulation is clearer?)

Thus we have

\[
(4.3) \quad \text{tr } \rho(f) = \sum_\pi m^G(\pi) \text{ tr } \pi(f)
\]

where \( m^G(\pi) \in \mathbb{Z}_{\geq 0} \), the sum is over all isomorphism classes of tensor products \( \pi = \pi_\infty \otimes \pi_f \) with \( \pi_\infty \) a unitary irreducible representation of \( \text{G}(\mathbb{R}) \) with central character \( \omega_\infty \) and \( \pi_f \) a smooth admissible unitary irreducible representation of \( \text{G}(\mathbb{A}_f) \) with central character \( \omega_f \), and all but countably many \( m^G(\pi) \) vanish. Note that the sum is absolutely convergent (by definition of trace class operators), but has infinitely many terms in general. Also note that each \( \pi_f \) decomposes as a restricted tensor product \( \bigotimes_f \pi_f \) of irreducible smooth representations of \( \text{G}(\mathbb{Q}_p) \), almost all of which are unramified (and endowed with a non-zero invariant under \( \text{G}(\mathbb{Z}_p) \ldots \)). We will say that \( \pi \) is an automorphic representation if \( m^G(\pi) > 0 \).
Remark 4.19. Recall from Gabriel Dospinescu’s course that if we fix a maximal compact subgroup \( K_\infty \) of \( G(\mathbb{R}) \) then we also have a decomposition of the space of square-integrable automorphic forms

\[
\mathcal{A}^2(\mathbb{G}(\mathbb{Q})\backslash \mathbb{G}(\mathbb{A}), \omega)^{K_f} \simeq \bigoplus_{\pi = \pi_\infty \otimes \pi_f} \left( \text{HC}(\pi_\infty) \otimes \pi_f^K \right)^{\otimes m_G(\pi)}
\]

where \( \text{HC}(\pi_\infty) \) is the \((\mathfrak{g}, K_\infty)\)-module \( \mathfrak{g} := \mathbb{C} \otimes_{\mathbb{R}} \text{Lie} \ G(\mathbb{R}) \) consisting of smooth \( K_\infty \)-finite vectors in \( \pi_\infty \). This decomposition contains the same information as the decomposition of \( L^2 \) above since any unitary irreducible representation \( \pi_\infty \) is determined by the \((\mathfrak{g}, K_\infty)\)-module \( \text{HC}(\pi_\infty) \). Recall that there is also a decomposition of the unitary representation \( L^2(\mathbb{G}(\mathbb{Q})\backslash \mathbb{G}(\mathbb{A}), \omega) \) of \( G(\mathbb{A}) \) (note that this is not a smooth representation of \( G(\mathbb{A}_f) \)), which also contains the same information, although the relation is non-trivial. Note that in this course we have not studied topological representations of \( p \)-adic groups.

In these notes we will only have to consider topological representations (even unitary on Hilbert spaces) of real groups, and smooth representations of \( p \)-adic groups.

Our first application of the trace formula is the existence of automorphic representations whose components at finitely many places are given (Theorem 4.22 below). An obvious necessary condition for existence is that the central characters are restrictions to \( \mathbb{Q}_p^\times \) of a continuous character \( \mathbb{Q}_p^\times \mathbb{A}_p^\times \to \mathbb{C}^\times \). This condition is made transparent by the following lemma.

Lemma 4.20. Let \( S' \) be a finite set of prime numbers. Let \( (\eta_p)_{p \in S'} \) be a family of (automatically unitary) continuous characters \( \eta_p : \mathbb{Z}_p^\times \to \mathbb{C}^\times \). Then there exists a continuous unitary character \( \omega : \mathbb{Q}_p^\times \mathbb{A}_p^\times \to \mathbb{C}^\times \) such that for any \( p \in S' \) we have \( \omega|_{\mathbb{Z}_p^\times} = \eta_p \).

Proof. Use once again \( \mathbb{A}_p^\times = \mathbb{Q}_p^\times \mathbb{R}_{>0} \mathbb{Z}_p^\times \). \( \square \)

Remark 4.21. The proof shows that we can even find \( \omega \) which is unramified at all primes not in \( S' \) and equal to a given unitary character on \( \mathbb{R}_{>0} \). The case of an arbitrary number field instead of \( \mathbb{Q} \) is more subtle, the statement is not as simple but it essentially reduces to Dirichlet’s unit theorem and [Che51].

Theorem 4.22. Let \( S \) be the finite set of places where \( D \) is not split. Let \( \omega : \mathbb{Q}_p^\times \mathbb{A}_p^\times \to \mathbb{C}^\times \) be a continuous unitary character. Let \( S' \) be a finite set of prime numbers, and \( (\sigma_p)_{p \in S} \) a collection of smooth irreducible representations of \( G(\mathbb{Q}_p) \) having central character \( \omega_p := \omega|_{\mathbb{Q}_p^\times} \). Assume that for any \( p \in S' \setminus S \) the representation \( \sigma_p \) is square-integrable. There exists an irreducible representation \( \pi = \bigotimes_v \pi_v \) in \( \lim_{K_f} \mathcal{L}^2(\mathbb{G}(\mathbb{Q})\backslash \mathbb{G}(\mathbb{A}), \omega)^{K_f} \) such that \( \pi_v \simeq \sigma_v \) for all \( p \in S' \).

Proof. Up to adding to \( S' \) a prime number which is not in \( S \), and taking for \( \sigma_p \) a supercuspidal representation of \( G(\mathbb{Q}_p) \) having central character \( \omega_p \) (such a representation exists by Theorem 3.34), we can assume that there exists \( p \in S' \setminus S \) such that \( \sigma_p \) is supercuspidal. Let \( \ell \) be a prime number which does not belong to \( S' \). We will apply the trace formula to a function \( f \in C^\infty_c(\mathbb{G}(\mathbb{A}), \omega^{-1}) \) which can be written as a product \( \prod_v f_v \).
• For \( v \) a place of \( \mathbb{Q} \) which does not belong to \( S' \cup \{ \ell \} \), pick \( f_v \in \mathcal{H}(G(\mathbb{Q}_v), \omega_v^{-1}) \) (for \( v \) the Archimedean place this means \( C_c^\infty(G(\mathbb{Q}_v), \omega_v^{-1}) \)) such that \( f_v(1) \neq 0 \), and \( f_p \) is the characteristic function of \( G(\mathbb{Z}_p) \) for almost all primes numbers \( p \).

• For each \( p \in S' \), choose a pseudo-coefficient \( f_p \in \mathcal{H}(G(\mathbb{Q}_p), \omega_p^{-1}) \) for the representation \( \sigma_p \). Recall that such pseudo-coefficients were constructed in Propositions 3.25 and 3.26 for \( p \not\in S \), and are easy to construct using finite group representation theory for \( p \in S \) (for example \( f_p = (\dim \sigma_p)^{-1} \mathbf{tr} \sigma_p \)).

• Finally, for \( K_\ell \) a compact open subgroup of \( \ker(\det : G(\mathbb{Q}_\ell) \rightarrow \mathbb{Q}_\ell^*) \), small enough so that \( K_\ell \cap Z(\mathbb{Q}_\ell) = \{1\} \), take \( f_\ell \in \mathcal{H}(G(\mathbb{Q}_\ell), \omega_\ell^{-1}) \) to be the function with support in \( K_\ell Z(\mathbb{Q}_\ell) \) and such that \( f(k) = \text{vol}(K_\ell Z(\mathbb{Q}_\ell)/Z(\mathbb{Q}_\ell))^{-1} \) for \( k \in K_\ell \).

Now we claim that if \( K_\ell \) is chosen sufficiently small, the only non-vanishing summand on the geometric side of the trace formula \((4.2)\) is for \( \gamma = 1 \). Start with an arbitrary \( K_\ell \). The set \( X(\text{supp}(f)) \) of conjugacy classes \([\gamma]\) in \( G_{ad}(\mathbb{Q}) \) having a non-zero contribution in the trace formula is finite. For any non-central \( \gamma \) we have \( \nu(\gamma) \neq 4 \) (\( \nu \) as in the proof of Theorem 4.18). By continuity of \( \nu \) there exists an open subgroup \( K_\ell' \) of \( K_\ell \) such that \( \nu(K_\ell') \cap \nu(X(\text{supp}(f)) \setminus \{ [1] \}) = \emptyset \). Thus up to replacing \( K_\ell \) by \( K_\ell' \), the claim holds true.

So the geometric side of the trace formula is simply \( \text{vol}(Z(\mathbb{A})G(\mathbb{Q}) \setminus G(\mathbb{A}))f(1) \). Since \( f(1) \neq 0 \), it does not vanish, and so the spectral side \((4.3)\) does not vanish either. In particular there exists \( \pi \) such that \( m^G(\pi) > 0 \) and \( \text{tr} \pi(f) \neq 0 \). We have \( \text{tr} \pi(f) = \prod_v \text{tr} \pi_v(f_v) \), so the property of pseudo-coefficients implies that for any \( p \in S' \), either \( \pi_p \simeq \sigma_p \) or \( p \not\in S \) and \( \pi_p \) is one-dimensional. Corollary 4.24 below shows that the second possibility contradicts the fact that there exists \( p' \in S' \setminus S \) such that \( \pi_{p'} \) is supercuspidal.

\[ \square \]

**Theorem 4.23** (Strong approximation). Let \( G' \) be the algebraic subgroup of \( G \) which is the kernel of the determinant morphism. Let \( v \) be a place of \( \mathbb{Q} \) which is not in \( S \), i.e. \( \mathbb{Q}_v \otimes_{\mathbb{Q}} D \simeq M_2(\mathbb{Q}_v) \). Then \( G'(\mathbb{Q})G'(\mathbb{Q}_v) \) is dense in \( G'(\mathbb{A}) \).

**Proof.** See [Kne65, §3].

**Corollary 4.24.** Let \( \pi = \pi_\infty \otimes \bigotimes_p^\prime \pi_p \) be an automorphic representation of \( G(\mathbb{A}) \) having central character \( \omega \). Assume that there exists a place \( v \) of \( \mathbb{Q} \) which is not in \( S \) and such that \( \pi_v \) is one-dimensional. Then \( \pi \) is one-dimensional, i.e. for every place \( w \) of \( \mathbb{Q} \) the representation \( \pi_w \) is one-dimensional.

**Proof.** Fix a maximal compact subgroup \( K_\infty \) of \( G(\mathbb{R}) \). Fix \( v_0 \in \text{HC}(\pi_\infty) \otimes \bigotimes_p^\prime \pi_p \setminus \{0\} \). For simplicity, assume that \( v_0 \) is a pure tensor. There exists a compact open subgroup \( K_f \) of \( G(\mathbb{A}_f) \) fixing \( v_0 \). Let \( \varphi : \pi_\infty \otimes \pi_f^{K_f} \rightarrow L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}), \omega)^{K_f} \) be a non-zero continuous \( G(\mathbb{R}) \)-equivariant linear map which is also equivariant for the action of the Hecke algebra \( \mathcal{H}(G(\mathbb{A}_f), K_f) \). (Note that this last property is equivalent to requiring that \( \varphi \) extends to a \( G(\mathbb{A}_f) \)-equivariant map \( \pi \rightarrow \lim_{\rightarrow K_f} L^2(\mathbb{Q} \setminus G(\mathbb{A}), \omega)^{K_f} \).) Let \( f = \varphi(v_0) \), then \( f \) is an automorphic form (this
non-trivial fact was proved in Gabriel Dospinescu’s course), in particular it is continuous. The group \( G'(\mathbb{Q}_v) \cong \text{SL}_2(\mathbb{Q}_v) \) is perfect, so \( G'(\mathbb{Q}_v) \subset \ker \pi_v \), and \( f \) is right \( G'(\mathbb{Q}_v) \)-invariant. Let \( x \in G(\mathbb{A}) \) and \( y \in G'(\mathbb{A}) \). There are sequences \( (\gamma_n)_n \) and \( (y_n)_n \) of elements of \( G'(\mathbb{Q}) \) and \( G'(\mathbb{Q}_v) \) such that \( (\gamma_n y_n)_n \) converges to \( xg^{-1} \), so \( f(xy) = \lim_{n \to +\infty} f(y_n x) = f(x) \). The representation \( \pi \) is irreducible and so \( \phi \) is injective, so we deduce that \( v_0 \) is fixed by \( G'(\mathbb{A}) \), and so for every place \( w \) of \( \mathbb{Q} \) there is a non-zero vector in \( \pi_w \) fixed by \( G'(\mathbb{Q}_w) \). Since \( G'(\mathbb{Q}_w) \) is distinguished in \( G(\mathbb{Q}_w) \) this implies that \( G'(\mathbb{Q}_w) \subset \ker \pi_w \) (for \( w = \infty \) we use the fact that \( \pi_{\infty} \) is topologically irreducible, whereas for finite \( w \) we use the fact that \( \pi_w \) is simply irreducible). \( \square \)

**Remark 4.25.**

1. *Theorem 4.23 and Corollary 4.24* (for discrete automorphic representations) are still valid if \( S = \emptyset \), i.e. if \( G = \text{GL}_2 \), with the same proof.

2. If we knew more about the classification of representations of \( \text{GL}_2(\mathbb{R}) \) and harmonic analysis for this group, including the existence of pseudo-coefficients for square-integrable representations, we could also include the Archimedean place in the set \( S' \) in Theorem 4.22. For the case of arbitrary reductive groups over number fields see [Clo86].

4.4. **The simple trace formula for \( \text{GL}_2 \)**. We would like to prove an analogous formula for \( \text{GL}_2 \). It turns out that this is much harder, due to the continuous part of the automorphic spectrum on the spectral side, and the contributions of non-elliptic elements on the geometric side (note that \( \text{vol}(G_\gamma(\mathbb{Q})Z(\mathbb{A}) \backslash G(\mathbb{A})) = +\infty \) for \( \gamma \) semi-simple regular hyperbolic). Under a simplifying assumption on the test function, we will get a reasonably simple trace formula for \( \text{GL}_2 \).

For the algebraic group \( \text{GL}_2 \) over \( \mathbb{Q} \) change the notation used in the first chapters for \( \text{GL}_2(\mathbb{Q}_p) \): the letters \( G, B, T, N \) will be used to denote the corresponding algebraic groups over \( \mathbb{Q} \).

We first recall the fundamental results on the cuspidal automorphic spectrum proved in Gabriel Dospinescu’s course. We first introduce cusp forms in the \( L^2 \) setting. Let \( \omega \) be a continuous unitary character of \( Z(\mathbb{Q}) \backslash Z(\mathbb{A}) \).

**Lemma 4.26.** Let \( \phi \in L^2(G(\mathbb{Q})Z(\mathbb{A}) \backslash G(\mathbb{A}), \omega) \). Then for almost all \( g \in G(\mathbb{A}) \), the integral on the RHS of

\[
\phi_B(g) := \int_{N(\mathbb{Q}) \backslash N(\mathbb{A})} \phi(n g) \, d\tilde{g}
\]

converges absolutely. Moreover if \( \text{vol} \left( Z(\mathbb{A}) \backslash \{ x \in G(\mathbb{Q}) \backslash G(\mathbb{A}) \mid \phi(x) \neq 0 \} \right) = 0 \) then \( \phi_B(g) = 0 \) for almost all \( g \in G(\mathbb{A}) \).

**Proof.** Let \( g_0 \in G(\mathbb{A}) \). There exists a continuous compactly supported function \( T_0 : Z(\mathbb{A}) \backslash G(\mathbb{A}) \to \mathbb{R}_{\geq 0} \) such that \( T_0(g_0) > 0 \). Let \( T : Z(\mathbb{A}) \backslash G(\mathbb{Q}) \backslash G(\mathbb{A}) \to \mathbb{R}_{\geq 0} \) be defined by \( T(g) = \sum_{\gamma \in Z(\mathbb{Q}) \backslash G(\mathbb{Q})} T(\gamma g) \). This function is clearly continuous and compactly supported, so it is bounded and

\[
\int_{Z(\mathbb{A}) \backslash G(\mathbb{Q}) \backslash G(\mathbb{A})} |\phi(g)|^2 T(g) \, d\tilde{g} < \infty.
\]
But this equals
\[ \int_{Z(\mathbb{A}) \backslash G(\mathbb{A})} |\phi(g)|^2 T_0(g) \, dg = \int_{Z(\mathbb{A}) N(\mathbb{A}) \backslash G(\mathbb{A})} |\phi(n g)|^2 T_0(n g) \, dn \, dg \]
\[ = \int_{Z(\mathbb{A}) N(\mathbb{A}) \backslash G(\mathbb{A})} |\phi(n g)|^2 \sum_{\gamma \in N(\mathbb{Q})} T_0(\gamma n g) \, dn \, dg. \]

The fact that this last integral converges implies both statements in the lemma, using Cauchy-Schwarz (note that \(N(\mathbb{Q}) \backslash N(\mathbb{A})\) is compact, so that the constant function 1 on it is square-integrable).

Let \(L^2_{\text{cusp}}(G(\mathbb{Q}) \backslash G(\mathbb{A}), \omega)\) be the subspace of \(L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}), \omega)\) consisting of all \(\phi\) such that for almost all \(g \in G(\mathbb{A})\) we have
\[ \int_{N(\mathbb{Q}) \backslash N(\mathbb{A})} \phi(n g) \, dn = 0. \]

**Theorem 4.27.** Let \(K_f\) be a compact open subgroup of \(G(\mathbb{A}_f)\). Then the unitary representation \(L^2_{\text{cusp}}(G(\mathbb{Q}) \backslash G(\mathbb{A}), \omega)^{K_f}\) of \(G(\mathbb{R})\) decomposes discretely.

Recall that this theorem is also proved using the theorem of Gelfand, Graev and Piatetski-Shapiro, that is by proving that for any \(f \in C^0_c(K_f \backslash G(\mathbb{A})/K_f, \omega^{-1})\), the operator \(\rho_{\text{cusp}}(f)\), which is the restriction of \(\rho(f)\) to \(L^2_{\text{cusp}}(G(\mathbb{Q}) \backslash G(\mathbb{A}), \omega)^{K_f}\), is compact, even Hilbert-Schmidt. Note that the proof did not exhibit an explicit kernel for this Hilbert-Schmidt operator. Nevertheless, we shall see that in the case of a cuspidal test function \(f\), essentially the same arguments, applied to the kernel instead of automorphic forms, do give an explicit kernel.

Before we can achieve this in Lemma 4.34 below, we need to recall two essential tools: reduction theory and the Poisson summation formula.

Let \(K_0\) be the maximal compact subgroup \(O_2(\mathbb{R}) \times G(\mathbb{Z})\) of \(G(\mathbb{A})\). Define
\[ B(\mathbb{A})^1 = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in B(\mathbb{A}) \mid |a| = |c| = 1 \right\}. \]

For \(\eta > 0\) define
\[ S(\eta) = \{ \text{diag}(x, y) \in G(\mathbb{R}) \mid x, y > 0 \text{ and } x/y \geq \eta \}. \]

We also introduce the function \(H : G(\mathbb{A}) \to \mathbb{R}\) defined using the Iwasawa decomposition by \(H(b k) = \log |x/y|\) if \(b = \begin{pmatrix} x & * \\ 0 & y \end{pmatrix} \in B(\mathbb{A})\) and \(k \in K_0\). Clearly \(H\) is left \(B(\mathbb{A})^1\) and right \(K_0\)-invariant. We have a “product formula” (actually a sum because of the logarithm . . .) \(H(g) = \sum_v H_v(g_v)\). Note that \(B(\mathbb{A})^1 S(\eta) K_0 = H^{-1}([\log \eta, +\infty[)\).

**Theorem 4.28.** There exists a compact subset \(\Omega\) of \(B(\mathbb{A})^1\) and \(\eta > 0\) such that \(G(\mathbb{A}) = G(\mathbb{Q}) \Omega S(\eta) K_0\).
Remark 4.29. This is a coarser version of the classical fundamental domain for the action of \( \text{SL}_2(\mathbb{Z}) \) on the Poincaré upper-half plane; explicitly we may take \( \eta = \sqrt{3}/2 \),

\[
\Omega = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \bigg| |x| \leq 1/2 \right\} \subset N(\mathbb{R}).
\]

See [Ser77, Ch. VII]. For arbitrary reductive groups over number fields the first part, together with the compactness criterion for arithmetic quotients, are theorems due to Borel, Harish-Chandra, Mostow, Tamagawa, Godement, Weil (see [God95] and [Spr94]; the latter also covers reductive groups of positive characteristic).

The first point in the following lemma gives “coordinates near the cusps” on \( G(\mathbb{Q}) \backslash G(\mathbb{A}) \). The second point is a slight generalization that will be useful later.

**Lemma 4.30.** (1) For any place \( v \) of \( \mathbb{Q} \), \( g \in G(\mathbb{Q}_v) \) and \( n \in N(\mathbb{Q}_v) \) we have \( H_v(nwg) \leq -H_v(g) \). In particular for \( g \in G(\mathbb{A}) \) and \( n \in N(\mathbb{A}) \) we have \( H(nwg) \leq -H(g) \). In particular for \( \kappa > 1 \) we have an embedding

\[
B(\mathbb{Q}) \backslash B(\mathbb{A})^1 S(\eta)K_0 \hookrightarrow G(\mathbb{Q}) \backslash G(\mathbb{A})
\]

(and similarly if we take quotients by \( Z(\mathbb{A}) \)).

(2) Let \( \eta > 0 \). Let \( C \) be a compact subset of \( G_{ad}(\mathbb{A}) \). There exists \( \kappa > 0 \) (depending on \( \eta \) and \( C \)) such that for any \( x \in B(\mathbb{A})^1 S(\kappa)K_0 \), \( \gamma \in G(\mathbb{Q}) \) and \( y \in B(\mathbb{A})^1 S(\eta)K_0 \) satisfying \( x^{-1} \gamma y \in C \), we have \( \gamma \in B(\mathbb{Q}) \).

**Proof.** (1) We consider the Archimedean and non-Archimedean cases separately. In any case we can assume that \( g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \), and that \( n = 1 \) since \( H_v \) is left \( Z(\mathbb{Q}_v)N(\mathbb{Q}_v) \)-invariant. We have \( wg = \begin{pmatrix} 0 & -1 \\ a & -b \end{pmatrix} \).

In the real case we compute \( wg^t(wg) = \begin{pmatrix} 1 & b \\ b & a^2 + b^2 \end{pmatrix} \) and solve for \( x^t x = wg^t(wg) \) with \( x \in B(\mathbb{Q}_v) \). We find \( x \in Z(\mathbb{Q}_v) \begin{pmatrix} a/(a^2 + b^2) & b/(a^2 + b^2) \\ 0 & 1 \end{pmatrix} \), and \( \log(a/(a^2 + b^2)) \leq \log(a/a^2) = - \log a \).

For a prime number \( p \), doing column operations on \( wg \) we find that if \( b/a \in \mathbb{Z}_p \) then \( H_p(wg) = -H_p(g) \) whereas if \( b/a \in \mathbb{Q}_p \backslash \mathbb{Z}_p \) then \( H_p(wg) = -H_p(g) - 2 \log(|b/a|) \).

The last assertion follows from the Bruhat decomposition for \( GL_2(\mathbb{Q}) \): if \( H(g) > 0 \) and \( \gamma \in GL_2(\mathbb{Q}) \backslash B(\mathbb{Q}) \) then \( H(\gamma g) \leq - \inf_{n \in N(\mathbb{Q})} H(wng) \leq 0 \).

(2) Up to replacing \( C \) by \( K_0CK_0 \) we may assume that \( C \) is bi-\( K_0 \)-invariant. Then \( B(\mathbb{A})^1 S(\kappa)K_0C = B(\mathbb{A})^1 S(\kappa)C \). There exists \( \epsilon > 0 \) such that \( C \subset B(\mathbb{A})^1 S(\epsilon)K_0 \), so that \( B(\mathbb{A})^1 S(\kappa)K_0C \subset B(\mathbb{A})^1 S(\epsilon \kappa)K_0 \). Assume that \( \kappa \) is large enough so that \( \kappa \epsilon \eta > 1 \). We will show that for any \( \gamma \in G(\mathbb{Q}) \), if \( \gamma B(\mathbb{A})^1 S(\eta)K_0 \cap B(\mathbb{A})^1 S(\kappa)K_0C \neq \emptyset \) then \( \gamma \in B(\mathbb{Q}) \). Since \( B(\mathbb{Q})B(\mathbb{A})^1 = B(\mathbb{A})^1 \) we may assume that \( \gamma = 1 \) or \( \gamma \in wN(\mathbb{Q}) \) (Bruhat decomposition).
The previous point shows that \( H(wN(Q)B(A)^{1}S(\gamma)K_{0}) \subset ] - \infty, -\log(\eta)] \), whereas \( H(B(A)^{1}S(\kappa)K_{0}C) \subset [\log(\kappa\epsilon), +\infty[. \)

\[ \square \]

I do not know of a reference for a generalization of the second part to arbitrary reductive groups, but there is no doubt that such a generalization exists . . .

We now recall the Poisson summation formula. First, the classical form, which is well-known. The assumptions we put are far from optimal.

**Proposition 4.31.** For \( f \in C^{1}_{c}(\mathbb{R}) \) we have \( \sum_{a \in \mathbb{Z}} f(u) = \sum_{v \in \mathbb{Z}} \hat{f}(v) \), where \( \hat{f}(x) = \int_{\mathbb{R}} f(t) e^{-2\pi xt} dt \) and the right-hand side is absolutely convergent.

**Proof.** The function \( t \mapsto \sum_{a \in \mathbb{Z}} f(t + u) \) on \( \mathbb{R} \) is \( \mathbb{Z} \)-periodic and \( C^{1} \) so it is the sum of its Fourier series, convergent for the sup norm. \[ \square \]

Recall that \( A = Q + \mathbb{R} + \mathbb{Z} \), and that the kernel of the surjection \( \mathbb{R} \times \mathbb{Z} \to Q \setminus A \) is \( \mathbb{Z} \). It follows that there is a unique continuous (automatically unitary) character \( \psi_{0} : Q \setminus A/\mathbb{Z} \to \mathbb{C}^{\times} \) whose restriction to \( \mathbb{R} \) is \( t \mapsto \exp(2i\pi t) \).

**Exercise 4.32.** Show that \( Q \to \text{Hom}_{cont}(Q \setminus A, \mathbb{C}^{\times}) \), \( \lambda \mapsto \psi_{0}(\lambda \cdot) \) is an isomorphism of topological groups.

For \( f \in L^{1}(A) \) define \( \hat{f} : A \to \mathbb{C} \) by \( \hat{f}(x) = \int_{A} f(t) \psi_{0}(-tx) dt \). Note that the Haar measure \(|dt|\) on \( A \) is characterized by the fact that \( \text{vol}(Q \setminus A) = 1 \).

**Proposition 4.33.** Let \( f \in C_{c}^{1}(A) \).

1. There exists an integer \( m > 0 \) such that for any \( a \in \mathbb{R}_{>0} \mathbb{Z}^{\times} \) (recall that \( A^{\times} = Q^{\times} \mathbb{R}_{>0} \mathbb{Z}^{\times} \)) we have \( \hat{f}(av) = 0 \) for any \( v \in Q \setminus m^{-1} \mathbb{Z} \), and \( (\hat{f}(av))_{v \in m^{-1} \mathbb{Z}} \in \ell^{1}(m^{-1} \mathbb{Z}) \).

2. For any \( a \in \mathbb{R}_{>0} \mathbb{Z}^{\times} \) we have \( \sum_{u \in \mathbb{Q}} f(a^{-1}u) = |a| \sum_{v \in \mathbb{Q}} \hat{f}(av) \).

3. If we assume further that \( f \in C_{c}^{2}(A) \) then there is a constant \( C > 0 \) such that for any \( a \in \mathbb{R}_{>0} \mathbb{Z}^{\times} \) we have \( |\hat{f}(av)| \leq C(1 + |av|)^{-2} \). In particular if \( \hat{f}(0) = 0 \) then \( \lim_{|a| \to +\infty} \sum_{u \in \mathbb{Q}} f(a^{-1}u) = 0 \).

**Proof.** Exercise: deduce the first two points from the classical Poisson summation formula.

For the third point, reduce to the real case again and use the relation \( \hat{f}''(x) = -4\pi^{2} x^{2} \hat{f}(x) \) to bound \( |\hat{f}| \) in terms of \( \|f\|_{L^{1}} \) (in \([-1, 1]\)) and \( \|f''\|_{L^{1}} \) (in \( \mathbb{R} \setminus [-1, 1] \)). The limit is then computed by dominated convergence. \[ \square \]

**Lemma 4.34.** As above \( K_{f} \) is a compact open subgroup of \( G(A_f) \) and \( \omega \) is a unitary continuous character of \( Z(Q) \setminus Z(A) \). Let \( f \in C_{c}^{2}(K_{f}\setminus G(A)/K_{f}, \omega^{-1}) \) be cuspidal, i.e. for any \( x, x \in G(A) \) we have \( \int_{N(A)} f(xny) dn = 0 \). Then the operator \( \rho(f) \) on \( L^{2}(G(Q)\setminus G(A), \omega) \) has image contained in \( L_{\text{cusp}}^{2}(G(Q)\setminus G(A), \omega) \) and is Hilbert-Schmidt with kernel \( K_{f} : (G(Q)\setminus G(A))^{2} \to \mathbb{C}, (x, y) \mapsto \sum_{\gamma \in Z(Q)\setminus Z(Q)} f(x^{-1}\gamma y) \).

This kernel is continuous and bounded.
Proof. Note that bounded implies square-integrable modulo $(Z(\mathbb{Q}) \setminus Z(\mathbb{A}))^2$. In fact this is how we will prove that $\rho_f$ is Hilbert-Schmidt. As in the anisotropic case the function $K_f$ is continuous, satisfies $K_f(z_1 x, z_2 y) = \omega(z_1 z_2^{-1}) K_f(x, y)$ for $z_1, z_2 \in Z(\mathbb{A})$ and for any $\phi \in L^2(G(\mathbb{Q}) \setminus G(\mathbb{A}), \omega)$ we have

$$(\rho(f) \phi)(x) = \int_{G_{\text{ad}}(\mathbb{Q}) \setminus G_{\text{ad}}(\mathbb{A})} \phi(y) K_f(x, y) \, dy.$$ 

If $X$ is a compact subset of $G_{\text{ad}}(\mathbb{A})$, for $x \in X$, for any $\gamma \in G_{\text{ad}}(\mathbb{Q})$ and $y \in G_{\text{ad}}(\mathbb{A})$ such that $x^{-1} \gamma y$ we have that $y$ belongs to the (compact) image of $X \text{supp}(f)$ in $G_{\text{ad}}(\mathbb{Q}) \setminus G_{\text{ad}}(\mathbb{A})$. This shows that for $x \in G_{\text{ad}}(\mathbb{Q}) \setminus G_{\text{ad}}(\mathbb{A}) X$ the support of $|K_f(x, \cdot)|$ is contained in a compact subset of $G_{\text{ad}}(\mathbb{Q}) \setminus G_{\text{ad}}(\mathbb{A})$ which does not depend on $x$ (of course it depends on $X$ and $f$), in particular it is bounded independently of $x$. The kernel $K_f$ is also cuspidal in the first variable: for any $x, y \in G(\mathbb{A})$ we have

$$\int_{N(\mathbb{Q}) \setminus N(\mathbb{A})} K_f(nx, y) \, dn = \int_{N(\mathbb{Q}) \setminus N(\mathbb{A})} \sum_{\gamma \in G_{\text{ad}}(\mathbb{Q})} \sum_{\alpha \in N(\mathbb{Q})} f(x^{-1} n^{-1} \alpha^{-1} \gamma y) \, dn$$

$$= \int_{N(\mathbb{Q}) \setminus N(\mathbb{A})} \sum_{\gamma \in N(\mathbb{Q}) \setminus G_{\text{ad}}(\mathbb{Q})} \sum_{\alpha \in N(\mathbb{Q})} f(x^{-1} n^{-1} \alpha^{-1} \gamma y) \, dn$$

$$= \sum_{\gamma \in N(\mathbb{Q}) \setminus G_{\text{ad}}(\mathbb{Q})} \int_{N(\mathbb{Q}) \setminus N(\mathbb{A})} \sum_{\alpha \in N(\mathbb{Q})} f(x^{-1} n^{-1} \alpha^{-1} \gamma y) \, dn$$

$$= \sum_{\gamma \in N(\mathbb{Q}) \setminus G_{\text{ad}}(\mathbb{Q})} \int_{N(\mathbb{A})} f(x^{-1} n^{-1} \gamma y) \, dn$$

$$= 0$$

where the third equality is justified by absolute convergence ($K_f$ is also continuous and $N(\mathbb{Q}) \setminus N(\mathbb{A})$ is compact so the first integral is finite) and the last equality follows from cuspidality of $f$. Now for any $x \in G(\mathbb{A})$ the image of $N(\mathbb{A}) x$ in $G_{\text{ad}}(\mathbb{Q}) \setminus G_{\text{ad}}(\mathbb{A})$ is compact, so $|K_f|$ is bounded on $N(\mathbb{A}) x \times (G_{\text{ad}}(\mathbb{Q}) \setminus G_{\text{ad}}(\mathbb{A})$. Since $|\phi|$ is integrable on $G_{\text{ad}}(\mathbb{Q}) \setminus G_{\text{ad}}(\mathbb{A})$ (it is square-integrable and vol($G_{\text{ad}}(\mathbb{Q}) \setminus G_{\text{ad}}(\mathbb{A})$) is finite) we have

$$\int_{N(\mathbb{Q}) \setminus N(\mathbb{A})} \int_{G_{\text{ad}}(\mathbb{Q}) \setminus G_{\text{ad}}(\mathbb{A})} |\phi(y) K_f(nx, y)| \, dy \, dn < \infty$$

so we can swap integral signs and deduce that we have

$$\int_{N(\mathbb{Q}) \setminus N(\mathbb{A})} (\rho(f) \phi)(nx) \, dn = 0.$$ 

This shows that the image of $\rho(f)$ is contained in $L_c^2(G(\mathbb{Q}) \setminus G(\mathbb{A}), \omega)$.

Let us now show that $|K_f|$ is bounded. Let $C \subset G_{\text{ad}}(\mathbb{A})$ be $K_0 \text{supp}(f) K_0$ and write $C = B_C K_0$ for some compact subset $B_C$ of $B_{\text{ad}}(\mathbb{A})$ which is right $B_{\text{ad}}(\mathbb{A}) \cap (K_0/(K_0 \cap Z(\mathbb{A})))$-invariant. Let $\Omega \subset B(\mathbb{A})^2$ and $\eta > 0$ be as in Theorem 4.28. Let $\kappa > 0$ be as in (2) of Lemma 4.30 (with respect to $\eta$ and $C$). Let $x = a_x \text{diag}(a_x, 1) k_x$ with $a_x \in \Omega$, $a_x \in \mathbb{R}_{\geq 0}$ and $k_x \in K_0$, and similarly $y = a_y \text{diag}(a_y, 1) k_y$. Assume that $a_x > \kappa$. By Lemma 4.30, if $\gamma \in G_{\text{ad}}(\mathbb{Q})$ is such that $x^{-1} \gamma y \in C$ then $\gamma \in B_{\text{ad}}(\mathbb{Q})$. 


We then also have (in $G_{\text{ad}}(\mathbb{A})$) $k_x x^{-1} \gamma y k_y^{-1} \in K_0 C K_0 = B_C K_0$, and moreover $k_x x^{-1} \gamma y k_y^{-1} \in B_{\text{ad}}(\mathbb{A})$ so writing $\gamma = \begin{pmatrix} a_\gamma & * \\ 0 & 1 \end{pmatrix}$ we obtain $a_x^{-1} a_x a_y \in C'$ where $C' \subset \mathbb{A}^\times$ is a compact subset which depends on $C$ and $\Omega$ (explicitly $C'$ is the image of $\Omega B_C \Omega^{-1}$). Using the decomposition $\mathbb{A}^\times = Q^\times \mathbb{R}_{>0} \mathbb{Z}^\times$ (which is a homeomorphism) we conclude that there exists $\epsilon > 0$ and a finite set $F \subset B_{\text{ad}}(\mathbb{Q})/N(\mathbb{Q})$ (depending on $\Omega$, $\eta$ and $C$) such that for $x \in \Omega S(\kappa) K_0$, $y \in \Omega S(\eta) K_0$ and $\gamma \in G_{\text{ad}}(\mathbb{Q})$, if $x^{-1} \gamma y \in C$ then $\gamma \in B_{\text{ad}}(\mathbb{Q})$, $a_x/a_y \in [\epsilon, \epsilon^{-1}]$ and the image of $\gamma$ in $B_{\text{ad}}(\mathbb{Q})/N(\mathbb{Q})$ lies in $F$.

This argument is symmetric in $x$ and $y$, up to replacing $C$ by the larger compact subset of $G_{\text{ad}}(\mathbb{A})$:

$$K_0 \{ g \in G_{\text{ad}}(\mathbb{A}) \mid g \in \text{supp}(f) \text{ or } g^{-1} \in \text{supp}(f) \} K_0.$$ 

Let $\tilde{F}$ be the preimage of $F$ in $\{ \text{diag}(a, 1) \mid a \in Q^\times \}$, naturally in bijection with $F$. We have shown that for $(x, y) \in (\Omega S(\eta) K_0)^2$, one of them in $\Omega S(\kappa) K_0$, we have $K_f(x, y) = \sum g \in \text{supp}(f) \sum n \in N(\mathbb{Q}) f(x^{-1} \gamma n y)$.

Note that the image of $(\Omega S(\eta) K_0 \times \Omega S(\kappa) K_0)^2$ in $(G(\mathbb{Q}) Z(\mathbb{A}) \setminus G(\mathbb{A}))^2$ is relatively compact (essentially because the interval $[\eta, \kappa]$ is compact). In order to bound $K_f$ on $(G(\mathbb{Q}) Z(\mathbb{A}))^2$, it is therefore enough to bound the sum over $n \in N(\mathbb{Q})$ when $x$ or $y$ belongs to $\Omega S(\kappa) K_0$. For this we will use the Poisson summation formula.

Write $\gamma = \text{diag}(a_\gamma, 1)$ and $n = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ with $u \in \mathbb{Q}$, so that

$$x^{-1} \gamma n y = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{a_x^{-1} a_x a_y} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_{y^{-1} u} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_y & 0 \\ 0 & 1 \end{pmatrix} k_y.$$

Observe that the set $\{ \text{diag}(a^{-1}, 1) \text{diag}(a, 1) \mid a \in \Omega, \ a \in \mathbb{R}_{>0} \}$ is relatively compact in $B(\mathbb{A})^1$. Together with the relation $a_x/a_y \in [\epsilon, \epsilon^{-1}]$ observed above, this implies that the function $\beta_1$ (resp. $\beta_2$) is bounded on $(\Omega S(\eta) K_0)^2 \times \tilde{F}$ (resp. $\Omega S(\eta) K_0$), in the sense that its image is relatively compact in $G(\mathbb{A})$. Define $\Xi_{f,x,y,\gamma}(u') = f(\beta_1(x, y, \gamma) \begin{pmatrix} 1 & u' \\ 0 & 1 \end{pmatrix} \beta_2(y))$. The Poisson summation formula (Proposition 4.33) reads

$$\sum_{n \in N(\mathbb{Q})} f(x^{-1} \gamma n y) = \text{constant} \times \sum_{v \in \mathbb{Q}} a_y \Xi_{f,x,y,\gamma}(a_y v).$$

We finally use the assumption that $f$ is cuspidal, which implies that $\Xi_{f,x,y,\gamma}(0) = 0$. Thanks to the boundedness of $\beta_1$ and $\beta_2$, the constant in (3) of Proposition 4.33 may be found independently of $x, y$ as above, and we get that $K_f$ goes to 0 at infinity, i.e. for any $\delta > 0$ there exists a compact subset $C_\delta$ of $(G_{\text{ad}}(\mathbb{Q}) \setminus G_{\text{ad}}(\mathbb{A}))^2$ such that for any $(x, y) \notin C_\delta$ we have $|K_f(x, y)| < \delta$. In particular $|K_f|$ is bounded. \qed
Theorem 4.35. Let $K_f$ be a compact open subgroup of $G(A_f)$ and $\omega$ a unitary continuous character of $Z(\mathbb{Q})\backslash Z(A)$. Let $f \in C_c^\infty(K_f \backslash G(A)/K_f, \omega^{-1})$ be cuspidal. Assume that for any $x \in G(A)$ and $\gamma \in G_{ad}(\mathbb{Q})$ such that $f(x^{-1}\gamma x) \neq 0$, $\gamma$ is semi-simple regular elliptic (over $\mathbb{Q}$). Then

$$\text{tr } \rho(f) = \text{tr } \rho_{\text{cusp}}(f) = \sum_{[\gamma]} t(\gamma)^{-1} \text{vol}(G_{\gamma}(\mathbb{Q})\backslash G(A))O_\gamma(f)$$

where the sum is over conjugacy classes of semi-simple regular elliptic elements $\gamma$ in $G(\mathbb{Q})$, and only finitely many terms in the sum are non-zero.

Proof. Recall from Gabriel Dospinescu’s course that the action $\rho_{\text{cusp}}(f)$ of any element $f$ of $C_c^\infty(G(A), \omega^{-1})$ on $L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(A), \omega)$ is a Hilbert-Schmidt operator (we proved this in the previous lemma for cuspidal $f$, but it holds for arbitrary $f$ if we restrict to the space of cusp forms). So we can argue as in the proof of Theorem 4.18 using a Dixmier-Malliavin expression for $f$ to conclude that $\rho_{\text{cusp}}(f)$ is trace class. Thanks to the previous lemma, for $f$ cuspidal $\rho(f)$ is also trace class and $\text{tr } \rho(f) = \text{tr } \rho_{\text{cusp}}(f)$ (note that $L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(A), \omega)$ and its orthogonal in $L^2(G(\mathbb{Q})\backslash G(A), \omega)$ are stable under $\rho(f)$).

The main difference with the anisotropic case is that it is not true that $K_{|f|}$ is also bounded ($|f|$ is not cuspidal . . . ), so while we still have

$$\text{tr } \rho(f) = \int_{G_{ad}(\mathbb{Q})\backslash G_{ad}(A)} K_f(x, x) \, d\hat{x}$$

thanks to Theorem 4.13, we cannot blindly insert the definition of $K_f$ and exchange sums and integrals. Nevertheless, the proof of the previous lemma shows that there exists a compact subset $C(f)$ of $G_{ad}(\mathbb{Q})\backslash G_{ad}(A)$ such that for $x \in G(A)$ which does not map to $C(f)$, if $\gamma \in G_{ad}(\mathbb{Q})$ is such that $f(x^{-1}\gamma x) \neq 0$ then $\gamma$ is conjugated (in $G_{ad}(\mathbb{Q})$) to an element of $B_{ad}(\mathbb{Q})$. Together with the assumption in the theorem, this implies that $K_f$ has compact support on the diagonal (even that each term in the sum defining $K_f(x, x)$ vanishes when $x$ is outside a compact subset of $G_{ad}(A)\backslash G_{ad}(A)$), and we can conclude as in the proof of Theorem 4.18 (including the last finiteness assertion).

Theorem 4.36. Let $\omega : \mathbb{Q}^\times \backslash A^\times \rightarrow \mathbb{C}^\times$ be a continuous unitary character. Let $S'$ be a finite set of prime numbers, and $(\sigma_p)_{p \in S}$ a collection of smooth irreducible square-integrable representations of $G(\mathbb{Q}_p)$ having central character $\omega_p := \omega|_{\mathbb{Q}_p^\times}$. There exists an irreducible representation $\pi = \bigotimes'_v \pi_v$ in $\lim_{\rightarrow K_f} L^2_{\text{cusp}}(G(\mathbb{Q})\backslash G(A), \omega)^{K_f}$ such that $\pi_p \simeq \sigma_p$ for all $p \in S'$.

Proof. Of course the idea is the same as in Theorem 4.22, but now our simple trace formula does not allow us to use functions $f$ satisfying $f(1) \neq 0$. Adding one prime to $S'$ if necessary and thanks to Theorem 3.34 we can assume that at least one $\sigma_p$ is supercuspidal.

First we fix, for each $p \in S$, a pseudo-coefficient $f_p$ for $\sigma_p$. Note that the fact that there exists $p \in S'$ such that $\sigma_p$ is cuspidal implies that $f_p$ is cuspidal, and so will be any product $\prod f_v$. Thanks to the elliptic orthogonality formula (Theorem
3.32) applied to \((\sigma_p, \sigma_p)\) we know that there exists a semi-simple regular elliptic conjugacy class \([\gamma_p]\) in \(G(\mathbb{Q}_p)\) such that \(O_{\gamma_p}(f_p) \neq 0\). It follows from Krasner’s lemma and smoothness of orbital integrals (Lemma 3.4) that there exists \(\epsilon > 0\) such that for any \(p \in S', a \in \mathbb{Q}_p\) and \(b \in \mathbb{Q}_p^\times\) satisfying \(|a - \text{tr} \gamma_p|_p < \epsilon\) and \(|b - \det \gamma_p|_p < \epsilon\), the conjugacy class in \(G(\mathbb{Q}_p)\) defined by the characteristic polynomial \(X^2 - aX + b\) contains an element \(\delta_p\) in the anisotropic maximal torus \(Q_p[\gamma_p]^\times\) of \(G(\mathbb{Q}_p)\) which is regular and sufficiently close to \(\gamma_p\) so that \(O_{\delta_p}(f_p) = O_{\gamma_p}(f_p)\). We can find \(a \in \mathbb{Q}\) and \(b \in \mathbb{Q}_p^\times\) in these \(p\)-adic balls for all \(p \in S'\): for \(a\) this is essentially the Chinese remainder theorem (we can even assume that \(a\) is integral at finite places not in \(S'\)), for \(b\) it follows from Dirichlet’s theorem on primes in arithmetic progressions. (These two existence results are known as weak approximation for the additive and multiplicative groups. In fact the additive group even has strong approximation.) Let \(\gamma\) be an element of \(G(\mathbb{Q})\) having characteristic polynomial \(X^2 - aX + b\). Note that \(\gamma\) is semi-simple regular, and elliptic over \(\mathbb{Q}\) since it is elliptic over \(\mathbb{Q}_p\) for some \(p\).

As in the proof of Theorem 4.22, fix \(\ell\) a prime number which does not belong to \(S'\). Fix \(f^{S' \setminus \ell}(\ell) = \bigoplus_{v \in S} f_v\) with \(f_v \in C_c^\infty(G(\mathbb{Q}_v), \omega_v^{-1})\) almost all trivial, such that for any \(v \notin S' \cup \{\ell\}\) we have \(O_v(f_v) \neq 0\). (Exercise: such a function exists.) Finally, take \(K_\ell\) an open compact subgroup of \(\text{SL}_2(\mathbb{Q}_\ell)\) such that \(K_\ell \cap \mathbb{Z}(\mathbb{Q}_\ell) = \{1\}\) and define \(f_\ell \in C_c^\infty(G(\mathbb{Q}_\ell), \omega_\ell^{-1})\) supported in \(\gamma K_\ell \mathbb{Z}(\mathbb{Q}_\ell)\), right \(K_\ell\)-invariant and such that \(f_\ell(\gamma) = 1\). By essentially the same argument as in the proof of Theorem 4.22 we see that if \(K_\ell\) is small enough then the assumption of Theorem 4.35 is satisfied and the only non-vanishing term in the sum on the geometric side is \(\iota(\gamma)^{-1} \text{vol}(G_\gamma(\mathbb{Q}) \backslash G(\mathbb{A})) O_\gamma(f),\) which does not vanish. We conclude as in the proof of Theorem 4.22, using Corollary 4.24. \(\square\)

5. Comparison of trace formulas

5.1. Separation of representations. To compare trace formulas, we start with a simplification lemma, to get rid of the infinite sums on the spectral side of trace formulas.

Recall the notation \(f^*(g) := \overline{f(g^{-1})}\).

**Lemma 5.1.** Let \(\omega : \mathbb{R}^\times \to \mathbb{C}^\times\) be a continuous unitary character. Let \((V_i, \pi_i)_{i \in I}\) be a family of irreducible unitary representations of \(\text{GL}_2(\mathbb{R})\) having central character \(\omega\) and pairwise non-isomorphic. Let \((\lambda_i)_{i \in I}\) be a family of complex numbers such that for any \(f \in C_c^\infty(\text{GL}_2(\mathbb{R}), \omega^{-1})\), the operator \(\bigoplus_{i \in I} \lambda_i \pi_i(f^* \ast f)\) on \(\bigoplus_{i \in I} V_i\) is trace class, and \(\sum_{i \in I} \lambda_i \text{tr} \pi_i(f^* \ast f) = 0\). Then all \(\lambda_i = 0\).

**Proof.** Assume that there exists \(i_0 \in I\) such that \(\lambda_{i_0} \neq 0\). Up to multiplying all \(\lambda_i\)’s by \(-\lambda_{i_0}^{-1}\), we can assume that \(\lambda_{i_0} = -1\), so that \(\text{tr} \pi_{i_0}(f^* \ast f) = \sum_{i \in I'} \lambda_i \text{tr} \pi_i(f^* \ast f)\) with \(I' = \{i \in I \mid i \neq i_0\text{ and } \lambda_i \neq 0\}\). For each \(i \in I'\), fix an orthonormal basis \((e_{i,j})_{j \in J_i}\) of \(V_i\). The trace class assumption implies that for any \(f \in C_c^\infty(\text{GL}_2(\mathbb{R}), \omega^{-1})\) we have \(\sum_{i \in I'} \sum_{j \in J_i} |\lambda_i| \|\pi_i(f)e_{i,j}\|_{V_i}^2 < \infty\). Let \(V\) be the completion of \(\bigoplus_{i \in I'} V_i\) for the Hermitian inner product

\[ \|(v_{i,j})_{i \in I'}\|_V^2 = \sum_{i \in I'} |\lambda_i| \sum_{j \in J_i} \|v_{i,j}\|_{V_i}^2. \]
It is naturally a representation of $G$, which is clearly continuous and unitary. Consider the subspace

$$W_0 = \{(\{\pi_i(f)e_{i,j}\}_{j \in J_i})_{i \in I'} \mid f \in C_c^\infty(GL_2(\mathbb{R}), \omega^{-1})\}$$

of $V$, and let $W$ be its closure in $V$, a subrepresentation of $V$. Let $v \in V_{i_0}$ be such that $\|v\|_{V_{i_0}} = 1$. Completing this to form an orthonormal basis of $V_{i_0}$ and writing traces in this basis, we obtain that for any $f \in C_c^\infty(GL_2(\mathbb{R}), \omega^{-1})$

$$\|\pi_{i_0}(f)v\|_{V_{i_0}}^2 \leq \text{tr} \pi_{i_0}(f^* \ast f) = \sum_{i \in I'} \lambda_i \text{tr} \pi_i(f^* \ast f) \leq \sum_{i \in I'} |\lambda_i| \sum_{j \in J_i} \|\pi_i(f)e_{i,j}\|^2_{V_i}.$$

This inequality implies the existence and uniqueness of a continuous linear map $\Xi : W \to V_{i_0}$ mapping $(\{\pi_i(f)e_{i,j}\}_{j \in J_i})_{i \in I'}$ to $\pi_{i_0}(f)v$. This characterization shows that $\Xi$ is $GL_2(\mathbb{R})$-equivariant. Moreover we know that there exists $f$ such that $\pi_{i_0}(f)v \neq 0$, thus $\Xi \neq 0$. We can uniquely extend $\Xi$ to a linear map $V \to V_{i_0}$, abusively still denoted $\Xi$, by imposing that $\Xi|_{W^\perp} = 0$ (here $W^\perp$ is the orthogonal of $W$ in $V$). This extension is clearly also continuous $GL_2(\mathbb{R})$-equivariant. But the restriction of $\Xi$ to each factor $((V_i)_{i \in I}, e_{i,j})_{i \in I'}$ is zero since $\pi_i \not\cong \pi_{i_0}$, so $\Xi = 0$ by definition of $V$. We have obtained a contradiction, so the assumption that there exists $i_0 \in I$ such that $\lambda_{i_0} \neq 0$ was absurd. □

5.2. Multiplicity one results. For the proof of Theorem 1.1 we will need to admit a few important theorems, which rely on theories which were not developed in this course.

**Theorem 5.2** (Multiplicity one). Let $\omega : Z(\mathbb{Q}) \backslash Z(\mathbb{A}) \to \mathbb{C}^\times$ be a unitary continuous character. Any cuspidal automorphic representation $\pi$ of $GL_2(\mathbb{A})$ occurs with multiplicity one in $\lim_{K_f} L^2(G(\mathbb{Q}) \backslash G(\mathbb{A}), \omega)^K_f$, i.e. $n_{cusp}^L(\pi) = 1$.

This was proved in [JL70] (over arbitrary global fields), and generalized to $GL_n$ in [Sha74]. The proof uses Whittaker models, in particular their local uniqueness (this generalizes to all quasi-split reductive groups) and the fact that a cusp form can be reconstructed from Whittaker functionals (this is particular to general linear groups).

**Theorem 5.3** (Strong multiplicity one). Let $\pi$ and $\pi'$ be cuspidal automorphic representations of $GL_2(\mathbb{A})$. Assume that there exists a finite set $S$ of prime numbers such that for all $p \notin S$ we have $\pi_p \simeq \pi'_p$. Then $\pi \simeq \pi'$.

See [PS79] for a proof using Kirillov models (related to the Whittaker models). Using Rankin-Selberg $L$-functions (again, relying on Whittaker models), a much more general result is proved in [JS81]. Morally, Čebotarev density theorem and linear independence of characters.

For inner forms, these methods do not adapt, essentially because there is Whittaker model (at all non-split places, and thus globally). Ultimately one can show strong multiplicity one results, but using the trace formula and after proving the local Jacquet-Langlands correspondence.

Nevertheless, Godement-Jacquet $L$-functions and $\epsilon$ factors [GJ72] (this theory generalizes the abelian case of Tate’s thesis and does not use Whittaker models).
can be used to prove the following weaker result, which will be crucial for the proof of the local Jacquet-Langlands correspondence. As usual we specialize to the cases relevant to this course.

**Theorem 5.4.** Let $D$ be a quaternion algebra over $\mathbb{Q}$, $G$ the associated inner form of $GL_2$. Let $S$ be a finite set of prime numbers, and $(\sigma_v)_{v \notin S}$ a collection of smooth irreducible representations of $G(\mathbb{Q}_v)$. Then $\sum \pi \mu^G(\pi) < \infty$ where the sum is over automorphic representations $\pi$ of $G(\mathbb{A})$ such that $\pi_v \simeq \sigma_v$ for all places $v \notin S$.

See [DKV84, Lemme B.1.e p. 80].

5.3. **Easy transfer.** To compare trace formulas we have to produce matching functions on different groups, i.e. functions which have the same orbital integrals (note that this requires an identification of conjugacy classes and of centralizers in the two groups).

**Lemma 5.5.** Let $p$ be a prime number. Let $\omega : Z(GL_2(\mathbb{Q}_p)) \to \mathbb{C}^\times$ be a smooth character. Recall that $T$ denotes a set of representatives for the (finitely many) conjugacy classes of maximal tori in $GL_2(\mathbb{Q}_p)$. Fix Haar measures on $PGL_2(\mathbb{Q}_p)$ and on each $T'/Z(GL_2(\mathbb{Q}_p))$ for $T' \in T$. Let $(F_{T'})_{T'T}$ be a family of smooth functions $T'_{G-reg} \to \mathbb{C}$ such that $F_{T'}$ is $\omega^{-1}$-equivariant, $N_{GL_2(\mathbb{Q}_p)}(T')$-invariant and compactly supported modulo $Z(GL_2(\mathbb{Q}_p))$. Then there exists $f \in C_c^\infty(GL_2(\mathbb{Q}_p), \omega^{-1})$ whose support is contained in the set of regular semisimple elements in $GL_2(\mathbb{Q}_p)$ such that for any $T' \in T$ and any $t \in T'_{G-reg}$ we have $O_t(f) = F(t)$, and for any $T' \in T$ such that $F_{T'}$ vanishes identically $f$ also vanishes on all elements conjugate to elements of $T'$.

Note that the assumption is that the support of $F_{T'}$ is a compact subset of $T'_{G-reg}/Z(GL_2(\mathbb{Q}_p))$, not just a relatively compact subset of $T'/Z(GL_2(\mathbb{Q}_p))$.

**Proof.** We use the functions $\phi_{T'}$ defined in Section 3.3. For $T' \in T$ let $U_{T'}$ be a non-empty compact open subset of $T'\backslash GL_2(\mathbb{Q}_p)$ such that $w'U_{T'} \cap U_{T'} = \emptyset$, where $w'$ is the non-trivial element of $N_{GL_2(\mathbb{Q}_p)}(T')$. Let $f(\phi_{T'}(t, \dot{g})) = \text{vol}(U_{T'})^{-1}F(t)/2$ if $\dot{g} \in U$, zero otherwise. Then

$$O_t(f) = \int_{T'\backslash GL_2(\mathbb{Q}_p)} f(g^{-1}tg) \, d\dot{g} = \int_{U_{T'} \cup w^*U_{T'}} f(g^{-1}tg) \, d\dot{g} = \frac{F(t) + F(tw')}{2} = F(t).$$

\[ \square \]

5.4. **Proof of the local Jacquet-Langlands correspondence.** We can finally prove Theorem 1.1. In this section $D$ will denote a non-split quaternion algebra over $\mathbb{Q}_p$. The first step is to associate to an essentially square-integrable representation $\tau_p$ of $GL_2(\mathbb{Q}_p)$ an irreducible representation of $D^\times$, satisfying the relation between Harish-Chandra characters. If $\tau_p \simeq (\chi_p \circ \det) \otimes \text{St}$, we have already seen (Corollary 3.22) that the representation $\chi \circ \det$ of $D^\times$ corresponds to $\tau_p$.

Thus we can assume that $\tau_p$ is supercuspidal. As in the Steinberg case, it is enough to prove the result with $\tau_p$ replaced by $(\chi_p \circ \det) \otimes \tau_p$ for some smooth character $\chi_p : \mathbb{Q}_p^\times \to \mathbb{C}^\times$. Let $\ell_1 \neq \ell_2$ be prime numbers distinct from $p$. Let $\omega : Z(\mathbb{Q}) \backslash Z(\mathbb{A}) \to \mathbb{C}^\times$ be a continuous unitary character such that $\omega|_{\mathbb{Q}_p^\times} = \omega_{\tau_p}$,
Let $\chi_p$ be one of the two unramified character of $\mathbb{Q}_p^\times$ such that $(\chi \circ \det |_{Z(\mathbb{Q}_p)}) \omega_{\ell_1} = \omega_p$. Let $\chi_{\ell_1} : \mathbb{Q}_p^\times \rightarrow \mathbb{C}^\times$ be one of the two unramified characters such that $\chi_{\ell_1} \circ \det |_{Z(\mathbb{Q}_p)} = \omega_{\ell_1}$. Choose an irreducible supercuspidal representation $\tau_{\ell_2}$ of $\text{GL}_2(\mathbb{Q}_p)$ having central character $\omega_{\ell_2}$ (Theorem 3.34). Let $D_{\text{glob}}$ be a quaternion algebra corresponding to $S = \{p, \ell_1\}$, and let $G$ be the associated inner form of $\text{GL}_2$. In particular we have an isomorphism $G(\mathbb{Q}_p) \simeq D^\times$, well-defined up to inner composing with an inner automorphism. Fix Haar measures on $G_{ad}(\mathbb{Q}_v)$ (for all places $v$, so that $G_{ad}(\mathbb{Z}_p)$ has volume 1 for almost all $p$) and endow $G_{ad}(\mathbb{A})$ with the product of Haar measures. We will apply the trace formula with functions $f_v \in C^\infty_c(G(\mathbb{Q}_v), \omega_v)$ as follows:

- $f_{\ell_1}$ is a coefficient for the representation $\chi_{\ell_1} \circ \det$ of $G(\mathbb{Q}_{\ell_1})$. To be explicit, we can take $f_{\ell_1}(g) = \text{vol}(G_{ad}(\mathbb{Q}_p))^{-1} \chi_{\ell_1}(\det g)^{-1}$.

- $f_{\ell_2}$ is a coefficient for $\tau_{\ell_2}$.

- $f_p$ is any smooth function which vanishes on $Z(\mathbb{Q}_p)$.

- for $v \notin \{p, \ell_1, \ell_2\}$, $f_v$ is arbitrary, we only impose that $f_p$ is the unit in $\mathcal{H}(G(\mathbb{Q}_p), G(\mathbb{Z}_p), \omega_p^{-1})$ for almost all prime numbers $p$.

Recall that there is a family $(\psi_v)_{v \notin \{p, \ell_1\}}$ of isomorphisms $\psi_v : G \times \mathbb{Q}_v \simeq \text{GL}_2(\mathbb{Q}_v)$, well-defined up to composition on the right with $\text{Ad}(g)$ for some $g \in \text{GL}_2(A(\mathbb{Q}_{\ell_1}))$ where $A(\mathbb{Q}_{\ell_1}) = \prod_{v \notin \{p, \ell_1\}} \mathbb{Q}_v = \mathbb{A}/\mathbb{Q}_p\mathbb{Q}_{\ell_1}$. Fix Haar measures on $\text{PGL}_2(\mathbb{Q}_v)$ for all places $v$, so that for any $v \notin \{p, \ell_1\}$, endow $\text{PGL}_2(\mathbb{Q}_v)$ with the Haar measure transported from that on $G_{ad}(\mathbb{Q}_v)$ via $\psi_v$. Endow $\text{PGL}_2(\mathbb{Q}_p)$ and $\text{PGL}_2(\mathbb{Q}_{\ell_1})$ with arbitrary Haar measures.

Now choose corresponding functions $f_v^\text{GL} \in C^\infty_c(\text{GL}_2(\mathbb{Q}_v), \omega_v^{-1})$ as follows:

- $f_{\ell_1}^\text{GL}$ is the pseudo-coefficient for the representation $(\chi_{\ell_1} \circ \det) \otimes \text{St}$ constructed in Proposition 3.26. Note that by Theorem 3.27 this implies that the orbital integrals of $f_{\ell_1}^\text{GL}$ are opposite to that of $f_{\ell_1}$. Note that there is an isomorphism between the centralizers $(\text{GL}_2)_{\ell_1}(Q_{\ell_1})$ and $G_{\ell_1}(Q_{\ell_1})$, well-defined up to normalizers, so we can transport Haar measures between $(\text{GL}_2)_{\ell_1}(Q_{\ell_1})/Z(Q_{\ell_1})$ and $G_{\ell_1}(Q_{\ell_1})/Z(Q_{\ell_1})$, and comparing orbital integrals makes sense.

- $f_p^\text{GL}$ is supported on the set of semisimple regular elliptic elements in $\text{GL}_2(\mathbb{Q})$ and such that for any semisimple regular $t \in \text{GL}_2(\mathbb{Q}_p)$ we have

\[
O_t(f_p^\text{GL}) = \begin{cases} -O_v(f_p) & \text{if } t' \in G(\mathbb{Q}_p) \text{ has same characteristic polynomial as } t \\ 0 & \text{if } t \text{ is hyperbolic.} \end{cases}
\]

The existence of such $f_p \in C^\infty_c(\text{GL}_2(\mathbb{Q}_p), \omega_p^{-1})$ follows from Lemma 5.5. The same remark as at the place $\ell_1$ applies for the comparison of Haar measures of centralizers.

- for any $v \notin \{\ell_1, p\}$, there is an isomorphism $\psi_v : \text{GL}_2(\mathbb{Q}_v) \simeq G \times \mathbb{Q}_v$, and we let $f_v^\text{GL} = f_v \circ \psi_v^{-1}$. In particular $f_v$ is trivial for almost all $v$, and the orbital integrals of $f_v$ do not depend on the choice of $\psi_v$. 

These choices were made so that the geometric sides of the traces formulas for $GL_2$ (Theorem 4.35) and $G$ (Theorem 4.18) are equal (again, the centralizers $G_\gamma$ and $(GL_2)_\gamma$ can be identified up to conjugation by the Weyl group, and the signs at $\ell_1$ and $p$ cancel each other so the global orbital integrals are equal). Therefore the spectral sides are equal:

$$\sum_{\pi} m^{GL_2}(\pi) \text{tr} \pi(f^{GL}) = \sum_{\pi'} m^{G}(\pi') \text{tr} \pi'(f).$$

Let $\sigma$ be a cuspidal automorphic representation of $GL_2(\mathbb{A})$ with central character $\omega$ such that

- $\sigma_p \simeq (\chi_p \circ \text{det}) \otimes \tau_p$,
- $\sigma_{\ell_1} \simeq (\chi_{\ell_1} \circ \text{det}) \otimes \text{St}$, and
- $\sigma_{\ell_2} \simeq \tau_{\ell_2}$.

The existence of such a $\sigma$ follows from Theorem 4.36.

Applying Lemma 5.1 (using $\psi_\infty$ to identify $GL_2(\mathbb{R})$ and $G(\mathbb{R})$), we obtain

$$\sum_{(\pi_q)_q} m^{GL_2}(\sigma_\infty \otimes q \otimes \pi_q) \prod_q \text{tr} \pi_q(f_q) = \sum_{(\pi'_q)_q} m^{G}(\sigma_\infty \otimes q \otimes \pi'_q) \prod_q \text{tr} \pi'_q(f_q)$$

where the products are over all prime numbers $q$. Note that both sides are traces in an admissible representation. As recalled at the beginning of Section 3.4, the theory of finite-dimensional representation of associative $\mathbb{C}$-algebras (for the Hecke algebra $\mathcal{H}(GL_2(\mathbb{A}^{\infty,p,\ell_1,\ell_2})),(\omega(\infty,p,\ell_1,\ell_2))^{-1})$ tells us that this implies

$$\sum_{(\pi_{p,\ell_1,\ell_2})} m^{GL_2}(\pi_p \otimes \pi_{\ell_1} \otimes \pi_{\ell_2} \otimes \bigotimes_{v \notin \{p,\ell_1,\ell_2\}} \sigma_v) \text{tr} \pi_p(f^{GL}) \text{tr} \pi_{\ell_1}(f^{GL}) \text{tr} \pi_{\ell_2}(f^{GL})$$

$$= \sum_{(\pi'_{p,\ell_1,\ell_2})} m^{G}(\pi'_{p} \otimes \pi'_{\ell_1} \otimes \pi'_{\ell_2} \otimes \bigotimes_{v \notin \{p,\ell_1,\ell_2\}} \sigma_v) \text{tr} \pi'_p(f) \text{tr} \pi'_{\ell_1}(f) \text{tr} \pi'_{\ell_2}(f)$$

Again this uses $(\psi_v)_{v \notin \{p,\ell_1,\ell_2,\infty\}}$. Since $f_{\ell_2}$ and $f_{\ell_2}^{GL}$ are coefficients for the supercuspidal representation $\pi_{\ell_2} \simeq \sigma_{\ell_2}$, this implies

$$\sum_{(\pi_{p,\ell_1})} m^{GL_2}(\pi_p \otimes \pi_{\ell_1} \otimes \bigotimes_{v \notin \{p,\ell_1\}} \sigma_v) \text{tr} \pi_p(f^{GL}) \text{tr} \pi_{\ell_1}(f^{GL})$$

$$= \sum_{(\pi'_{p,\ell_1})} m^{G}(\pi'_p \otimes \pi'_{\ell_1} \otimes \bigotimes_{v \notin \{p,\ell_1\}} \sigma_v) \text{tr} \pi'_p(f) \text{tr} \pi'_{\ell_1}(f)$$

By the same argument as in Theorem 4.36, the analogue of Corollary 4.24 for $GL_2$ implies that for any non-vanishing term on the left-hand side, $\pi_{\ell_1} \simeq \sigma_{\ell_1}$ (that is, $\pi_{\ell_1}$
is not isomorphic to $\chi_{\ell_1} \circ \det$, and so $\text{tr} \pi_{\ell_1}(f_{\ell_1}^{GL}) = 1$. On the right-hand side, any non-vanishing term has $\pi'_{\ell_1} \simeq \chi_{\ell_1} \circ \det$, and so $\text{tr} \pi'_{\ell_1}(f_{\ell_1}) = 1$. Therefore

$$\sum_{\pi_p} m_{\text{cusp}}^{GL_2} \left( \pi_p \otimes \bigotimes_{v \neq p} \sigma_v \right) \text{tr} \pi_p(f_p^{GL}) = \sum_{\pi_p'} m^{G} \left( \pi_p' \otimes \left( \chi_{\ell_1} \circ \det \right) \bigotimes_{v \notin \{p, \ell_1\}} \sigma_v \right) \text{tr} \pi'_p(f_p)$$

Thanks to the strong multiplicity one theorem (Theorem 5.3), we know that the left-hand side is simply $\text{tr} \sigma_p(f_p^{GL})$. The right-hand side is the trace of $f_p$ on a semisimple admissible smooth representation of $G(\mathbb{Q}_p)$. Thanks to Theorem 5.4 we know that it is in fact a finite length (i.e. finite-dimensional) representation, that we denote $\text{JL}(\sigma_p)$. It is indeed determined by $\sigma_p$ up to isomorphism because its trace is (there is a restriction on $f_p$, but $G(\mathbb{Q}_p) \setminus \mathbb{Z}(\mathbb{Q}_p)$ is an open and dense subset of $G(\mathbb{Q}_p)$). Namely, we have $\Theta_{\sigma_p}(t) = -\Theta_{\text{JL}(\sigma_p)}(t')$ for any semisimple regular $t \in GL_2(\mathbb{Q}_p)$ and $t' \in D^\times$ having the same characteristic polynomial (note the minus sign which comes from the definition of $f_p^{GL}$). We need to show that $\text{JL}(\sigma_p)$ is irreducible. This follows from elliptic orthogonality (Theorem 3.32 and its easier analogue for $D^\times$, which follows from the analogous Weyl integration formula): if $\text{JL}(\sigma_p) \simeq \bigoplus_i \rho_i^{\oplus m_i}$ with distinct irreducible $\rho_i$’s, comparing the two elliptic orthogonality relations we obtain $\sum_i m_i^2 = 1$. Elliptic orthogonality also implies that the map $\text{JL}$ is injective on isomorphism classes (an irreducible essentially square-integrable representation of $GL_2(\mathbb{Q}_p)$ is determined by the restriction of its Harish-Chandra character to the semisimple regular elliptic locus).

Finally we need to show that for any irreducible smooth representation $\rho$ of $D^\times$, there exists an irreducible essentially square-integrable representation $\tau_p$ of $GL_2(\mathbb{Q}_p)$ such that $\text{JL}(\tau_p) \simeq \rho$. The argument is almost the same as above, except that we start with an automorphic representation $\sigma$ of $G(\mathbb{A})$ such that $\sigma_p$ is a twist of $\rho$, $\sigma_{\ell_1} \simeq \chi_{\ell_1} \circ \det$ and $\sigma_{\ell_2}$ is supercuspidal (such a $\sigma$ exists thanks to Theorem 4.22). We obtain

$$\sum_{\pi_p} m_{\text{cusp}}^{GL_2} \left( \pi_p \otimes \left( \chi_{\ell_1} \circ \det \right) \otimes \text{St} \right) \otimes \bigotimes_{v \notin \{p, \ell_1\}} \sigma_v \text{tr} \pi_p(f_p^{GL}) = \sum_{\pi_p'} m^{G} \left( \pi_p' \otimes \bigotimes_{v \neq p} \sigma_v \right) \text{tr} \pi'_p(f_p)$$

By the strong multiplicity one theorem, the left-hand side has at most one non-zero term, for which the multiplicity is 1. The right-hand side is the trace of $f_p$ on a non-zero representation of $G(\mathbb{Q}_p)$ of finite length, in particular there exists $f_p$ such that the right-hand side does not vanish. Thus there exists a unique $\pi_p$ contributing to the left-hand side, and going back to the previous argument we have $\text{JL}(\pi_p) = \rho$. 

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